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Part 5. Finance and econometrics

OPTIMAL CLAIMS WITH FIXED PAYOFF STRUCTURE

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OPTIMAL CLAIMS WITH FIXED PAYOFF STRUCTURE

BY CAROLE BERNARD, LUDGER RÜSCHENDORF AND STEVEN VANDUFFEL

Abstract

Dybvig (1988) introduced the interesting problem of how to construct in the cheapest possible way a terminal wealth with desired distribution. This idea has induced a series of papers concerning generality, consequences, and applications. As the optimized claims typically follow the trend in the market, they are not useful for investors who wish to use them to protect an existing portfolio. For this reason, Bernard, Moraux, Rüschendorf and Vanduffel (2014b) imposed additional state-dependent constraints as a way of controlling the payoff structure. The present paper extends this work in various ways. In order to obtain optimal claims in general models we allow in this paper for extended contracts. We deal with general multivariate price processes and dispense with several of the regularity assumptions in the previous work (in particular, we omit any continuity assumption). State dependence is modeled by requiring terminal wealth to have a fixed copula with a benchmark wealth. In this setting, we are able to characterize optimal claims. We apply the theoretical results to deal with several hedging and expected utility maximization problems of interest.

Keywords: Cost-efficient payoff; optimal portfolio; state-dependent utility

2010 Mathematics Subject Classification: Primary 91G10; 91B16

1. Introduction

We consider optimal investment problems in a financial market given by a market model $S = \{S_t\}_{0\le t\le T}$ in a filtered probability space $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{0\le t\le T}, \mathbb{P})$. Here S may consist of several stocks and also bank accounts. Our basic assumption is that state prices at time t are determined by a pricing process $\xi = \{\xi_t\}_{0\le t\le T}$ that is adapted to the filtration. Typically, ξ is the (discounted) pricing density process of a martingale pricing rule. Typical examples include exponential Lévy models that use an Esscher pricing measure for which the pricing process has the form $\xi_t = g_t(S_t)$; note that this covers the case of the multidimensional Black–Scholes market. But we could also think of a stochastic volatility model in which ξ_t is a function of the process $\{S_u\}_{0\le u\le t}$ and some additional volatility process $\{\sigma_u\}_{0\le u\le t}$.

Let X_T be a payoff at time T (i.e. X_T is $\mathcal{A}_T$-measurable) with payoff distribution F and cost $c(X_T) := \mathbb{E}[\xi_T X_T]$. The aim of an investor with law-invariant (state-independent) preferences—as in many classical behavioral theories, including mean-variance optimization (see Markovitz (1952)), expected utility theory (see von Neumann and Morgenstern (1947)), dual theory (see Yaari (1987)), rank-dependent utility theory (see Quiggin (1993)), cumulative prospect theory (see Tversky and Kahneman (1992)), and sp/a theory (see Shefrin and Statman (2000))—is to construct a payoff X_T^* with the same payoff distribution F at the lowest possible cost, i.e.

$$c(X_T^*) = \inf\{\mathbb{E}[\xi_T Y_T]\colon Y_T \sim F\}, \tag{1.1}$$

where $Y_T \sim F$ means that Y_T has the same payoff distribution F as X_T. Note that the right-hand side of the problem in (1.1) depends only on the distribution F and not on the specific form of the payoff X_T; in what follows, X_T denotes any generic payoff with distribution function F.

The optimization problem in (1.1) represents the static version of the optimal portfolio problem (see He and Pearson (1991a, 1991b)). The optimal payoff can (in a second step) always be attained by hedging strategies in complete markets. Characterizations and sufficient conditions for representation of the optimal claims in incomplete markets by continuous-time trading strategies have been established in the literature and are related to the optional decomposition theorem (see Jacka (1992), Ansel and Stricker (1994), Delbaen and Schachermayer (1995), Goll and Rüschendorf (2001), and Rheinländer and Sexton (2011)).

The cost minimization problem in (1.1) has been stated and solved under various assumptions on the distributions in Dybvig (1998), Bernard *et al.* (2014a), Carlier and Dana (2011), Rüschendorf (2012), and elsewhere. Several explicit calculations of optimal claims (in this paper we use the notions of payoffs and claims synonymously) have been given in the framework of the Black–Scholes model (see Bernard *et al.* (2011, 2014a)) and in exponential Lévy models (see von Hammerstein *et al.* (2013) and Vanduffel *et al.* (2009)). In Section 2 we introduce the class of extended payoffs; these payoffs are based on the market information $\mathcal{A}_T$ up to time T but also allow for external randomization. We refer to them as randomized payoffs. The use of randomization allows us to construct optimal claims explicitly without imposing regularity conditions as in Bernard *et al.* (2014b). Indeed, we provide a simple proof showing that optimal claims are dependent only on ξ_T and possibly some independent randomization V. In the particular case of Lévy models this result implies path independence of optimal claims, i.e. optimal claims are of the form $X_T^* = f(S_T)$ or $f(S_T, V)$.

Bernard *et al.* (2014a, 2014c) pointed out that solutions to the cost minimization problem (1.1) are not suitable for investors who are exposed to some external risk against which they want protection. These investors are prepared to pay more to obtain a certain distribution, simply because they want the optimal payoff to pay out more in some desired states. For example, a put option gives its best outcomes in the worst states of the market and thus allows investors to protect the value of an existing investment portfolio that is long with the market. The same observation is also at the core of insurance business. People buy a fire insurance contract and not a cheaper financial contract with identical distribution ('digital option') because the insurance contract provides wealth when it is actually needed; see also Bernard and Vanduffel (2014a). In other words, two payoffs with the same distribution do not necessarily present the same 'value' for an investor; see also the discussion in Vanduffel *et al.* (2012). Therefore, in Section 3, following the development in Bernard *et al.* (2014b), we introduce and discuss additional restrictions on the form of the payoffs. These restrictions are determined by fixing the desired copula of the claim with a random benchmark A_T. This type of constraint allows an investor to control the states of the economy in which he/she wants to receive payments. Note that, when A_T is deterministic, no restriction is imposed and we again obtain the optimal payoffs in the classical context of no constraint. Our main result is to determine payoffs with minimal price and given payoff distribution F under state-dependent constraints in general markets. In comparison to the results in Bernard *et al.* (2014b), by using the extended notion of (randomized) claims we obtain optimal solutions that are functions of ξ_T, A_T, and some independent randomization. This characterization extends the concept of 'twins' as optimal solutions, as in Bernard *et al.* (2014b).

We use this characterization result to deal with several hedging and investment problems of interest. In Section 4 we provide the optimal claim for an expected utility maximizer with state-dependent constraints. In Section 5 we solve some optimal hedging problems, and also determine the optimal contract for an expected return maximizer with constraints on the minimum and maximum desired returns.

2. Randomized claims and cost-efficient payoffs

Denote by $\mathcal{L}(\mathcal{A}_T)$ the class of all $\mathcal{A}_T$-measurable claims (payoffs) at time T. For the construction of optimal claims, it is useful to extend the notion of claims (payoffs) to randomized claims (randomized payoffs). We generally assume that the underlying probability space $(\Omega, \mathcal{A}_T, \mathbb{P})$ is rich enough to allow us to construct, for each element Y_T, a random variable V that is independent of Y_T and uniformly distributed on $(0, 1)$.

A 'randomized claim' is a claim of the form $f(Y_T, V)$ involving a randomization V that is independent of Y_T. The use of randomized claims is an essential point in this paper; it allows us to solve portfolio optimization problems in general market models. Under the continuity assumptions used in Bernard *et al.* (2014b), we can avoid this additional randomization. At first glance, it may seem strange to an investor to use an independent randomization for the construction of an investment. A similar objection also concerns the use of randomized tests in classical testing theory. As in testing theory, where one obtains existence of optimal tests only in the class of randomized tests, we can expect optimal claims to exist only within the more general class of randomized claims. In some market models it may be possible to use that model to construct this independent randomization. This underlies the concept of twins in Bernard *et al.* (2014b), but in general the investor should be prepared to roll a die in order to be better off. In what follows we use randomized claims without further ado.

For a given payoff distribution F, a claim $X_T^* \in \mathcal{L}(\mathcal{A}_T)$ with payoff distribution F is called *cost efficient* if it minimizes the cost $c(Y_T)$ over all claims Y_T with payoff distribution F, i.e. if X_T^* solves (1.1) (see Bernard *et al.* (2014a)). For the construction of cost-efficient payoffs, we use the following two classical results.

Result 2.1. (Hoeffding–Fréchet bounds (see Hoeffding (1940) and Fréchet (1940, 1951)).) *Let X and Y be random variables with distribution functions F and G, and let $U \sim U(0, 1)$ be uniformly distributed on* $(0, 1)$. *Then*

$$\mathbb{E}[F^{-1}(U)G^{-1}(1-U)] \leq \mathbb{E}[XY] \leq \mathbb{E}[F^{-1}(U)G^{-1}(U)]. \tag{2.1}$$

The upper bond is attained only if $(X, Y) \sim (F^{-1}(U), G^{-1}(U))$, where '$\sim$' refers to equality in distribution, i.e. X and Y are co-monotonic. The lower bound is attained only if $(X, Y) \sim (F^{-1}(U), G^{-1}(1-U))$, i.e. X and Y are anti-monotonic.

Result 2.2. (Distributional transform (see Rüschendorf (1981, 2009)).) *For a random variable $X \sim F$ and a random variable $V \sim U(0, 1)$ that is independent of X, the distributional transform τ_X is defined by*

$$\tau_X = F(X, V), \tag{2.2}$$

where, with a slight abuse of notation, $F(x, \lambda) := \mathbb{P}\{X < x\} + \lambda\,\mathbb{P}\{X = x\}$; note that $F(x, \lambda) = F(x)$ when F is continuous. Then

$$\tau_X \sim U(0, 1) \quad \textit{and} \quad X = F^{-1}(\tau_X) \quad \textit{almost surely (a.s.)}. \tag{2.3}$$

The variable τ_X can thus be seen as a uniformly distributed variable that is associated to (or, transformed from) X.

For a payoff distribution function F, denote by $\mathcal{K}(F)$ the class of all claims that have payoff distribution F:

$$\mathcal{K}(F) = \{Y_T \in \mathcal{L}(\mathcal{A}_T) \colon Y_T \sim F\}.$$

Combining the Hoeffding–Fréchet bounds in (2.1) and the distributional transform in (2.2) allows us to obtain in a straightforward way the following general form of the cost-efficient claim.

Theorem 2.1. (Cost-efficient claim.) *For a given payoff distribution F, the claim*

$$X_T^* = F^{-1}(1 - \tau_{\xi_T}) \tag{2.4}$$

is cost efficient, i.e.

$$c(X_T^*) = \inf_{Y_T \in \mathcal{K}(F)} c(Y_T). \tag{2.5}$$

Proof. By (2.3), the distributional transform $\tau_{\xi_T} = F(\xi_T, V)$ is uniformly distributed on $(0, 1)$ and $\xi_T = F^{-1}(\tau_{\xi_T})$ a.s. This implies that the pair (ξ_T, X_T^*) is anti-monotonic and, thus, (2.5) is a consequence of the Hoeffding–Fréchet lower bound in (2.1).

Remark 2.1. When F_{ξ_T} is continuous, the additional randomization V can be omitted and (2.4) coincides with the classical result on cost-efficient claims (see Dybvig (1988) and Bernard *et al.* (2014a)). When $\xi_T = g_T(S_T)$ for an appropriate function g_T, we deduce that

$$X_T^* = h(S_T)$$

for some function h. Thus, any path-dependent option can be improved by a path-independent option. For this observation, see Bernard *et al.* (2014a).

Remark 2.2. Several explicit results on lookback options, Asian options, and related path-dependent options have been given in the context of Black–Scholes models and Lévy models in Bernard *et al.* (2011, 2014a) and von Hammerstein *et al.* (2013).

Remark 2.3. It is not difficult to see (cf. the proof of Theorem 4.1) that optimal claims that follow from optimizing a law-invariant objective (e.g. expected utility) at a given horizon T must be cost efficient.

3. Payoffs with fixed payoff structure

If ξ_T is a decreasing function of S_T (this property is predicted by economic theory and confirmed by many popular pricing models, including increasing exponential Lévy-type models) then an optimized payoff X_T^* is increasing in S_T. The optimal payoff can thus be quite different from the initial payoff X_T and may perform poorly when the market asset S_T reaches low levels. These qualitative features do not demonstrate a defect of the solution, but rather show that portfolio optimization which considers only distributional properties of terminal wealth is not suitable in all situations. For example, some investors buy put options to protect their existing portfolio (as a source of benchmark risk) and they are not interested in the cost-efficient alternatives because these are long with the market and do not offer the protection that is sought. These observations led Bernard *et al.* (2014b) to include constraints in the optimization problem that can depend on the states in which payments are received. In this paper we build further on this development. We restrict the class of admissible options in the portfolio optimization problem by requiring admissible claims to pay out more in some desired states (e.g. when S_T is low) and less in other states, i.e. we consider only state-dependent constraints.

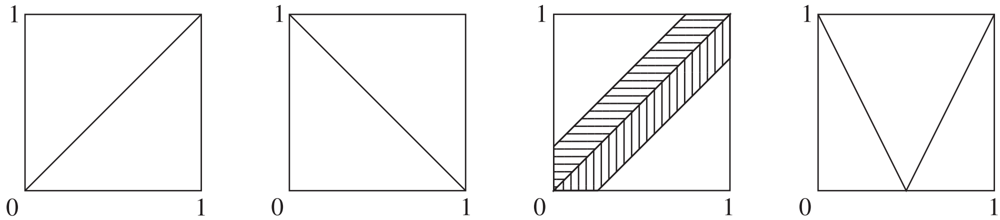

FIGURE 1: Various dependence prescriptions.

To model state-dependent constraints, we use a random benchmark A_T and couple the admissible claims Y_T to the behavior of A_T. Specifically, let A_T be some random benchmark such as, e.g. $A_T = S_T$ or $A_T = (S_T - K)_+$, or some other available claim in the market, and let C denote a copula describing the desired payoff structure of admissible claims. The copula C is not necessarily the copula of a given initial claim with the benchmark A_T, but rather a tool to describe those states of the benchmark in which the investor wants to receive income (or protection). We consider a claim Y_T to be admissible if the copula of the pair (Y_T, A_T) is C, i.e.

$$C_{(Y_T, A_T)} = C.$$

The copula C determines how the payoff structure of Y_T is coupled to the benchmark A_T. In this way we can prescribe that payoffs should be (approximately) increasing or decreasing in A_T or take place for either large or small A_T (see Figure 1).

Modify the portfolio optimization problem in (1.1) so as to include a fixed payoff structure, namely, determine $X_T^* \sim F$ with copula $C_{(X_T^*, A_T)} = C$ such that

$$c(X_T^*) = \inf\{c(Y_T) \colon Y_T \sim F,\ C_{(Y_T, A_T)} = C\}. \tag{3.1}$$

Since the joint distribution function G of (Y_T, A_T) is given by

$$G = C(F, F_{A_T}),$$

problem (3.1) is equivalent to the cost minimization problem when fixing the joint distribution of (Y_T, A_T) to be equal to G, i.e.

$$c(X_T^*) = \inf\{c(Y_T) \colon (Y_T, A_T) \sim G\}. \tag{3.2}$$

For the construction of the solution of the portfolio optimization problem in (3.1) or (3.2), we use the concept of the conditional distributional transform.

Definition 3.1. The *conditional distributional transform* of X given Y is defined by

$$\tau_{X \mid Y} = F_{X \mid Y}(X, V),$$

where, for all y, V is independent of $(X \mid Y = y)$.

It is clear that, by the distributional transform property (2.3),

$$\tau_{X \mid Y} \sim U(0, 1) \text{ and } \tau_{X \mid Y} \text{ is stochastically independent of } Y. \tag{3.3}$$

In the following theorem we determine the optimal solution of the portfolio optimization problem in (3.1) or (3.2). Based on the concept of randomized claims, it gives an extension

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of Theorem 3.3 of Bernard *et al.* (2014b) to the case of general market models, avoiding the regularity conditions imposed in that paper. Let X_T be a payoff with distribution F and such that (X_T, A_T) has copula C, or, equivalently, $(X_T, A_T) \sim G$.

Theorem 3.1. (Cost-efficient claim with fixed payoff structure.) *Let X_T be a claim with $(X_T, A_T) \sim G$. Then*

$$X_T^* := F^{-1}_{X_T \mid A_T}(1 - \tau_{\xi_T \mid A_T})$$

is a cost-efficient claim with fixed dependence structure, i.e. X_T^* is a solution of the portfolio optimization problem with fixed payoff structure

$$c(X_T^*) = \inf\{c(Y_T) \colon (Y_T, A_T) \sim G\}.$$

Proof. Let $U = \tau_{\xi_T \mid A_T}$, and write $X_T^* = F^{-1}_{X_T \mid A_T}(1 - U)$. Then

$$(X_T^* \mid A_T = a) = (F^{-1}_{X_T \mid A_T = a}(1 - U) \mid A_T = a) = F^{-1}_{X_T \mid A_T = a}(1 - U)$$

because U and A_T are independent (see (3.3)). Consequently, we obtain $(X_T^*, A_T) \sim (X_T, A_T) \sim G$ and, thus, X_T^* is admissible. Furthermore, since, conditionally on $A_T = a$,

$$((X_T^*, \xi_T) \mid A_T = a) \sim ((F^{-1}_{X_T \mid A_T = a}(1 - F_{\xi_T \mid A_T = a}(\xi_T)), \xi_T) \mid A_T = a),$$

we conclude that X_T^* and ξ_T are anti-monotonic conditionally on $A_T = a$. From the Hoeffding–Fréchet bounds in (2.1), this then implies that

$$\mathbb{E}[X_T^* \xi_T] = \mathbb{E}[\,\mathbb{E}[X_T^* \xi_T \mid A_T]\,] \leq \mathbb{E}[\,\mathbb{E}[X_T \xi_T \mid A_T]\,] = \mathbb{E}[X_T \xi_T],$$

i.e. X_T^* is cost efficient in the class of portfolios with fixed dependence structure.

Remark 3.1. The proof shows that the cost-efficient claim with fixed dependence structure is characterized by the property that, conditionally on A_T, it is anti-monotonic with state price ξ_T. Note that Theorem 3.1 holds in the case that C is any copula (not necessarily the copula of a given initial claim X_T with A_T). The construction of X_T^* depends only on F, A_T, and the copula C, i.e. the asserted payoff structure.

Remark 3.2. When the state price $\xi_T = g_T(S_T)$ is a decreasing function of the stock S_T, as occurs in increasing exponential Lévy models, we deduce that a cost-efficient claim X_T^* is characterized by the property that, conditionally on A_T, X_T^* and S_T are co-monotonic.

Remark 3.3. In the case that the independent randomization V can be generated from the market pair (S_t, S_T) by a transformation we obtain a cost-efficient claim of the form $f(S_t, S_T)$ if $A_T = S_T$, or $f(S_t, S_T, A_T)$ in the general case. Claims of this form are called 'twins' in Bernard *et al.* (2014b). It was shown in that paper that, under some conditions, cost-efficient payoffs are given by twins. With the notion of extended payoffs in this paper we conclude that, generally, optimal payoffs are of the form $f(S_T, V)$ or $f(S_T, A_T, V)$ for some independent randomization V.

4. Utility optimal payoffs with fixed payoff structure

The basic optimization problem of maximizing the expected utility of final wealth X_T at a given horizon T with an initial budget w, i.e.

$$\max_{c(X_T) = w} \mathbb{E}[u(X_T)], \tag{4.1}$$

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was solved in varying degrees of generality in classical papers by Merton (1971), Cox and Huang (1989), and He and Pearson (1991a, 1991b). The optimal solution for differentiable increasing concave utility functions u on (a, b) is of the form

$$X_T^* = (u')^{-1}(\lambda \xi_T), \tag{4.2}$$

where λ is such that $c(X_T^*) = w$. For the existence of λ such that $c(X_T^*) = w$, it is assumed that u' is strictly decreasing, and $u'(a+) = \infty$ and $u'(b-) = 0$.

An extension of the utility optimization problem to the case with a fixed payoff structure was introduced in Bernard *et al.* (2014b) as

$$\max_{c(X_T)=w,\, C_{(X_T, A_T)}=C} \mathbb{E}[u(X_T)]. \tag{4.3}$$

To deal with problem (4.3), define

$$Z_T = C_{1\,|\,A_T}^{-1}(1 - \tau_{\xi_T\,|\,A_T}),$$

where $C_{1\,|\,A_T} = C_{1\,|\,\tau_{A_T}}$ is the conditional distribution function (with respect to C) of the first component given that the second component is the distributional transform τ_{A_T}. Then $Z_T \sim U(0, 1)$, Z_T has copula C with A_T, and the pair (Z_T, ξ_T) is anti-monotonic conditionally on A_T (see also (3.3)).

Next, we introduce the following condition.

(D) $H_T = \mathbb{E}[\xi_T \mid Z_T] = \varphi(Z_T)$ is a decreasing function of Z_T.

Condition (D) does not always hold, but is natural since Z_T and ξ_T are anti-monotonic conditionally on A_T. Strictly speaking, it needs some regularity condition to be satisfied.

The following theorem describes the utility optimal payoff with fixed payoff structure and given budget w under condition (D).

Theorem 4.1. (Utility optimal payoff with given payoff structure.) *Under condition (D), the solution of the restricted portfolio optimization problem (4.3) is given by*

$$X_T^* = (u')^{-1}(\lambda H_T), \tag{4.4}$$

where λ is such that $c(X_T^*) = w$.

Proof. The utility optimal payoff must be a cost-efficient claim with fixed payoff structure (with cost w) as in Theorem 3.1. Otherwise, it is possible to construct a strictly cheaper solution which yields the same utility while respecting the dependence constraint. Consequently, the solution X_T (when it exists) is characterized by the property that, conditionally on A_T, it is anti-monotonic with state price ξ_T and, therefore,

$$X_T = F_{X_T\,|\,A_T}^{-1}(1 - \tau_{\xi_T\,|\,A_T}).$$

The payoff $F_{X_T}^{-1}(Z_T)$ has distribution function F_{X_T}, has copula C with A_T, and, conditionally on A_T, is anti-monotonic with state price ξ_T. By the uniqueness property of cost-efficient claims, this implies that

$$X_T = F_{X_T}^{-1}(Z_T) \quad \text{a.s.}$$

In particular, the optimal solution is increasing in Z_T and the constraint on its cost can be written as

$$c(X_T) = \mathbb{E}[\xi_T F_{X_T}^{-1}(Z_T)] = \mathbb{E}[H_T F_{X_T}^{-1}(Z_T)],$$

where $H_T = \mathbb{E}[\xi_T \mid Z_T] = \varphi(Z_T)$ is decreasing in Z_T by condition (D).

The utility optimization problem of interest can thus be rewritten as

$$\max_{\mathbb{E}[X_T H_T = w],\ X_T = k(Z_T),\ \text{increasing } k} \mathbb{E}[u(X_T)].$$

By considering the relaxed problem

$$\max_{\mathbb{E}[X_T H_T = w]} \mathbb{E}[u(X_T)],$$

we obtain a utility optimization problem in standard form with price density H_T instead of ξ_T. By (4.2), its solution is given by

$$X_T^* = (u')^{-1}(\lambda H_T) = (u')^{-1}(\lambda \varphi(Z_T)),$$

where $\lambda > 0$ is chosen such that $\mathbb{E}[H_T X_T^*] = w$. Since, by condition (D), φ is decreasing, it follows that X_T^* is increasing in Z_T and, thus, it also solves the restricted portfolio optimization problem (4.4).

Bernard and Vanduffel (2014b, Propositions 5.1 and 5.2) derived optimal mean-variance efficient portfolios in the presence of a stochastic benchmark. Their results also follow from Theorem 4.1. An application of Theorem 4.1 in the univariate Black–Scholes model can be found in Bernard *et al.* (2014b). These authors used a Gaussian copula to fix the portfolio structure and verified that condition (D) is satisfied. Note that this example can be extended to the multivariate Black–Scholes model (cf. Section 5).

Interestingly, Theorem 4.1 can be extended to the general case without assuming condition (D) holds. When this is done, the optimal claim is slightly more complex. For the extension, we need to project the function φ from the representation of H_T to the convex cone of decreasing L^2-functions $M_\downarrow$ on [0, 1], where

$$M_\downarrow = \{f \in L^2[0, 1]\colon f \text{ nonincreasing}\}.$$

Let $\varphi \in L^2[0, 1]$, supplied with Lebesgue measure and the Euclidean norm, and let $\widehat{\varphi} = \pi_{M_\downarrow}(\varphi)$ denote the projection of φ on $M_\downarrow$. Then we obtain the following result.

Theorem 4.2. (Utility optimal payoff.) *Assume that* $H_T = \mathbb{E}[\xi_T \mid Z_T] = \varphi(Z_T)$ *with* $\varphi \in L^2[0, 1]$. *Then the solution to the restricted utility optimization problem (4.3) is given by*

$$X_T^* = (u')^{-1}(\lambda \widehat{H}_T),$$

where $\widehat{H}_T = \widehat{\varphi}(Z_T)$ *and* λ *is such that* $c(X_T^*) = w$.

Proof. The proof is analogous to that of Theorem 5.2 of Bernard *et al.* (2014b). It is based on properties of projection on convex cones; these can be found in Barlow *et al.* (1972).

Remark 4.1. The projection $\widehat{\varphi}$ of φ on $M_\downarrow$ is given as the slope of the smallest concave majorant $\mathrm{SCM}(\varphi)$ of φ, i.e. $\widehat{\varphi} = (\mathrm{SCM}(\varphi))'$. Fast algorithms exist for determining $\widehat{\varphi}$.

Remark 4.2. The condition $\varphi \in L^2[0, 1]$ is implied by the condition $\xi_T \in L^2(\mathbb{P})$.

5. Optimal hedging and quantile hedging

In this section we use the results of Sections 2–4 to solve various forms of static partial hedging problems. Let L_T be a financial derivative (liability), and let $w_L = c(L_T)$ denote the price of L_T with respect to the underlying pricing measure. If the available budget w is smaller than w_L then it is of interest to find a best possible partial hedge (cover) of L_T with cost w under various optimality criteria. This leads to the following basic static partial hedging problems.

The *quantile (super-)hedging problem* is defined as

$$\max_{c(X_T)=w} \mathbb{P}\{X_T \geq L_T\}.$$

The *utility optimal hedging problem* is a natural variant of (5.1) and is stated as

$$\max_{c(X_T)=w} \mathbb{E}[u(X_T - L_T)],$$

where u is a given concave utility function which is defined on $\mathbb{R}$ and satisfies the same regularity conditions as in Section 4.

A more general version of the hedging problem in (5.2) is obtained by replacing the expected utility by some law-invariant, convex risk measure Ψ (Ψ monotonic in the natural order), i.e.

$$\max_{c(X_T)=w} \Psi(X_T - L_T).$$

We also consider (state-dependent) variants of the partial hedging problems (5.1)–(5.3) in which the excess $X_T - L_T$ satisfies additional restrictions, allowing us to control its excess structure. For example, we may want $X_T - L_T$ to have a certain copula C with a benchmark A_T, i.e. $C_{(X_T - L_T, A_T)} = C$, or we may impose certain additional boundedness conditions on X_T.

We start with the unconstrained optimal hedging problem (5.2). Its solution is given in the following proposition.

Proposition 5.1. (Utility optimal hedge.) *Let L_T be a financial claim with price $c(L_T) = w_L$, and let $w < w_L$ be the budget available for hedging. Then the optimal hedge for the utility optimal hedging problem (5.2) is given by*

$$X_T^* = L_T + (u')^{-1}(\lambda \xi_T),$$

where $\lambda \geq 0$ is such that

$$c((u')^{-1}(\lambda \xi_T)) = w - w_L.$$

Proof. By the classical portfolio optimization result (see (4.2)), it follows that the optimal solution of the utility optimization problem

$$\max_{c(X_T)=w-w_L} \mathbb{E}[u(X_T)]$$

is given by

$$\widehat{X}_T = (u')^{-1}(\lambda \xi_T)$$

with λ chosen in such a way that $c(\widehat{X}_T) = \mathbb{E}[\xi_T \widehat{X}_T] = w - w_L$. The claim $X_T^* := \widehat{X}_T + L_T$ therefore has price w, $c(X_T^*) = w$. By definition, X_T^* solves the utility optimal hedging problem in (5.2).

In the following two extensions we fix the joint dependence structure of the excess $X_T - L_T$ with a given benchmark A_T.

Proposition 5.2. (Utility optimal hedge with dependence restriction on the excess.) *Let L_T be a financial claim with price $c(L_T) = w_L$, let $w < w_L$ be the budget available for hedging, and let A_T be a given benchmark. Then the restricted utility optimal hedging problem*

$$\max_{c(X_T)=w,\, C_{(X_T-L_T, A_T)}=C} \mathbb{E}[u(X_T - L_T)]$$

has the solution

$$X_T^* = L_T + (u')^{-1}(\lambda \widehat{H}_T),$$

where $\widehat{H}_T = \widehat{\varphi}(Z_T)$ and $\lambda \geq 0$ is such that

$$c((u')^{-1}(\lambda \widehat{H}_T)) = w - w_L.$$

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Proof. The optimality of X_T^* is a consequence of Theorem 4.2 and is based on a simple replacement strategy, as in the proof of Proposition 5.1.

In the following variant of the hedging problem we aim to avoid super hedging L_T.

Proposition 5.3. (Utility optimal hedge with negative excess.) *Let L_T be a financial claim with price $c(L_T) = w_L$, and let $w < w_L$ be the budget available for hedging. The optimal hedge with a lower-bound constraint, i.e. the solution of*

$$\max_{X_T \le L_T,\, c(X_T)=w} \mathbb{E}[u(L_T - X_T)],$$

is given by

$$X_T^* = \min\{L_T, L_T + (u')^{-1}(\lambda \xi_T)\}, \tag{5.5}$$

where $\lambda \ge 0$ is such that

$$c((u')^{-1}(\lambda \xi_T)) = w - w_L.$$

Proof. When $Y_T = L_T - X_T$, the hedging problem in (5.4) is reduced to the classical utility optimization problem in (4.1) with the additional constraint $Y_T \ge 0$. The solution of this problem is easily shown to be the classical solution Y_T^* restricted to this boundary. Consequently, we obtain $X_T^* = \min\{L_T, Y_T^* + L_T\}$, as in (5.5).

When enough financial resources $w > w_L$ are available, it might be of interest to obtain the best super hedge $X_T \ge L_T$. We omit details of the proof for this case.

Proposition 5.4. (Utility optimal hedge with boundedness restriction on the excess.) *Let L_T be a financial claim with price $c(L_T) = w_L$, and let $w > w_L$ be the budget available. The optimal super hedge, i.e. the solution of*

$$\max_{L_T \le X_T,\, c(X_T)=w} \mathbb{E}[u(X_T - L_T)],$$

is given by

$$X_T^* = \max\{L_T, L_T + (u')^{-1}(\lambda \xi_T)\},$$

where $\lambda \ge 0$ is such that

$$c((u')^{-1}(\lambda \xi_T)) = w - w_0.$$

The quantile super-hedging problem (5.1) was introduced in Browne (1999) for a deterministic target L_T in a Black–Scholes model. This result was extended in Bernard *et al.* (2014c, Theorem 5.6) to random targets $L_T \ge 0$ under regularity conditions. The following proposition solves this problem without posing any regularity conditions.

Proposition 5.5. (Quantile super hedging.) *Let $L_T \ge 0$ be a financial claim with price $c(L_T) = w_L$, and let $w \le w_L$ be the budget available. Then the solution to the quantile super-hedging problem in (5.1), i.e.*

$$\max_{0 \le X_T,\, c(X_T)=w} \mathbb{P}\{X_T \ge L_T\},$$

is given by

$$X_T^* = L_T \mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}},$$

where λ is such that $c(X_T^*) = w$.

Proof. The optimal solution X_T^* of (5.1) has a joint distribution G with the 'benchmark' L_T. Then Theorem 3.1 implies that X_T^* is an optimal claim with fixed payoff structure. Therefore, conditionally on L_T, X_T^* is anti-monotonic with state price ξ_T and is of the form $X_T^* = f(\xi_T, L_T, V)$, where V is some independent randomization.

Define the sets $A_0 = \{f(\xi_T, L_T, V) = 0\}$ and $A_1 = \{f(\xi_T, L_T, V) = L_T\}$ so that $\mathbb{P}\{A_0 \cup A_1\} = 1$, because otherwise it would be possible to construct an improved solution. Consequently, we deduce that f can be represented in the form

$$f(\xi_T, L_T, V) = L_T \mathbf{1}_{\{h(\xi_T, L_T, V) \in A\}}$$

for some function h and measurable set A. Define $\lambda > 0$ such that

$$\mathbb{P}\{h(\xi_T, L_T, V) \in A\} = \mathbb{P}\{\tau_{L_T \xi_T} < \lambda\}.$$

Then $\mathbf{1}_{\{h(\xi_T, L_T, V) \in A\}}$ and $\mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}}$ have the same distribution while $L_T \xi_T$ and $\mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}}$ are anti-monotonic. Then by using the Hoeffding–Fréchet lower bound in (2.1) we obtain

$$c(L_T \mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}}) = \mathbb{E}[L_T \xi_T \mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}}] \le \mathbb{E}[L_T \xi_T \mathbf{1}_{\{h(\xi_T, L_T, V) \in A\}}],$$

and, thus, $L_T \mathbf{1}_{\{\tau_{L_T \xi_T} < \lambda\}}$ is optimal.

Remark 5.1. It was pointed out to the authors by a reviewer that the optimization results in Proposition 5.5 and Proposition 5.7 can be cast as unconstrained optimization problems in Lagrangian form. For Proposition 5.5, this takes the form

$$\sup_{X_T \ge 0} \mathbb{E}[\mathbf{1}_{\{X_T \ge L_T\}} - \lambda \xi_T X_T] + \lambda w. \tag{5.6}$$

Under a continuity assumption, a solution of (5.6) is achieved by $X_T = L_T \mathbf{1}_{\{\lambda \xi L_T < 1\}}$ with λ chosen suitably. The Lagrangian form for the case of Proposition 5.7 is similar.

As a last application on hedging problems, we extend Proposition 5.5 by considering the combined case of a random claim L_T that needs to be hedged and the requirement that the hedging portfolio has some copula C with a random benchmark A_T.

Proposition 5.6. (Quantile hedging with fixed payoff structure.) *For a random claim $L_T \ge 0$, benchmark A_T, and given copula C, the solution to the restricted hedging problem*

$$\max_{0 \le X_T,\ c(X_T) = w,\ C_{(X_T, A_T)} = C} \mathbb{P}\{X_T \ge L_T\}$$

is given by

$$X_T^* = L_T \mathbf{1}_{\{Z_T \ge \lambda\}},$$

where $Z_T = C^{-1}_{1 \mid A_T}(1 - \tau_{L_T \xi_T \mid A_T})$ and λ is such that $c(X_T^*) = w$.

Proof. Let G be the joint distribution of the optimal claim X_T^* with A_T. Then, by the randomization technique in Section 2, there exists a claim of the form $f(\xi_T, A_T, V)$ with a randomization V independent of (ξ_T, A_T), such that

$$(f(\xi_T, A_T, V), A_T) \sim (X_T^*, A_T) \sim G \quad \text{and} \quad c(f(\xi_T, A_T, V), A_T) = c(X_T^*) = w.$$

Thus, the claim $f(\xi_T, A_T, V)$ is also optimal.

As in the proof of Proposition 5.5, we define $A_0 = \{f(\xi_T, A_T, V) = 0\}$ and $A_1 = \{f(\xi_T, A_T, V) = L_T\}$ for which $\mathbb{P}\{A_0 \cup A_1\} = 1$, so there exists a measurable set A and a function h such that

$$f(\xi_T, A_T, V) = L_T \mathbf{1}_{\{h(\xi_T, A_T, V) \in A\}}.$$

Define $\lambda > 0$ by the equation

$$\mathbb{P}\{h(S_T, A_T, V) \in A\} = \mathbb{P}\{Z_T \ge \lambda\}.$$

Then $\mathbf{1}_{\{h(\xi_T, A_T, V)\in A\}}$ and $\mathbf{1}_{\{Z_T\geq\lambda\}}$ have the same distribution. By the Hoeffding–Fréchet inequalities in (2.1), this implies that

$$c(L_T\mathbf{1}_{\{Z_T\geq\lambda\}}) = \mathbb{E}[\xi_T L_T \mathbf{1}_{\{Z_T\geq\lambda\}}] \leq c(L_T\mathbf{1}_{\{h(\xi_T, A_T)\in A\}})$$

because, conditionally on A_T, Z_T is anti-monotonic with $\xi_T L_T$ and, thus, $\mathbf{1}_{\{Z_T\geq\lambda\}}$ is anti-monotonic with $\xi_T L_T$. This implies optimality of X_T^*.

In the final application we consider the related problem of maximizing expected return with given cost and target bounds. For given bounds a and b with $a < b$, assume the existence of a claim X_T such that $a \leq X_T \leq b$ and $c(X_T) = w$.

Proposition 5.7. (Maximizing expected return with given target bounds.) *The solution of the expected returns maximization problem*

$$\max_{a\leq X_T\leq b,\, c(X_T)=w} \mathbb{E}[X_T]$$

is given by the payoff

$$X_T^* = a\mathbf{1}_{\{\tau_{\xi_T}>\lambda\}} + b\mathbf{1}_{\{\tau_{\xi_T}\leq\lambda\}},$$

where λ is such that $c(X_T^*) = w$.

Proof. Assume that X_T^* is not an optimal payoff. Then there exists an admissible payoff Y_T such that $\mathbb{E}[Y_T] > \mathbb{E}[X_T^*]$. We can then also find a strategy Y_T^* of the form

$$Y_T^* = a\mathbf{1}_{\{\tau_{\xi_T}>d\}} + b\mathbf{1}_{\{\tau_{\xi_T}\leq d\}}$$

with d chosen such that $\mathbb{E}[Y_T^*] = \mathbb{E}[Y_T]$. Since $\mathbb{E}[Y_T^*] > \mathbb{E}[X_T^*]$, it follows that $d > \lambda$ and, therefore,

$$c(Y_T^*) = \mathbb{E}[\xi_T Y_T^*] > c(X_T^*) = w.$$

On the other hand, because Y_T^* has the same expectation and shifts all mass to the boundaries, $Y_T \leq_{\mathrm{cx}} Y_T^*$, where '$\leq_{\mathrm{cx}}$' denotes the convex order inequality. Let $\widehat{Y}_T = F_{Y_T}^{-1}(1-\tau_{\xi_T})$ be the random variable with $\widehat{Y}_T \sim Y_T$ and such that $\widehat{Y}_T$ and ξ_T are anti-monotonic. Then, from the Hoeffding inequality and the Lorentz ordering theorem (see Rüschendorf (2013)), we obtain

$$c(Y_T) = \mathbb{E}[\xi_T Y_T] \geq \mathbb{E}[\xi_T \widehat{Y}_T] \geq \mathbb{E}[\xi_T Y_T^*] = c(Y^*),$$

where we have used the inequality $\widehat{Y}_T \sim Y_T$, implying that $\widehat{Y}_T \leq_{\mathrm{cx}} Y_T^*$. This in turn implies that $c(Y_T) > w$, and, thus, Y_T is not admissible. This contradiction implies the result.

Example 5.1. (*Maximizing expected return with target bounds in a Black–Scholes market.*) In the n-dimensional Black–Scholes market there is a (risk-free) bond with price process $\{S_t^0\} = \{S_0^0 \mathrm{e}^{rt}\}$ for some (small) $r > 0$ and n risky assets $S^1, S^2, \dots, S^n$ with price processes

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_t^i, \qquad i = 1, \dots, n,$$

where the $\{B_t^i\}$ are (correlated) standard Brownian motions, with constant correlation coefficients $\rho_{ij} := \mathrm{corr}(B_t^i, B_{t+s}^j)$, $t, s \geq 0$ and $i, j = 1, \dots, n$. Let $\boldsymbol{\mu}^\top = (\mu_1, \dots, \mu_n)$ and $(\boldsymbol{\Sigma})_{ij} = \rho_{ij}\sigma_i\sigma_j$, and assume that $\mu_i \neq r$ for some i. Let $\boldsymbol{\Sigma}$ be positive definite. Then the state price takes the form (see, e.g. Bernard *et al.* (2011))

$$\xi_t = c\left(\frac{S_t}{S_0}\right)^{-d},$$

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where, in terms of the parameters $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)^\top$, m and σ^2 defined by

$$\boldsymbol{\pi} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\,\mathbf{1})}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\,\mathbf{1})}, \qquad m = \boldsymbol{\pi}^\top \boldsymbol{\mu}, \qquad \sigma^2 = \boldsymbol{\pi}^\top \boldsymbol{\Sigma}\,\boldsymbol{\pi},$$

$$d = \frac{m - r}{\sigma^2}, \quad \text{and} \quad c = \exp\biggl\{-\frac{1}{2}\biggl[d - 1 + \biggl(1 + \frac{r}{\sigma^2}\biggr)^2\biggr]\sigma^2 t\biggr\},$$

and the process $\{S_t\}$ satisfies the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = m\,\mathrm{d}t + \sigma\,\mathrm{d}B_t,$$

where $\{B_t\}$ is a standard Brownian motion satisfying $B_t = \sum_{i=1}^n \pi_i \sigma_i B_t^i / \sigma$. The process $\{S_t\}$ is the price process that corresponds to a so-called constant-mix trading strategy (at each time $t > 0$ a fixed proportion π_i is invested in the ith risky asset).

We make the (economic appealing) assumption that $m > r$. From Proposition 5.7, since τ_{ξ_T} is decreasing in S_T, the optimal payoff is of the form

$$X_T^* = a\mathbf{1}_{\{S_T < \alpha\}} + b\mathbf{1}_{\{S_T \geq \alpha\}},$$

where α is such that $\mathbb{E}_Q[\mathbf{1}_{\{S_T \geq \alpha\}}] = (w\mathrm{e}^{rT} - a)/(b - a)$ in which $\mathrm{d}Q/\mathrm{d}\mathbb{P} = \mathrm{e}^{rT}\xi_T$. It follows that α is given by

$$\alpha = \exp\biggl[\biggl(r - \frac{1}{2}\sigma^2\biggr)T - \sigma\sqrt{T}\,\Phi^{-1}\biggl(\frac{w\mathrm{e}^{rT} - a}{b - a}\biggr)\biggr].$$

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