2 Rigid-Body Configuration Space

This chapter considers a freely moving rigid body, $\mathcal{B}$, surrounded by stationary rigid bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The body $\mathcal{B}$ represents an object being grasped by a robot hand. The surrounding bodies represent fingertips, or segments of a robot hand supporting an object $\mathcal{B}$ against gravity. The chapter introduces the notion of rigid-body configuration space, or *c-space*, which is essential for analyzing the mobility and stability of the object $\mathcal{B}$ with respect to the surrounding finger bodies. The chapter begins with a parametrization of $\mathcal{B}$'s c-space in terms of Euclidean coordinates, effectively transforming c-space from an abstract manifold to a familiar Euclidean space. Configuration space obstacles, or *c-obstacles*, are then introduced and several of their properties are described. The chapter proceeds to describe the first- and second-order geometric properties of the c-obstacles, as this geometry plays a key role in subsequent chapters.

2.1 The Notion of Configuration Space

The object $\mathcal{B}$ is assumed to be a *rigid body* that moves freely in Euclidean space $\mathbb{R}^n$, where $n = 2$ or 3. Rigidity implies that the distance between the object's points remains fixed as $\mathcal{B}$ moves in $\mathbb{R}^n$. The object's *configuration* specifies the position of its points in $\mathbb{R}^n$. The parametrization of $\mathcal{B}$'s configurations requires a selection of two frames, depicted in Figure 2.1. The first is a fixed *world frame*, denoted $\mathcal{F}_W$, which establishes a coordinate system for the physical space $\mathbb{R}^n$. The second is a *body frame*, denoted $\mathcal{F}_B$, which is rigidly attached to $\mathcal{B}$. The configuration of $\mathcal{B}$ is specified by a vector $d \in \mathbb{R}^n$

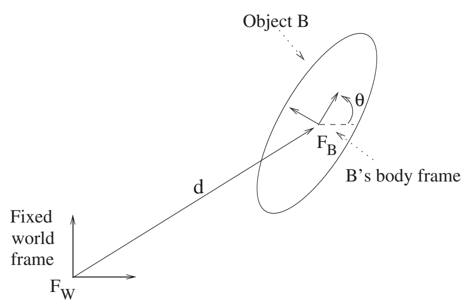

Figure 2.1 The basic setup for the c-space representation of a rigid object $\mathcal{B}$.

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describing the position of $\mathcal{F}_B$'s origin with respect to $\mathcal{F}_W$, and an *orientation matrix*, $R \in \mathbb{R}^{n\times n}$, whose columns describe the orientation of $\mathcal{F}_B$'s axes with respect to those of $\mathcal{F}_W$. The orientation matrices form a group under matrix multiplication, which is defined as follows (see Exercises).

DEFINITION 2.1 (**orientation matrices**) *The* $n \times n$ **orientation matrices** *form the special orthogonal group:*

$$SO(n) = \left\{R \in \mathbb{R}^{n\times n} : R^T R = I \quad \textit{and} \quad \det(R) = 1\right\},$$

where I *is the* $n \times n$ *identity matrix.*

The orientation matrices possess two important properties. First, every orientation matrix acts as a rotation on vectors $v \in \mathbb{R}^n$. That is, R preserves the norm of v: $\|Rv\| = (v^T R^T R v^T)^{1/2} = \|v\|$ for $v \in \mathbb{R}^n$. Second, $SO(n)$ is a smooth manifold of dimension $\frac{1}{2}n(n-1)$ in the space of $n \times n$ matrices.[1] In particular, $SO(2)$ is a compact *one-dimensional* manifold in the space of 2×2 matrices, while $SO(3)$ is a compact *three-dimensional* manifold in the space of 3×3 matrices. The practical meaning of this topological property is that one needs a single scalar, $\theta \in \mathbb{R}$, to parametrize $SO(2)$, and three scalars, $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$, to parametrize $SO(3)$.

Manifold structure of $SO(3)$: The manifold $SO(3)$ is topologically equivalent to a canonical three-dimensional manifold, the *projective space* RP^3. One way to construct RP^3 is to take the unit ball centered at the origin of $\mathbb{R}^3$ and identify antipodal points on its bounding sphere. The manifold RP^3 is path connected, compact and orientable. ○

The configuration space, or *c-space*, of the freely moving object $\mathcal{B}$ consists of pairs (d, R) as stated in the following definition.

DEFINITION 2.2 (**configuration space**) *The* **c-space** *of an* n*-dimensional rigid body* $\mathcal{B}$ *is the smooth manifold* $\mathcal{C} = \mathbb{R}^n \times SO(n)$*, consisting of pairs* (d, R) *such that* $d \in \mathbb{R}^n$ *and* $R \in SO(n)$*, where* $n = 2$ *or* 3.

The dimension of $\mathcal{C}$ is the sum: $m = n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$, giving $m = 3$ when $\mathcal{B}$ is two-dimensional (2-D) and $m = 6$ when $\mathcal{B}$ is three-dimensional (3-D). Every position and orientation of $\mathcal{B}$ is represented by a *point* in $\mathcal{C}$, while every continuous motion of $\mathcal{B}$ is represented by a *curve* in $\mathcal{C}$. However, practical analysis requires coordinates for the c-space manifold. Therefore, we shall introduce *global* coordinates for $\mathcal{C}$ in terms of a Euclidean space $\mathbb{R}^m$, with some periodicity rules for the coordinates representing the orientation matrices.

First consider the coordinates for the orientation matrices, which form a *matrix group*. A standard means for parametrizing such groups is by *exponential coordinates*:

$$R(\theta) = e^{[\theta\times]} \qquad \theta \in \mathbb{R}^{\frac{1}{2}n(n-1)},$$

[1] An m-dimensional manifold is a hypersurface, $\mathcal{M}$, such that at each point $p \in \mathcal{M}$ the manifold can be locally represented by Euclidean coordinates $\mathbb{R}^m$.

where the matrix exponential is defined by the series: $e^A = I + A + \frac{1}{2!}A^2 + \ldots$, and $[\theta\times]$ is the following skew-symmetric matrix.[2] In the case of $SO(2)$, the parameter $\theta \in \mathbb{R}$ represents the orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$ in $\mathbb{R}^2$. The 2×2 matrix $[\theta\times]$ is given by

$$[\theta\times] = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \qquad \theta \in \mathbb{R}.$$

In the case of $SO(3)$, the parameters $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ represent the orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$ in $\mathbb{R}^3$. The 3×3 matrix $[\theta\times]$ is given by

$$[\theta\times] = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \qquad \theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3.$$

The skew symmetric matrix $[\theta\times]$ acts as a *cross-product* on vectors $v \in \mathbb{R}^3$: $[\theta\times]v = \theta \times v$ for $v \in \mathbb{R}^3$. Hence, it is called the *cross-product matrix*. When the skew symmetric matrices are substituted into the exponential series, one obtains the following parametrization of $SO(n)$.

THEOREM 2.1 (**exponential coordinates for $\boldsymbol{SO(n)}$**) *The 2×2 orientation matrices are globally parametrized by $\theta \in \mathbb{R}$, according to the formula:*

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta \in \mathbb{R},$$

where θ is measured using the right-hand rule (Figure 2.1). The 3×3 orientation matrices are globally parametrized by $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ according to Rodrigues' formula:

$$R(\theta) = I + \sin(\|\theta\|)[\hat{\theta}\times] + \big(1 - \cos(\|\theta\|)\big)[\hat{\theta}\times]^2 \qquad \theta \in \mathbb{R}^3$$

where I is the 3×3 identity matrix, $[\hat{\theta}\times]$ is a 3×3 cross-product matrix, and $\hat{\theta} = \theta/\|\theta\|$.

The parametrization of $SO(2)$ in terms of $\theta \in \mathbb{R}$ is periodic in 2π, with each 2π interval parametrizing the entire manifold $SO(2)$ (Figure 2.2). In Rodrigues' formula, the unit vector $\hat{\theta}$ represents the axis of rotation of $R(\theta)$, while the scalar $\|\theta\|$ corresponds to the angle of rotation about this axis according to the right-hand rule. The parametrization of $SO(3)$ in terms of $\theta \in \mathbb{R}^3$ satisfies the following periodicity rule. The origin $\theta = \vec{0}$ is mapped to the identity orientation matrix I. Similarly, all concentric spheres of radius $\|\theta\| = 2\pi, 4\pi, \ldots$ are mapped to I. Each pair of antipodal points on the sphere of radius $\|\theta\| = \pi$ is mapped to the same matrix R, since $R(\pi\hat{\theta}) = R(-\pi\hat{\theta})$ for all $\hat{\theta}$ (similarly, antipodal points on the spheres of radius $\|\theta\| = 3\pi, 5\pi, \ldots$ are identified). Since $\hat{\theta}$ can have any direction in $\mathbb{R}^3$, the manifold $SO(3)$ is fully parametrized by the closed ball of radius π centered at the origin, with antipodal points on its bounding sphere identified. Note that this rule matches the definition of RP^3, thus providing a constructive proof that $SO(3)$ is topologically equivalent to RP^3.

The pair (d, θ) provides a *global* parametrization of the c-space manifold $\mathcal{C} = \mathbb{R}^n \times SO(n)$ in terms of c-space coordinates, $q \in \mathbb{R}^m$, as stated in the following definition.

[2] The skew-symmetric matrices $[\theta\times]$ form the *Lie algebra* of the matrix group $SO(n)$.

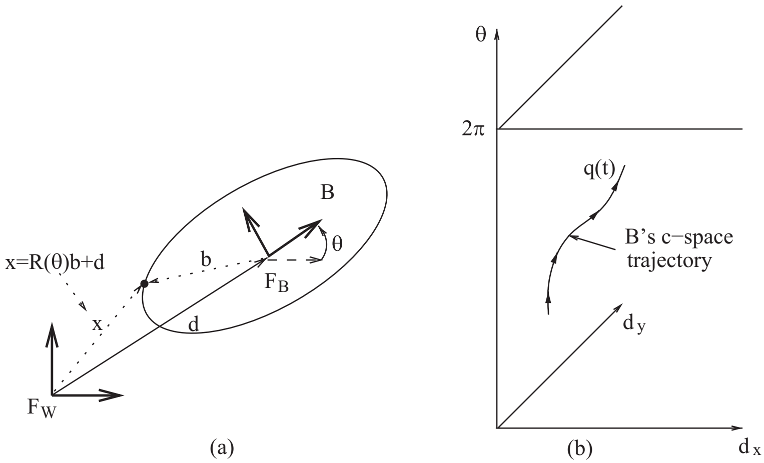

Figure 2.2 (a) The c-space coordinates $q = (d_x, d_y, \theta)$ of a 2-D object $\mathcal{B}$, where θ is periodic in 2π. (b) A physical motion of $\mathcal{B}$ is represented by a c-space trajectory in $\mathbb{R}^m$.

DEFINITION 2.3 (**c-space coordinates**) *When $\mathcal{B}$ is a 2-D body, its* **c-space coordinates** *are $q = (d, \theta) \in \mathbb{R}^3$, where $d = (d_x, d_y) \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$. When $\mathcal{B}$ is a 3-D body, its* **c-space coordinates** *are $q = (d, \theta) \in \mathbb{R}^6$, where $d = (d_x, d_y, d_z) \in \mathbb{R}^3$ and $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$.*

The c-space coordinates allow us to model the physical motions of $\mathcal{B}$ as trajectories of a *point* in $\mathbb{R}^m$, where $m = 3$ for a 2-D object and $m = 6$ for a 3-D object. For simplicity we will refer to $\mathbb{R}^m$ as the *c-space* of $\mathcal{B}$. Before we discuss how the finger bodies appear as forbidden regions in $\mathcal{B}$'s c-space, let us review the key notion of *rigid-body transformation.*

DEFINITION 2.4 (**rigid-body transformation**) *When a rigid body $\mathcal{B}$ is located at a configuration q, the position of its point $b \in \mathcal{B}$ expressed in $\mathcal{F}_B$ relative to $\mathcal{F}_W$ is given by the* **rigid-body transformation**, *$X(q, b) : \mathbb{R}^m \times \mathcal{B} \to \mathbb{R}^n$, according to the formula* (Figure 2.2(a)):

$$X(q, b) = R(\theta)b + d \qquad q = (d, \theta) \in \mathbb{R}^m,\ b \in \mathcal{B}$$

where $m = 3$ in 2-D and $m = 6$ in 3-D.

We will occasionally use the notation $X_b(q)$ to specify the rigid-body transformation with the point $b \in \mathcal{B}$ held fixed on $\mathcal{B}$. In this case, $X_b(q)$ gives the world position of the fixed point b as a function of $\mathcal{B}$'s configuration q.

2.2 Configuration Space Obstacles

Rigid bodies cannot interpenetrate in physical space. Hence, when a rigid object $\mathcal{B}$ is surrounded by stationary rigid finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$, the finger bodies form obstacles that constrain the object's possible motions. The finger bodies induce the following forbidden regions in $\mathcal{B}$'s c-space, called *c-obstacles.*

DEFINITION 2.5 (**c-obstacle**) *Let $\mathcal{B}(q)$ be the set of points in $\mathbb{R}^n$ occupied by $\mathcal{B}$ at a configuration q. The* **c-obstacle** *induced by a stationary finger body $\mathcal{O}$, denoted $\mathcal{CO}$, is the set of configurations at which $\mathcal{B}(q)$ intersects $\mathcal{O}$:*

$$\mathcal{CO} = \left\{ q \in \mathbb{R}^m : \mathcal{B}(q) \cap \mathcal{O} \neq \emptyset \right\}$$

where $m = 3$ for a 2-D object $\mathcal{B}$ and $m = 6$ for a 3-D object $\mathcal{B}$.

When $\mathcal{B}$ is a full body with nonempty interior, its c-obstacle $\mathcal{CO}$ occupies an m-dimensional set in $\mathcal{B}$'s c-space $\mathbb{R}^m$, even when $\mathcal{O}$ is a point obstacle. The c-obstacle boundary, denoted $\mathcal{S}$, forms a piecewise smooth $(m-1)$-dimensional manifold whose points satisfy the following property.

LEMMA 2.2 (**c-obstacle boundary**) *The* **c-obstacle boundary** *consists of configurations q at which $\mathcal{B}(q)$ touches $\mathcal{O}$ strictly from the outside:*

$$\mathcal{S} = \{ q \in \mathbb{R}^m : \mathcal{B}(q) \cap \mathcal{O} \neq \emptyset \ \text{ and } \ \mathsf{int}(\mathcal{B}(q)) \cap \mathsf{int}(\mathcal{O}) = \emptyset \},$$

where int *denotes set interior.*

In the case of 2-D bodies, one can conceptually construct the c-obstacle surface as follows. First fix the orientation of $\mathcal{B}$'s reference frame to a particular θ. Then slide $\mathcal{B}$ along the perimeter of $\mathcal{O}$ with fixed orientation, making sure that $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$. The trace of $\mathcal{B}$'s frame origin during this circumnavigation forms a closed curve. The resulting curve is the fixed-θ slice of $\mathcal{S}$. When this process is repeated for all θ, the resulting stack of loops forms the c-obstacle boundary.

Example: Figure 2.3 shows an elliptical object $\mathcal{B}$ moving in a planar environment in the presence of a stationary disc finger $\mathcal{O}$. The c-obstacle $\mathcal{CO}$ is depicted in Figure 2.3(a) for a choice of $\mathcal{F}_B$'s origin at the ellipse's center. The c-obstacle forms a spiraling stack of 2-D ovals. Each oval is formed by sliding the object $\mathcal{B}$ about $\mathcal{O}$

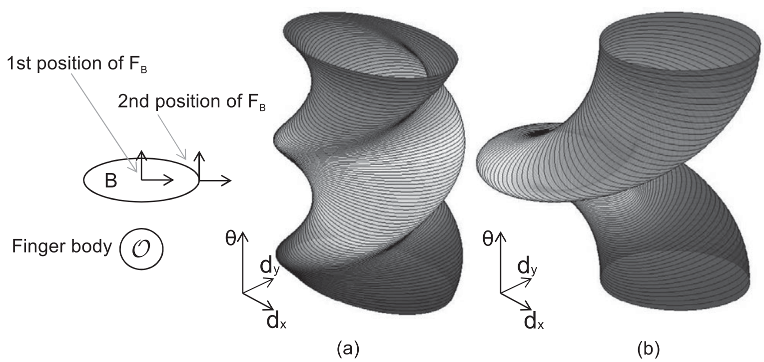

Figure 2.3 The c-obstacle surface induced by a stationary disc representing a finger body, shown for two choices of $\mathcal{B}$'s reference frame: (a) at the ellipse's center, and (b) at the ellipse's tip.

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with fixed orientation, and the spiraling ovals represent the 2π periodicity of the θ axis. The same c-obstacle is depicted in Figure 2.3(b) for a choice of $\mathcal{F}_B$'s origin at the tip of the ellipse's major axis. While the c-obstacle geometric shape (surface normal and curvature) has changed, it is topologically equivalent to the c-obstacle depicted in Figure 2.3(a). This observation holds under all choices of the reference frames $\mathcal{F}_W$ and $\mathcal{F}_B$. ○

A detailed discussion of the c-obstacles can be found in on robot motion planning texts; see Bibliographical Notes. The following are some key properties of the c-obstacles.

1. **Connectivity and compactness propagate.** When $\mathcal{B}$ is compact and connected, a compact and connected finger body $\mathcal{O}$ induces a compact and connected c-obstacle $\mathcal{CO}$.
2. **Union propagates.** When $\mathcal{B}$ is a union of two sets, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, the c-obstacle induced by a finger body $\mathcal{O}$ is a union of the c-obstacles corresponding to the two sets, $\mathcal{CO} = \mathcal{CO}_1 \cup \mathcal{CO}_2$.
3. **Convexity propagates.** A set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be *convex* when every pair of points in $\mathcal{A}$ can be connected by a line segment lying in $\mathcal{A}$. When $\mathcal{B}$ and $\mathcal{O}$ are convex bodies, each fixed-orientation slice of $\mathcal{CO}$ forms a convex set.
4. **Polygonality propagates.** When $\mathcal{B}$ and $\mathcal{O}$ are polygonal 2-D bodies, each fixed-orientation slice of $\mathcal{CO}$ forms a polygonal set. When $\mathcal{B}$ and $\mathcal{O}$ are polyhedral 3-D bodies, each fixed-orientation slice of $\mathcal{CO}$ forms a polyhedral set.

Parametrization of c-obstacle boundary: Explicit parametrization of the c-obstacle boundary is available when $\mathcal{B}$ and $\mathcal{O}$ are convex bodies. This example considers a 2-D smooth convex object $\mathcal{B}$ and a disc finger $\mathcal{O}$ centered at x_0 of radius r. Let $\beta(s)$ for $s \in \mathbb{R}$ be a counterclockwise parametrization of $\mathcal{B}$'s perimeter in its body frame $\mathcal{F}_B$, such that the tangent $\beta'(s)$ is a unit vector. Note that $J\beta'(s)$ is the unit *outward* normal to $\mathcal{B}$ in $\mathcal{F}_B$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. When $\mathcal{B}$ touches $\mathcal{O}$ at a point $x(s,q) = R(\theta)\beta(s) + d$ where $q = (d, \theta)$, the two bodies share collinear contact normals:

$$\frac{1}{r}(x - x_0) = -R(\theta)J\beta'(s). \tag{2.1}$$

It follows from Eq. (2.1) that $x(s,q) = x_0 - rR(\theta)J\beta'(s)$. Substituting for x in the rigid-body transformation, then solving for d in terms of s and θ gives

$$d(s,\theta) = x(s,\theta) - R(\theta)\beta(s) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big).$$

The function $\varphi(s,\theta) : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(s,\theta) = \begin{pmatrix} d(s,\theta) \\ \theta \end{pmatrix} \qquad d(s,\theta) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big) \tag{2.2}$$

parametrizes the c-obstacle boundary in terms of s and θ. The curves depicted on the c-obstacle surfaces in Figure 2.3 are fixed-θ curves of the parametrization (2.2). ○

The c-obstacle boundary, $\mathcal{S}$, inherits smoothness properties from those of $\mathcal{B}$ and $\mathcal{O}$. When both bodies are smooth and convex, $\mathcal{S}$ forms a smooth $(m-1)$-dimensional

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manifold in $\mathbb{R}^m$. More generally, $\mathcal{S}$ is locally smooth at any configuration $q \in \mathcal{S}$ at which $\mathcal{B}(q)$ touches $\mathcal{O}$ at a single point, such that the two bodies possess smooth boundaries at the contact point. When the two bodies are piecewise smooth, for instance when $\mathcal{B}$ and $\mathcal{O}$ are polygons, $\mathcal{S}$ becomes piecewise smooth. In the latter case $\mathcal{S}$ consists of smooth $(m-1)$-dimensional "patches," meeting along lower-dimensional manifolds. For instance, when $\mathcal{B}$ is a convex polygon and $\mathcal{O}$ is a disc, $\mathcal{S}$ consists of surface patches generated by an edge of $\mathcal{B}$ sliding on $\mathcal{O}$, and surface patches generated by a vertex of $\mathcal{B}$ sliding on $\mathcal{O}$. Similar observations hold for the five-dimensional boundary of $\mathcal{CO}$ in the 3-D case.

Example – Smoothness of c-obstacle boundary: Consider the previous example of a 2-D convex object $\mathcal{B}$ and a disc finger $\mathcal{O}$. Using the boundary parametrization of Eq. (2.2), let us verify that the c-obstacle boundary forms a smooth surface in $\mathcal{B}$'s c-space. It suffices to show that the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial\theta}\varphi(s,\theta)$ are linearly independent and thus span a well defined tangent plane at every point $q=\varphi(s,\theta)\in\mathcal{S}$. The tangent vector $\frac{\partial}{\partial s}\varphi(s,\theta)$ is given by

$$\frac{\partial}{\partial s}\varphi(s,\theta)=\begin{pmatrix}-R(\theta)\big(\beta'(s)+rJ\beta''(s)\big)\\0\end{pmatrix}=-(1+r\kappa_{\mathcal{B}}(s))\begin{pmatrix}R(\theta)\beta'(s)\\0\end{pmatrix},$$

where we used the fact that $J\beta''(s)$ is collinear with $\beta'(s)$, and that $\kappa_{\mathcal{B}}(s)=\beta'(s)\cdot J\beta''(s)$ is the *curvature* of $\mathcal{B}$ at $\beta(s)$ (more details on curvature are provided in Section 2.4). The tangent vector $\frac{\partial}{\partial\theta}\varphi(s,\theta)$ is given by

$$\frac{\partial}{\partial\theta}\varphi(s,\theta)=\begin{pmatrix}JR(\theta)\big(\beta(s)+rJ\beta'(s)\big)\\1\end{pmatrix}=\begin{pmatrix}JR(\theta)\beta(s)\\1\end{pmatrix}-r\begin{pmatrix}R(\theta)\beta'(s)\\0\end{pmatrix},$$

where we used the identities $R'(\theta)=-JR(\theta)=-R(\theta)J$ and $J^2=-I$. Since $\kappa_{\mathcal{B}}(s)>0$ for a convex object, $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial\theta}\varphi(s,\theta)$ are linearly independent and the c-obstacle boundary forms a smooth surface in c-space (see Figure 2.3). ○

2.3 The C-Obstacle Normal

When a rigid object $\mathcal{B}$ is contacted by stationary rigid finger bodies $\mathcal{O}_1,\ldots,\mathcal{O}_k$, the object's configuration q lies at the intersection of the finger c-obstacle boundaries. We will see in Part II of the book that the free motions of $\mathcal{B}$ are determined in this case by the first and second-order geometry of the c-obstacle boundaries – i.e., by their normals and curvatures at q. This section derives a formula for the *c-obstacle normal,* while the next section derives a formula for the *c-obstacle curvature.*

We shall assume that the object $\mathcal{B}$ touches a stationary body $\mathcal{O}$ at a *single point,* such that the two bodies have locally smooth boundaries at the contact point. Under this assumption, the c-obstacle boundary is locally smooth and has a well-defined normal.[3]

[3] The c-obstacle normal is well defined even when one of the contacting bodies is non-smooth at the contact point, provided that the other body has a smooth boundary at this point.

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In order to compute the c-obstacle normal, consider the following inter-body distance function.

DEFINITION 2.6 (**C-obstacle distance function**) *Let $\mathcal{O}$ be a stationary body in $\mathbb{R}^n$, and let $\mathsf{dst}(x,\mathcal{O}) : \mathbb{R}^n \to \mathbb{R}$ be the minimal distance of a point x from $\mathcal{O}$: $\mathsf{dst}(x,\mathcal{O}) = \min_{y\in\mathcal{O}}\{\|x-y\|\}$. The* **c-obstacle function,** *$o(q) : \mathbb{R}^m \to \mathbb{R}$, is the minimal distance between $\mathcal{B}(q)$ and $\mathcal{O}$:*

$$o(q) = \min_{x\in\mathcal{B}(q)} \{\mathsf{dst}(x,\mathcal{O})\},$$

where $\mathcal{B}(q)$ is the set of points $\mathbb{R}^n$ occupied by $\mathcal{B}$ at a configuration q.

The c-obstacle distance function $o(q)$ is identically zero in the interior of the c-obstacle $\mathcal{CO}$ and is strictly positive outside $\mathcal{CO}$. Hence $\mathcal{CO} = \{q \in \mathbb{R}^m : o(q) \leq 0\}$. If $o(q)$ would have been differentiable at $q \in \mathcal{S}$, its gradient $\nabla o(q)$ would be collinear with the c-obstacle outward normal at q. But $o(q)$ is identically zero inside $\mathcal{CO}$ and monotonically increasing away from $\mathcal{CO}$, implying that it is *non-differentiable* on the c-obstacle boundary $\mathcal{S}$. However, $o(q)$ is Lipschitz continuous and can be analyzed with tools that resemble the classical ones (Lipschitz continuity and other relevant aspects of non-smooth analysis are reviewed in Appendix A). Lipschitz continuous functions are piecewise smooth, and they possess a *generalized gradient* at points were the function is non-differentiable. The *generalized gradient* of f at x, denoted $\partial f(x)$, is the convex combination of the gradients $\nabla f(y)$ at points y that approach x from all sides (see Appendix A). In particular, $\partial f(x)$ reduces to the usual gradient $\nabla f(x)$ at points where f is differentiable.

Let us compute the generalized gradient of $o(q)$ and see how it determines the c-obstacle normal. To emphasize that only q is a free variable in $o(q)$, we write:

$$o(q) = \min_{b\in\mathcal{B}} \left\{\mathsf{dst}\big(X_b(q),\mathcal{O}\big)\right\},$$

where $X_b(q)$ is the rigid-body transformation specified in Definition 2.4. According to Property (3) in Appendix A, $o(q)$ is Lipschitz continuous when its constituent functions, $\mathsf{dst}\big(X_b(q),\mathcal{O}\big)$ for $b \in \mathcal{B}$, are Lipschitz continuous. The rigid-body transformation $X_b(q)$ is smooth and therefore Lipschitz continuous in q. The minimal distance function, $\mathsf{dst}(x,\mathcal{O})$, is shown in Appendix A to be Lipschitz continuous in x. Since Lipschitz continuity is preserved under function composition, the functions $\mathsf{dst}(X_b(q),\mathcal{O})$ are Lipschitz continuous. Hence, $o(q)$ is Lipschitz continuous and, therefore, possesses a *generalized gradient,* $\partial o(q)$, which determines the c-obstacle normal. As discussed in Appendix A, $\partial o(q)$ is the convex combination of the generalized gradient of the functions $\mathsf{dst}\big(X_b(q),\mathcal{O}\big)$ that attain the minimum distance at q. When $\mathcal{B}(q)$ touches $\mathcal{O}$ at a *single point,* one obtains the following corollary.

COROLLARY 2.3 *Let $\mathcal{B}(q)$ contact a stationary body $\mathcal{O}$ at a single point $b_0 \in \mathcal{B}$. Then $o(q)$ is attained by a single function, $o(q) = \mathsf{dst}(X_{b_0}(q),\mathcal{O})$, and the generalized gradient of $o(q)$ is given by $\partial o(q) = \partial\mathsf{dst}(X_{b_0}(q),\mathcal{O})$.*

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The computation of $\partial o(q)$ thus reduces to the computation of $\partial \mathsf{dst}(X_{b_0}(q),\mathcal{O})$. The function $\mathsf{dst}(X_{b_0}(q),\mathcal{O})$ is a composition of $\mathsf{dst}(x,\mathcal{O})$ with $x(q)=X_{b_0}(q)$. The generalized gradient of such a composition can be computed using a *generalized chain rule,* described in Appendix A. It specifies that $\partial \mathsf{dst}\big(X_{b_0}(q),\mathcal{O}\big)=\partial \mathsf{dst}(x_0,\mathcal{O})\cdot DX_{b_0}(q)$, where $x_0=X_{b_0}(q)$ is the position of the contact point b_0 in the world frame $\mathcal{F}_W$, and $DX_{b_0}(q)$ is the Jacobian matrix of $X_{b_0}(q)$. The resulting formula for $\partial o(q)$ is as follows.

PROPOSITION 2.4 (**generalized gradient of $\boldsymbol{o(q)}$**) *The generalized gradient of $o(q)$ at $q\in\mathcal{S}$ is a line segment based at q:*

$$\partial o(q)=s\cdot\begin{pmatrix} n(x_0)\\ R(\theta)b_0\times n(x_0)\end{pmatrix}\qquad 0\le s\le 1, \tag{2.3}$$

where $n(x_0)$ is the unit normal pointing out of $\mathcal{O}$ and into $\mathcal{B}$ at the contact point x_0.

Proof: Based on the generalized chain rule:

$$\partial o(q)=\partial \mathsf{dst}\big(X_{b_0}(q),\mathcal{O}\big)=\partial \mathsf{dst}(x_0,\mathcal{O})\cdot DX_{b_0}(q). \tag{2.4}$$

The Jacobian $DX_{b_0}(q)$ is given by (see Exercises): $DX_{b_0}(q)=\big[I\ \ -R(\theta)b_0\times\big]$. As shown in Appendix A, the generalized gradient of $\mathsf{dst}(x_0,\mathcal{O})$ is a line segment based at x_0:

$$\partial \mathsf{dst}(x_0,\mathcal{O})=s\cdot n(x_0)\qquad 0\le s\le 1,$$

where $n(x_0)$ is $\mathcal{O}$'s outward unit normal at x_0. Substituting for $DX_{b_0}(q)$ and $\partial \mathsf{dst}(x_0,\mathcal{O})$ in Eq. (2.4), then taking the transpose so that $\partial o(q)$ would become a column vector, gives Formula (2.3) for $\partial o(q)$. □

The generalized gradient of $o(q)$ forms a line segment based at $q\in\mathcal{S}$. This line segment points *outward* with respect to $\mathcal{CO}$, since $o(q)$ increases away from $\mathcal{CO}$. The c-obstacle normal is simply the *direction* of this line segment, as summarized in the following theorem, which uses the notation b instead of b_0 and x instead of x_0.

THEOREM 2.5 (**c-obstacle normal**) *Let a freely moving rigid body $\mathcal{B}$ located at q contact a stationary rigid body $\mathcal{O}$. The* **c-obstacle outward normal** *at $q\in\mathcal{S}$, denoted $\eta(q)$, is given by*

$$\eta(q)=DX_b^T(q)n(x)=\begin{pmatrix} n(x)\\ R(\theta)b\times n(x)\end{pmatrix}, \tag{2.5}$$

where b is $\mathcal{B}$'s contact point with $\mathcal{O}$ expressed in $\mathcal{F}_B$, $x=X_b(q)$ is the contact point expressed in $\mathcal{F}_W$, and $n(x)$ is the unit contact normal at x, pointing outward with respect to $\mathcal{O}$ at x.[4]

Proof: Consider a c-space path $\alpha(t)$ that lies in $\mathcal{S}$, such that $\alpha(0)=q$. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along α, $o(\alpha(t))=0$ along this path. By the generalized chain rule, $\frac{d}{dt}o(\alpha(t))=\partial o(\alpha(t))\cdot\frac{d}{dt}\alpha(t)=0$ for $t\in\mathbb{R}$. Since $\frac{d}{dt}\big|_{t=0}\alpha(t)$ is an arbitrary

[4] In the 2-D case, $Rb\times n=(Rb)^TJn$, where $J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

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tangent vector to $\mathcal{S}$ at q, the line segment $\partial o(q)$ is perpendicular to the tangent space $T_q\mathcal{S}$. This line segment points outward with respect to $\mathcal{CO}$. The vector $\eta(q)$ specified in Eq. (2.5), obtained by substituting $s = 1$ in $\partial o(q)$, therefore forms the c-obstacle outward normal at q. □

We will see in the next chapter that the c-obstacle normal $\eta(q)$ can be interpreted as the generalized force, or *wrench*, generated by a unit-magnitude normal force acting on $\mathcal{B}$ at x. The vanishing of the product: $\partial o(\alpha(t)) \cdot \frac{d}{dt}\alpha(t) = 0$, reflects the physical fact that a normal contact force does no work along any contact preserving motion of $\mathcal{B}$ as it slides along the boundary of the stationary body $\mathcal{O}$.

Example: Consider the parametrization $\varphi(s,\theta)$ of the c-obstacle boundary $\mathcal{S}$, obtained in Eq. (2.2). We already verified that the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial \theta}\varphi(s,\theta)$ span the tangent plane to $\mathcal{S}$ at $q = \varphi(s,\theta)$. The cross-product of the two tangent vectors should therefore be collinear with $\eta(q)$. A straightforward calculation gives

$$\frac{\partial}{\partial s}\varphi(s,\theta) \times \frac{\partial}{\partial \theta}\varphi(s,\theta) = \begin{pmatrix} -JR(\theta)\beta'(s) \\ (JR(\theta)\beta'(s)) \cdot JR(\theta)\beta(s) \end{pmatrix} = \begin{pmatrix} n(x) \\ (R(\theta)b)^T J n(x) \end{pmatrix},$$

where $b = \beta(s)$, and $n(x) = -JR(\theta)\beta'(s)$ (since $JR(\theta) = R(\theta)J$, and $-J\beta'(s)$ is $\mathcal{B}$'s inward unit normal at b expressed in $\mathcal{F}_B$). The c-obstacle normal computed from $\varphi(s,\theta)$ thus matches formula (2.5) for $\eta(q)$. ○

2.4 The C-Obstacle Curvature

As mentioned in the previous section, when a rigid object $\mathcal{B}$ is held by stationary rigid finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$, the free motions of $\mathcal{B}$ are determined by the first- and second-order geometry of the finger c-obstacles. The first-order geometry corresponds to the c-obstacle normal; the second-order geometry corresponds to the c-obstacle curvature, which is studied in this section.

We will develop the *c-obstacle curvature formula* for a freely moving 2-D object $\mathcal{B}$ and a stationary 2-D body $\mathcal{O}$. The 3-D version of the c-obstacle curvature formula is summarized at the end of this section. In the 2-D case, the c-obstacle boundary forms a surface in $\mathbb{R}^3$. The curvature of this surface depends on the curvature of the contacting bodies for which we need to introduce notation. First consider the stationary body $\mathcal{O}$. As before, n denotes the outward unit normal to $\mathcal{O}$ at the contact point x. Let $x(t)$ parametrize the boundary of $\mathcal{O}$, such that $x(0) = x$ and $\frac{d}{dt}\big|_{t=0} x(t) = \dot{x}$. The *curvature* of $\mathcal{O}$ at x, denoted $\kappa_{\mathcal{O}}(x)$, is the scalar measuring the change of the unit normal $n(x)$ along $x(t)$:

$$\left.\frac{d}{dt}\right|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}(x)\dot{x}.$$

Note that the change in $n(x)$ is tangent to $\mathcal{O}$'s boundary at x, since $\|n(x)\| = 1$ along $x(t)$. The sign of $\kappa_{\mathcal{O}}(x)$ is positive when $\mathcal{O}$ is convex at x, negative when $\mathcal{O}$ is concave at x and zero when $\mathcal{O}$ is flat at x. The *radius of curvature* of $\mathcal{O}$ at x, denoted $r_{\mathcal{O}}(x)$, is

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the reciprocal of the curvature, $r_{\mathcal{O}}(x) = 1/\kappa_{\mathcal{O}}(x)$. The *circle of curvature* of $\mathcal{O}$ at x is the circle of radius $|r_{\mathcal{O}}(x)|$ tangent to $\mathcal{O}$ at x. It forms the boundary's second-order approximation at x. The curvature of $\mathcal{B}$ is similarly defined with respect to its body frame $\mathcal{F}_{\mathcal{B}}$. The curvature of $\mathcal{B}$ at a boundary point b is the signed scalar $\kappa_{\mathcal{B}}(b)$, and its radius of curvature is $r_{\mathcal{B}}(b) = 1/\kappa_{\mathcal{B}}(b)$. In the following discussion, $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ will be used for $\kappa_{\mathcal{B}}(b)$ and $\kappa_{\mathcal{O}}(x)$.

The curvature of the c-obstacle boundary is defined as follows. Denote by $T_q\mathcal{S}$ the tangent plane to the c-obstacle boundary $\mathcal{S}$ at q, and recall that $\eta(q)$ denotes the c-obstacle outward normal at $q \in \mathcal{S}$. Let $\hat{\eta}(q) = \eta(q)/\|\eta(q)\|$ be the unit-magnitude outward normal, and let $q(t)$ be a c-space path that lies in $\mathcal{S}$, such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. The *curvature form* of $\mathcal{S}$ at q, denoted $\kappa(q, \dot{q})$, measures the change in $\hat{\eta}(q)$ along tangent directions $\dot{q} \in T_q\mathcal{S}$:

$$\kappa(q,\dot{q}) = \dot{q} \cdot \frac{d}{dt}\bigg|_{t=0} \hat{\eta}(q(t)) = \dot{q} \cdot D\hat{\eta}(q)\dot{q} \qquad \dot{q} \in T_q\mathcal{S}. \tag{2.6}$$

The curvature $\kappa(q, \dot{q})$ is a *quadratic form* that acts on tangent vectors $\dot{q} \in T_q\mathcal{S}$. Since $T_q\mathcal{S}$ is a two-dimensional subspace of the ambient tangent space $T_q\mathbb{R}^3$, $D\hat{\eta}(q)$ acts as a 2×2 symmetric matrix on $T_q\mathcal{S}$. The eigenvalues and eigenvectors of $D\hat{\eta}(q)$ are the *principal curvatures* and *principal directions of curvature* of $\mathcal{S}$ at q. The principal curvatures are analogous to the curvature of a planar curve. That is, the principal curvatures can be used to construct a quadratic surface tangent to $\mathcal{S}$ at q, which forms the second-order approximation of $\mathcal{S}$ at q.

We now turn to the task of computing the c-obstacle curvature. When the object $\mathcal{B}$ moves along a c-space path $q(t)$ that lies in $\mathcal{S}$, it maintains continuous contact with the stationary body $\mathcal{O}$. Let $x(t)$ be the contact point of $\mathcal{B}$ with $\mathcal{O}$ along $q(t)$, expressed in the world frame $\mathcal{F}_W$. Since $\eta(q) = DX_b^T(q)n(x)$ according to Theorem 2.5, the derivative of $\hat{\eta}(q)$ along $q(t)$ involves the contact point velocity, $\dot{x}(t)$, along the c-space path $q(t)$. The contact point velocity depends on the curvatures of $\mathcal{B}$ and $\mathcal{O}$, as stated in the following proposition.

PROPOSITION 2.6 (**contact point velocity**) *Let $q(t)$ be a c-space path that lies in $\mathcal{S}$, and let $x(t)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$ along $q(t)$, expressed in the world frame $\mathcal{F}_W$. The* **contact point velocity** *along $q(t)$ is given by*

$$\dot{x}(t) = \tfrac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I & -JR(\theta)b_c \end{bmatrix} \dot{q}(t), \tag{2.7}$$

where $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at $x(t)$, b_c is $\mathcal{B}$'s center of curvature at $x(t)$ expressed in $\mathcal{F}_{\mathcal{B}}$, I is a 2×2 identity matrix and $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

The proof of the proposition appears in the chapter's appendix. The object's *center of curvature, b_c*, is the center of $\mathcal{B}$'s circle of curvature at x. The denominator, $\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}$, is positive semi-definite. For instance, when a concave body $\mathcal{O}$ touches a convex object $\mathcal{B}$ at x, $r_{\mathcal{B}} \le |r_{\mathcal{O}}|$; otherwise the two bodies would interpenetrate. In this case, $|\kappa_{\mathcal{O}}| \le \kappa_{\mathcal{B}}$ and, indeed, $\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}} \ge 0$. The quantity $\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}$ is strictly positive when the bodies' second-order approximations maintain point contact at x. Since we assume a single point contact, we may as well assume that $\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}} > 0$.

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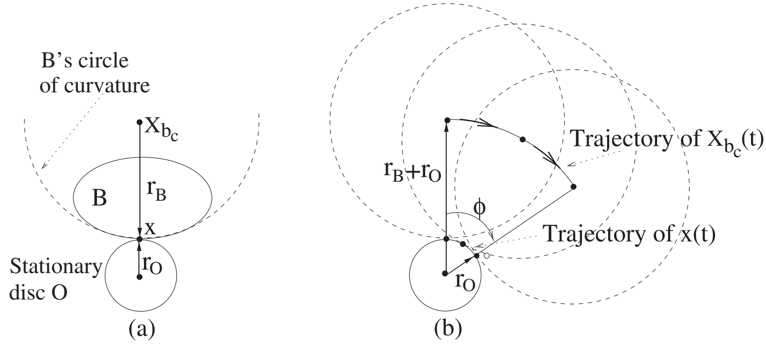

Figure 2.4 (a) The contacting bodies are replaced by their circles of curvature at x. (b) The $\mathcal{B}$-circle executes a contact preserving motion along the boundary of the stationary disc $\mathcal{O}$.

Geometric interpretation of the contact point velocity formula: Let $X_{b_c}(q)$ be the position of $\mathcal{B}$'s center of curvature at x, expressed in the world frame $\mathcal{F}_W$. Then $X_{b_c}(q) = R(\theta)b_c + d$. Assuming that b_c is held fixed on $\mathcal{B}$, $\dot{X}_{b_c} = \begin{bmatrix} I & -JR(\theta)b_c \end{bmatrix} \dot{q}$. Eq. (2.7) can thus be written as $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \dot{X}_{b_c}$. In order to justify this formula, replace the object $\mathcal{B}$ by its circle of curvature at x, and assume that $\mathcal{O}$ is a stationary disc (Figure 2.4(a)). As the $\mathcal{B}$-circle executes a contact preserving motion along $\mathcal{O}$ (Figure 2.4(b)), the $\mathcal{B}$-circle's center, X_{b_c}, moves along a circular arc of radius $|r_{\mathcal{B}} + r_{\mathcal{O}}|$. The circles' contact point, x, moves along a concentric circular arc of radius $|r_{\mathcal{O}}|$ during this motion. Since x and X_{b_c} lie on a common radius vector emanating from $\mathcal{O}$'s center, the two points move with identical angular velocities about $\mathcal{O}$'s center. Moreover, the two points move in the same direction when $r_{\mathcal{B}} \geq 0$. Assuming this case, let $\dot{\phi}$ be the common angular velocity of the two points (Figure 2.4(b)). Then $\dot{x} = |r_{\mathcal{O}}|\dot{\phi}$ while $\dot{X}_{b_c} = |r_{\mathcal{B}} + r_{\mathcal{O}}|\dot{\phi}$. Equating $\dot{\phi}$ in both expressions gives $\dot{x} = \frac{|r_{\mathcal{O}}|}{|r_{\mathcal{B}}+r_{\mathcal{O}}|} \dot{X}_{b_c}$. Finally, $\frac{r_{\mathcal{O}}}{r_{\mathcal{B}}+r_{\mathcal{O}}} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}$, giving the contact point velocity formula of Eq. (2.7). ○

Based on the contact point velocity formula, the c-obstacle curvature form is as follows.

THEOREM 2.7 (**c-obstacle curvature form in 2-D**) *Let $\mathcal{CO}$ be a c-obstacle associated with 2-D bodies $\mathcal{B}$ and $\mathcal{O}$. The* **curvature form** *of the c-obstacle boundary at $q \in \mathcal{S}$ is given by*

$$\kappa(q,\dot{q}) = \frac{1}{\|\eta(q)\|} \cdot \frac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \dot{q}^T \times \begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}} I & -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}} JRb_c \\ -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T & (\kappa_{\mathcal{O}}Rb - n(x))^T(\kappa_{\mathcal{B}}Rb + n(x)) \end{bmatrix} \dot{q} \qquad \dot{q} \in T_q\mathcal{S},$$

where $\eta(q)$ is the c-obstacle outward normal at q, $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at the contact point $x = X_b(q)$, $n(x)$ is $\mathcal{B}$'s inward unit normal at x, and b_c is $\mathcal{B}$'s center of curvature at x expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The proof of the theorem appears in the chapter's appendix. The key step in the proof is based on the contact point velocity formula. Consider a c-space path $q(t)$ that lies in $\mathcal{S}$, such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. According to Theorem 2.5, the c-obstacle

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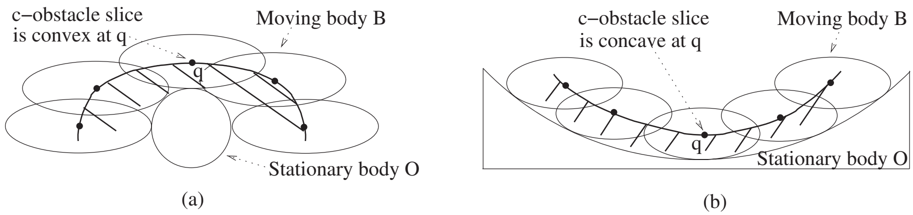

Figure 2.5 The c-obstacle slice is (a) convex when $\mathcal{B}$ and $\mathcal{O}$ are convex at x, and (b) concave when one of the two bodies is concave at x.

normal is given by $\eta(q) = DX_b^T(q)n(x)$, where $n(x)$ is $\mathcal{B}$'s inward unit normal at x. The curvature of $\mathcal{S}$ along $\dot{q}$ is determined by the derivative:

$$\frac{d}{dt}\bigg|_{t=0} \eta(q(t)) = DX_b^T(q)\, \frac{d}{dt}\bigg|_{t=0} n(x(t)) + \left(\frac{d}{dt}\bigg|_{t=0} DX_b^T(q(t))\right) n(x). \tag{2.8}$$

Since $\mathcal{B}$ maintains continuous contact with the stationary body $\mathcal{O}$ along $q(t)$, the contact point $x(t)$ moves along $\mathcal{O}$'s boundary during this motion. Hence $\frac{d}{dt}\big|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}\dot{x}$ in the first summand of Eq. (2.8). Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I \quad -JRb_c]\dot{q}$ according to Formula (2.7), then substituting the Jacobian formula $DX_b = [I \; -JRb]$ (see Exercises), gives

$$\begin{aligned} DX_b^T(q)\, \frac{d}{dt}\bigg|_{t=0} n(x(t)) &= \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I \\ (-JRb)^T \end{bmatrix} [I \quad -JRb_c]\,\dot{q} \\ &= \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I & -JRb_c \\ (-JRb)^T & b\cdot b_c \end{bmatrix} \dot{q}, \end{aligned}$$

where we used the identities $R^TR = I$ and $J^TJ = I$. The remainder of the proof appears in the chapter's appendix.

Curvature of c-obstacle slices: Consider the curvature of the fixed-θ slices of the c-obstacle boundary $\mathcal{S}$, denoted $\mathcal{S}|_\theta$. Each slice $\mathcal{S}|_\theta$ is a planar curve embedded in a fixed-θ plane in c-space $\mathbb{R}^3$. The vector tangent to $\mathcal{S}|_\theta$ at $q = (d, \theta)$ is $\dot{q} = (\dot{d}, 0)$, such that $\dot{d}$ is orthogonal to $n(x)$. This tangent vector corresponds to instantaneous translation of $\mathcal{B}$ along the tangent to $\mathcal{O}$'s boundary at x. Based on Theorem 2.7, the c-obstacle curvature along $\dot{q} = (\dot{d}, 0)$ is given by

$$\kappa\big(q, (\dot{d}, 0)\big) = \frac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}}\|\dot{d}\|^2.$$

The coefficient preceding $\|\dot{d}\|^2$ is the curvature of the c-obstacle slice $\mathcal{S}|_\theta$ at q. Its reciprocal is the slice's radius of curvature at q:

$$\left(\frac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}}\right)^{-1} = r_{\mathcal{B}} + r_{\mathcal{O}},$$

where $r_{\mathcal{B}}$ and $r_{\mathcal{O}}$ are the radii of curvature of $\mathcal{B}$ and $\mathcal{O}$ at x. We see that the radius of curvature of $\mathcal{S}|_\theta$ is the *algebraic sum* of the contacting bodies' radii of curvature. In particular, $\mathcal{S}|_\theta$ is convex at q when the bodies are convex at x (Figure 2.5(a)), while $\mathcal{S}|_\theta$ is concave at q when one of the two bodies is concave at x (Figure 2.5(b)). ○

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C-obstacle curvature in 3-D: The c-obstacle curvature in the 3-D case depends on the same geometric data as in the 2-D case, with the bodies' surface curvatures replacing the curvatures $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$. Let $\mathcal{S}$ be the five-dimensional boundary of $\mathcal{CO}$, let $q \in \mathcal{S}$, and let $x = X(q,b)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$. We assume that $\mathcal{S}$ is locally smooth at q, and denote by $T_q\mathcal{S}$ the five-dimensional tangent space of $\mathcal{S}$ at q. Let $q(t)$ be a c-space path that lies in $\mathcal{S}$, such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. The *curvature form* of $\mathcal{S}$ at q is given by $\kappa(q,\dot{q}) = \dot{q} \cdot \frac{d}{dt}\big|_{t=0} \hat{\eta}(q(t))$, where $\dot{q} \in T_q\mathcal{S}$ and $\hat{\eta}(q(t))$ is the unit outward normal to $\mathcal{S}$ along $q(t)$. The surface curvatures of $\mathcal{B}$ and $\mathcal{O}$ at x are determined by the linear maps $L_{\mathcal{B}}$ and $L_{\mathcal{O}}$. These linear maps act on the tangent plane of the respective surface to yield the change in the surface normal along a given tangent direction. The *curvature form* of $\mathcal{S}$ at q is given by (see Bibliographical Notes):

$$\kappa\big(q,\dot{q}\big) = \frac{1}{\|\eta(q)\|}\dot{q}^T \left(\begin{bmatrix} I & -[Rb\times] \\ O & [n(x)\times] \end{bmatrix}^T \begin{bmatrix} L_{\mathcal{B}}\,[L_{\mathcal{B}} + L_{\mathcal{O}}]^{-1}\,L_{\mathcal{O}} & -L_{\mathcal{O}}\,[L_{\mathcal{B}} + L_{\mathcal{O}}]^{-1} \\ -\,[L_{\mathcal{B}} + L_{\mathcal{O}}]^{-1}\,L_{\mathcal{O}} & -\,[L_{\mathcal{B}} + L_{\mathcal{O}}]^{-1} \end{bmatrix} \begin{bmatrix} I & -[Rb\times] \\ O & [n(x)\times] \end{bmatrix} + \begin{bmatrix} O & O \\ O & -\big([Rb\times]^T[n(x)\times]\big)_s \end{bmatrix} \right)\dot{q} \qquad \dot{q} \in T_q\mathcal{S},$$

where $\eta(q)$ is the c-obstacle outward normal at $q \in \mathcal{S}$, $n(x)$ is $\mathcal{O}$'s outward unit normal at x, I is a 3×3 identity matrix, O is a 3×3 matrix of zeroes and $(A)_s = \frac{1}{2}(A^T + A)$. Two comments are in order here. First, $L_{\mathcal{O}} + L_{\mathcal{B}} \geq 0$, otherwise the two bodies would interpenetrate at the contact. In particular, $L_{\mathcal{O}} + L_{\mathcal{B}} > 0$ in the generic case where the second-order approximations to the contacting surfaces of $\mathcal{B}$ and $\mathcal{O}$ maintain point contact at x. Second, the tangent vector $\dot{q} = (Rb \times n(x), n(x)) \in T_q\mathcal{S}$ is an eigenvector with zero eigenvalue of the matrix associated with the curvature form. This tangent vector corresponds to instantaneous rotation of $\mathcal{B}$ about its contact normal with $\mathcal{O}$. Hence, the c-obstacle $\mathcal{CO}$ always possesses *zero curvature* along instantaneous rotation of $\mathcal{B}$ about its contact normal with $\mathcal{O}$.

Bibliographical Notes

The notion of *configuration space* originated during a collaboration between two MIT doctoral students, assigned to develop one of the first robotic assembly stations [1–3]. While discussing the automation of the peg-in-a-hole insertion task, they observed that the best insertion approach would be to slide the peg along the horizontal surface at an oblique angle rather than at the vertical angle required for insertion (Figure 2.6). Once the oblique peg wedges itself into the hole, a rotational motion aligns the peg with the hole while completing the insertion task. This approach makes perfect sense when the c-obstacles induced by the hole's two sides are considered. The θ slice of the c-obstacles at a vertical angle contains only a *thin segment* of collision-free configurations (Figure 2.6(a)). In contrast, the θ slice of the c-obstacles at any oblique angle contains a full notch of collision-free configurations (Figure 2.6(b)). This observation led to the formulation of c-space as a framework for planning the motions of bodies in contact, a framework that has served virtually all motion planning algorithms. Some

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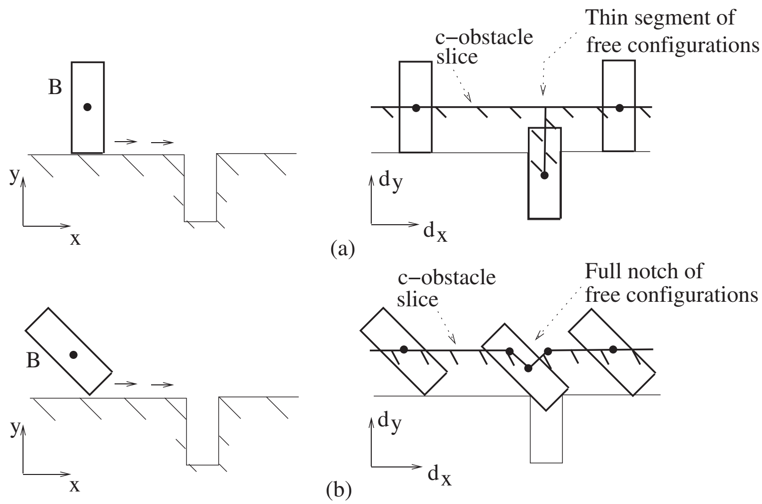

Figure 2.6 (a) At a vertical angle the peg's c-space contains only a thin segment of free configurations inside the hole. (b) At an oblique angle the peg's c-space contains a full notch of free configurations inside the hole.

recommended texts on robot motion planning are by Canny [4], Latombe [5], Lavalle [6] and, most recently, Lynch and Park [7].

The formulas for the c-obstacle normal and the c-obstacle curvature form are based on Rimon and Burdick [8, 9]. These papers contain the detailed derivation of the c-obstacle curvature form associated with 3-D bodies, which has been summarized here without any formal derivation. The contact point velocity formula of Proposition 2.6, which plays an important role in the derivation of the c-obstacle curvature form was derived by Montana [10].

Appendix: Details of Proofs

This appendix contains a derivation of the c-obstacle curvature formula in the 2-D case. We begin with the contact point velocity formula.

Proposition 2.6 *Let $q(t)$ be a c-space path that lies in $\mathcal{S}$, and let $x(t)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$ along $q(t)$, expressed in the world frame $\mathcal{F}_W$. The* **contact point velocity** *along $q(t)$ is given by*

$$\dot{x}(t) = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I & -J\,R(\theta)b_c \end{bmatrix} \dot{q}(t), \tag{2.9}$$

where $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at $x(t)$, b_c is $\mathcal{B}$'s center of curvature at $x(t)$ expressed in $\mathcal{F}_B$, I is a 2×2 identity matrix, and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

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Proof: When the object $\mathcal{B}$ moves along a c-space path $q(t)=(d(t),\theta(t))$ which lies in $\mathcal{S}$, the contact point position is given by

$$x(t)=X(q(t),b(t))=R(\theta(t))b(t)+d(t).$$

Taking the time derivative of both sides: $\dot{x}=DX_b(q)\dot{q}+R(\theta)\dot{b}$. In order to obtain an expression for $\dot{x}$ as a function of $\dot{q}$, we need a second equation relating $(\dot{x},\dot{b})$ to $\dot{q}$. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, $\mathcal{O}$'s *outward* unit normal at x, $n(x)$, must match $\mathcal{B}$'s *inward* unit normal at x. Denote by $\bar{n}(b)$ the outward unit normal of $\mathcal{B}$ at b, expressed in $\mathcal{F}_B$. Then $-R(\theta)\bar{n}(b)$ is the direction of $\mathcal{B}$'s *inward* unit normal at x in the world frame $\mathcal{F}_W$, and, therefore, $n(x(t))=-R(\theta(t))\bar{n}(b(t))$ along $q(t)$. Taking the time derivative of both sides gives

$$\tfrac{d}{dt}n(x(t))=JR(\theta)\bar{n}(b)\dot{\theta}-R(\theta)\tfrac{d}{dt}\bar{n}(b(t)),$$

where we used the formula $\dot{R}(\theta)=-JR(\theta)\dot{\theta}$. The curvature of $\mathcal{B}$ satisfies the relation $\frac{d}{dt}\bar{n}(b(t))=\kappa_{\mathcal{B}}\dot{b}$. The curvature of $\mathcal{O}$ satisfies the relation $\frac{d}{dt}n(x(t))=\kappa_{\mathcal{O}}\dot{x}$. Therefore,

$$\kappa_{\mathcal{O}}\dot{x}=JR(\theta)\bar{n}(b)\dot{\theta}-\kappa_{\mathcal{B}}R(\theta)\dot{b}.$$

Substituting $R(\theta)\dot{b}=\dot{x}-DX_b(q)\dot{q}$ in the latter equation gives

$$(\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}})\dot{x}=\kappa_{\mathcal{B}}DX_b(q)\dot{q}+JR(\theta)\bar{n}(b)\dot{\theta} \qquad \dot{q}=(\dot{d},\dot{\theta}).$$

Substituting $DX_b(q)\dot{q}=\dot{d}-JRb\dot{\theta}$, then pulling $\kappa_{\mathcal{B}}$ as a common factor gives

$$(\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}})\dot{x}=\kappa_{\mathcal{B}}\left\{\dot{d}-JR\big(b-r_{\mathcal{B}}\bar{n}(b)\big)\dot{\theta}\right\} \qquad r_{\mathcal{B}}=1/\kappa_{\mathcal{B}}.$$

The term $b-r_{\mathcal{B}}\bar{n}(b)$ is the position of $\mathcal{B}$'s center of curvature in $\mathcal{F}_B$: $b_c=b-r_{\mathcal{B}}\bar{n}(b)$. This gives $\dot{x}=\frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\left[I \quad -JR(\theta)b_c\right]\dot{q}$, where $\dot{q}=(\dot{d},\dot{\theta})$. □

The following theorem specifies the c-obstacle curvature formula in the 2-D case.

Theorem 2.7 *Let $\mathcal{CO}$ be a c-obstacle associated with 2-D bodies $\mathcal{B}$ and $\mathcal{O}$. The* **curvature form** *of the c-obstacle boundary at $q\in\mathcal{S}$ is given by*

$$\kappa(q,\dot{q})=\frac{1}{\|\eta(q)\|}\cdot\frac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\dot{q}^T\begin{bmatrix}\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}I & -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}JRb_c\\ -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T & (\kappa_{\mathcal{O}}Rb-n(x))^T(\kappa_{\mathcal{B}}Rb+n(x))\end{bmatrix}\dot{q}$$
$$\dot{q}\in T_q\mathcal{S},$$

where $\eta(q)$ is the c-obstacle outward normal at q, $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at the contact point $x=X_b(q)$, $n(x)$ is $\mathcal{B}$'s inward unit normal at x, and b_c is $\mathcal{B}$'s center of curvature at x expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

Proof: Let $q(t)$ be a c-space curve that lies in $\mathcal{S}$, such that $q(0)=q$ and $\frac{d}{dt}\big|_{t=0}q(t)=\dot{q}$. Based on the definition of $\kappa(q,\dot{q})$, we have to compute the derivative:

$$\frac{d}{dt}\Big|_{t=0}\hat{\eta}(q(t))=\frac{d}{dt}\Big|_{t=0}\frac{1}{\|\eta(q(t))\|}\eta(q(t))=\frac{1}{\|\eta(q)\|}\left[I-\hat{\eta}(q)\hat{\eta}(q)^T\right]\frac{d}{dt}\Big|_{t=0}\eta(q(t)).$$

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Since $\left[I - \hat{\eta}(q)\hat{\eta}(q)^T\right]\dot{q} = \dot{q}$ on $T_q\mathcal{S}$, the curvature form can be equivalently written as

$$\kappa(q,\dot{q}) = \frac{1}{\|\eta(q)\|}\,\dot{q}\cdot \left.\frac{d}{dt}\right|_{t=0}\eta(q(t)) \qquad \dot{q}\in T_q\mathcal{S}.$$

The c-obstacle outward normal is given by $\eta(q) = DX_b^T(q)n(x)$, where $DX_b(q) = [I \;\; -JRb]$ and $n(x)$ is $\mathcal{O}$'s outward unit normal at x (Theorem 2.5). Thus, we have to compute the derivative:

$$\left.\frac{d}{dt}\right|_{t=0}\eta(q(t)) = DX_b^T(q)\left.\frac{d}{dt}\right|_{t=0} n(x(t)) + \left(\left.\frac{d}{dt}\right|_{t=0} DX_b^T(q(t))\right)n(x).$$

Since $\mathcal{B}$ maintains continuous contact with the stationary body $\mathcal{O}$ along $q(t)$, the contact point $x(t)$ moves along $\mathcal{O}$'s boundary. Hence $\frac{d}{dt}\big|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}\dot{x}$ in the first summand. Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I \;\; -JRb_c]\dot{q}$ according to Proposition 2.6 gives

$$\begin{aligned}DX_b^T(q)\left.\frac{d}{dt}\right|_{t=0} n(x(t)) &= \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I \\ (-JRb)^T\end{bmatrix}[I \;\; -JRb_c]\,\dot{q}\\ &= \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I & -JRb_c \\ (-JRb)^T & b\cdot b_c\end{bmatrix}\dot{q},\end{aligned}$$

where we used the identities $R^TR = I$ and $J^TJ = I$. In the second summand, $\frac{d}{dt}\big|_{t=0} DX_b^T(q) = \left[O \;\; -J\frac{d}{dt}\big|_{t=0}(Rb)\right]^T$, where O is a 2×2 matrix of zeroes. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, the contact point satisfies the equation: $x(t) = R(\theta(t))b(t) + d(t)$. Taking the time derivative of both sides: $\dot{x} = \frac{d}{dt}(Rb) + \dot{d}$. Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I \;\; -JRb_c]\dot{q}$ according to Proposition 2.6 gives

$$\frac{d}{dt}(Rb) = \tfrac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I \;\; -JRb_c]\,\dot{q} - \dot{d} = \tfrac{-1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[\kappa_{\mathcal{O}}I \;\; \kappa_{\mathcal{B}}JRb_c]\,\dot{q} \qquad \dot{q} = (\dot{d},\dot{\theta}).$$

The second summand thus has the form:

$$\begin{aligned}\left(\left.\tfrac{d}{dt}\right|_{t=0} DX_b^T(q(t))\right)n(x) &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O \\ n^T(x)J\,[\kappa_{\mathcal{O}}I \;\; \kappa_{\mathcal{B}}JRb_c]\,\dot{q}\end{bmatrix}\\ &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O & \vec{0} \\ \kappa_{\mathcal{O}}n^T(x)J & -\kappa_{\mathcal{B}}n^T(x)Rb_c\end{bmatrix}\dot{q},\end{aligned}$$

where we used the identity $J^2 = -I$. Substituting for the two summands in the derivative $\frac{d}{dt}\big|_{t=0}\eta(q(t))$ gives

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\eta(q(t)) &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O & \vec{0} \\ \kappa_{\mathcal{O}}n^T(x)J & -\kappa_{\mathcal{B}}n^T(x)Rb_c\end{bmatrix}\dot{q}\\ &\quad + \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I & -JRb_c \\ (-JRb)^T & b\cdot b_c\end{bmatrix}\dot{q}\\ &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}I & -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}JRb_c \\ \kappa_{\mathcal{O}}n^T(x)J + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(-JRb)^T & -\kappa_{\mathcal{B}}n^T(x)Rb_c + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}b\cdot b_c\end{bmatrix}\dot{q}.\end{aligned}$$

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The expression on the lower left simplifies as follows. Denote by $\bar{n}(b)$ the outward unit normal to $\mathcal{B}$ at b, expressed in $\mathcal{F}_{\mathcal{B}}$. Then $n(x) = -R(\theta)\bar{n}(b)$, and, therefore,

$$\kappa_{\mathcal{O}} n^T(x) J + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(-JRb)^T = \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}\big(-r_{\mathcal{B}}\bar{n}(b) + b\big)^T R^T J = \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(-JRb_c)^T,$$

where $b_c = b - r_{\mathcal{B}}\bar{n}(b)$ is $\mathcal{B}$'s center of curvature at x. The expression on the lower right simplifies as follows:

$$-\kappa_{\mathcal{B}} n^T(x) R b_c + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}} b \cdot b_c = (\kappa_{\mathcal{O}} Rb - n(x))^T(\kappa_{\mathcal{B}} R b_c) = (\kappa_{\mathcal{O}} Rb - n(x))^T(\kappa_{\mathcal{B}} Rb + n(x)),$$

where we substituted $\kappa_{\mathcal{B}} R b_c = \kappa_{\mathcal{B}} R(b - r_{\mathcal{B}}\bar{n}(b)) = \kappa_{\mathcal{B}} Rb + n(x)$. Substituting the simplified terms in the derivative $\frac{d}{dt}\Big|_{t=0} \eta(q(t))$ gives

$$\frac{d}{dt}\bigg|_{t=0} \eta(q(t)) = \frac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}} I & -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}} J R b_c \\ -\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(J R b_c)^T & (\kappa_{\mathcal{O}} Rb - n(x))^T(\kappa_{\mathcal{B}} Rb + n(x)) \end{bmatrix} \dot{q}. \tag{2.10}$$

Pre-multiplying both sides of Eq. (2.10) by $1/\|\eta(q)\|$ and by the row vector $\dot{q}$ gives the c-obstacle curvature form. □

Exercises

Section 2.1

Exercise 2.1: Justify the definition of $SO(3)$ by the conditions $R^T R = I$ and $\det(R) = 1$.

Solution: Write $R = [c_1\, c_2\, c_3]$. The condition $R^T R = I$ ensures that $\|c_i\| = 1$ while $c_i \cdot c_j = 0$ for $1 \le i, j \le 3$, implying that the columns of R describe an orthonormal triplet. The condition $\det[c_1\, c_2\, c_3] = 1$ is equivalent to the condition $(c_1 \times c_2) \cdot c_3 = 1$, implying that the columns of R form a right-handed triplet. ○

Exercise 2.2: The matrix group $SO(3)$ forms a three-dimensional manifold topologically equivalent to the projective space RP^3. The points of RP^3 correspond to lines passing through the origin in $\mathbb{R}^4$. Explain why RP^3 is topologically equivalent to a three-dimensional unit ball centered at the origin of $\mathbb{R}^3$, with antipodal points on its bounding sphere identified.

Solution: The collection of lines passing through the origin in $\mathbb{R}^4$ can be identified with antipodal pairs of points on the unit three-dimensional sphere, S^3, embedded in $\mathbb{R}^4$.[5] The latter set can be identified with unique points on the upper hemisphere of S^3, together with antipodal pairs of points on the "equator" of S^3, which is the unit sphere S^2. The upper hemisphere with its equator S^2 is topologically equivalent to the unit three-dimensional ball embedded in $\mathbb{R}^3$, with antipodal points on its bounding unit sphere identified. Note that Rodrigues' formula parametrizes $SO(3)$ in terms of a radius π ball and its bounding sphere. ○

[5] When one uses *quaternions,* the global parametrization of $SO(3)$ is in terms of S^3 embedded in $\mathbb{R}^4$.

Exercise 2.3: When the matrix group $SO(3)$ is viewed as a manifold, it contains two classes of loops: those that can be contracted to a point and those that cannot be contracted to a point within $SO(3)$ (such a manifold is *not* simply connected). Identify the non-shrinkable loops in $SO(3)$, using the topological model obtained in the previous exercise.

Solution: Rodrigues' formula parametrizes $SO(3)$ in terms of a radius π ball with identified antipodal points on its bounding sphere. Consider a loop that starts at the origin, moves to the radius-π sphere, and then wraps through the antipodal point back to the origin. An attempt to contract this loop within the radius π ball would break it. ○

Exercise 2.4: Verify that $R(\theta)$ in Rodrigues' formula satisfies the conditions $R^T R = I$ and $\det(R) = 1$.

Exercise 2.5: Using Rodrigues' formula, verify that $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ is an eigenvector of $R(\theta) \in SO(3)$, and $v \cdot (R(\theta)v) = \cos(\|\theta\|)$ for any unit vector $v \in \mathbb{R}^3$ orthogonal to θ.

Exercise 2.6: Show that Rodrigues' formula gives the 2×2 orientation matrices when $\hat{\theta} = (0, 0, 1)$.

Exercise 2.7*: Verify that the rigid-body transformation, $X(q, b) = R(\theta)b + d$, is the general form of distance and orientation preserving embedding of a rigid body $\mathcal{B}$ in $\mathbb{R}^3$.

Solution: This basic property is discussed in geometry texts, such as Rees' *Notes on Geometry* [11]. ○

Section 2.2

Exercise 2.8: Explain the characterization of the c-obstacle boundary specified in Lemma 2.2.

Exercise 2.9: Prove that when $\mathcal{B}$ and $\mathcal{O}$ are path-connected bodies, the c-obstacle $\mathcal{CO}$ is also path connected.

Solution: A graphical proof of this property appears in Latombe [5](Proposition 2.6).

Exercise 2.10: A real-valued function forms a *convex function* when its epigraph (the set of points on or above the graph of the function) forms a convex set. Prove that each θ-slice of the c-obstacle associated with a convex object $\mathcal{B}$ and a convex stationary body $\mathcal{O}$ is convex, based on the fact that $\mathsf{dst}(x, \mathcal{O})$ is a convex function when $\mathcal{O}$ is convex.

Exercise 2.11: Let $\mathcal{B}$ be a 2-D smooth convex body and $\mathcal{O}$ a stationary disc of radius r and center x_0. Prove that the boundary of $\mathcal{CO}$ is parametrized by the formula: $\varphi(s, \theta) = (d(s, \theta), \theta)$, where $d(s, \theta) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big)$.

Solution: Let θ_0 be a particular orientation of $\mathcal{B}$. When $\mathcal{B}$ traces $\mathcal{O}$'s perimeter at a fixed orientation θ_0, the curve traced by $\mathcal{B}$'s contact point in $\mathcal{F}_W$ is $x(s) = R(\theta_0)b(s) + d(s)$ for $s \in \mathbb{R}$. The c-obstacle boundary is the curve traced by $\mathcal{B}$'s origin during this motion:

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$d(s) = x(s) - R(\theta_0)b(s)$. Since the contact normals of $\mathcal{O}$ and $\mathcal{B}$ are collinear at $x(s)$, $\mathcal{O}$'s center point satisfies the equation, $x_0 = x(s) + rR(\theta_0)J\beta'(s)$, since $J\beta'(s)$ points into $\mathcal{O}$. Substituting for $x(s)$ in the expression for $d(s)$ gives $d(s) = x_0 - rR(\theta_0)J\beta'(s) - R(\theta_0)b(s) = x_0 - R(\theta_0)\big(b(s) + rJ\beta'(s)\big)$. ○

Exercise 2.12: Show that the boundary of the c-obstacle $\mathcal{CO}$ associated with 2-D smooth convex bodies $\mathcal{B}$ and $\mathcal{O}$ forms a single smooth surface in $\mathcal{B}$'s c-space.

Exercise 2.13: When $\mathcal{B}$ is a polygon and $\mathcal{O}$ a stationary disc, the boundary of $\mathcal{CO}$ forms a piecewise smooth surface in $\mathcal{B}$'s c-space. What types of two-dimensional patches form the c-obstacle boundary? Write the (s,θ) parametrization of the patch generated by an edge of $\mathcal{B}$.

Solution: There are two types of smooth patches on the c-obstacle surface. An edge patch, generated by an edge of $\mathcal{B}$ sliding on $\mathcal{O}$, and a vertex patch, generated by a vertex of $\mathcal{B}$ sliding on $\mathcal{O}$. Let $\mathcal{O}$ have a radius r and center x_0. Consider now an edge of $\mathcal{B}$ having endpoints b_1 and b_2 and length L. Let $v = (b_2 - b_1)/L$ be the edge's direction. Then $\beta(s) = b_1 + sv$ for $0 \le s \le L$ parametrizes the edge in $\mathcal{F}_B$. Since the edge can touch $\mathcal{O}$ from the outside at any orientation θ, the parameter θ varies freely in $\mathbb{R}$. Following the solution approach of Exercise 2.11, $d(s,\theta) = x_0 - R(\theta)(b_1 + sv + rJv)$ for $0 \le s \le L$ and $\theta \in \mathbb{R}$. Note that $d(s,\theta)$ is linear in s, implying that the patch $\varphi(s,\theta) = (d(s,\theta),\theta)$ forms a *ruled surface* in this case. ○

Exercise 2.14*: Use the fact that Lipschitz continuous functions are piecewise smooth to conclude that the c-obstacle boundary is a piecewise smooth surface.

Section 2.3

Exercise 2.15: Verify the formula for the 3-D Jacobian of the rigid-body transformation, $DX_b(q) = \frac{d}{dq}X_b(q)$, which appears in the proof of Proposition 2.4. Obtain the 2-D Jacobian as a special case of the 3-D formula.

Exercise 2.16: Consider the c-obstacle normal, $\eta(q)$, at a configuration q at which $\mathcal{F}_B$'s origin lies along the contact normal. Verify that the tangent plane to the c-obstacle boundary, $T_q\mathcal{S}$, is vertical in this case, implying that instantaneous rotations of $\mathcal{B}$ about $\mathcal{F}_B$'s origin are tangent to $\mathcal{S}$ at q.

Section 2.4

Exercise 2.17: Prove that each fixed-θ slice of the c-obstacle boundary, $\mathcal{S}|_\theta$, is convex at $q = (d,\theta)$ when the contacting bodies are convex at x (Figure 2.5(a)).

Solution: Since $r_{\mathcal{B}}(x) \ge 0$ and $r_{\mathcal{O}}(x) \ge 0$ for convex bodies, $r_{\mathcal{B}}(x) + r_{\mathcal{O}}(x) \ge 0$, implying that $\mathcal{S}|_\theta$ is convex at $q = (d,\theta)$. ○

Exercise 2.18: Prove that each fixed-θ slice of the c-obstacle boundary, $\mathcal{S}|_\theta$, is concave at $q = (d,\theta)$ when one of the contacting bodies is concave at x (Figure 2.5(b)).

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Solution: Suppose that the stationary body $\mathcal{O}$ is concave while the moving body $\mathcal{B}$ is convex at the contact point x. Then $r_{\mathcal{B}}(x) \geq 0$ while $r_{\mathcal{O}}(x) < 0$. Since $|r_{\mathcal{O}}(x)| > r_{\mathcal{B}}(x)$ (otherwise the bodies would interpenetrate), $r_{\mathcal{B}}(x) + r_{\mathcal{O}}(x) < 0$, implying that $\mathcal{S}|_{\theta}$ is concave at $q = (d, \theta)$. ○

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