Geometrische ordnungen, by O. Haupt and H. Künneth. Springer-Verlag, Berlin, Heidelberg, New York, 1967. viii + 429 pages. $17.00.

This publication is unique in many respects. It is the first book which deals with order geometry. Many of the proofs in it appear for the first time. The presentation is very complete and up to date. (The bibliography at the end covers almost thirteen pages.)

The basic order concept is illustrated in the opening pages by two pioneer results of the subject. A curve in the real projective plane is defined to have order three if no line contains more than three of its points. Juel's structure theorem for such curves is quoted. The second result is the 4-vertex theorem dealing with Jordan curves with continuous curvature. A vertex is defined to be a curve point every curve neighbourhood of which contains at least four points of some circle and thus has local order of at least four. The line and the circle, thus used, lead naturally to the concept of order characteristics. An order characteristic is defined initially on a closed disk either as a simple arc with end points on the disk boundary or a Jordan curve which satisfies a certain axiom system. The two connected sets of the disk which remain after an order characteristic K is removed are known as the "sides" of K. Among the defining axioms of the order characteristics is the existence of a fundamental number k, $k \geq 1$, with the property that at most one order characteristic contains k distinct points $x_1, x_2, \ldots, x_k$ of the disk. Moreover if $x_1, x_2, \ldots, x_k$ are on an order characteristic, then there is an order characteristic which contains points $x'_1, x'_2, \ldots, x'_k$ provided only x'_i is sufficiently close to x_i, $1 \leq i \leq k$. As closed sets the order characteristics are elements of the Hausdorff topology of the closed sets of the disk.

The component order of a continuum C of a disk G with respect to an order characteristic K is defined to be the (cardinal) number of the components of $C \cap K$. This number, in the case in which each component of $C \cap K$ is a single point, is said to be the point order of C with respect to K. If the component (point) order of C is finite for each K, C is said to have finite component (point) order. If the order is bounded for all K then the minimum bound is defined to be the order of C with respect to system of order characteristics. The first few pages of the book contain results of great generality dealing with continua C of finite component order.

A short section is devoted to a new proof of the contraction theorem. This deals with a curve C an arc A of which contains $k + 1$ points of an order characteristic K (k is the fundamental number of the order characteristic system) which occur in the same order on both C and K. A process is defined whereby a characteristic K' is found which contains $k + 1$ points of C in an arbitrarily small subarc of A. A related expansion theorem is developed for curves of order $k + 1$.

In the next section, which includes a quarter of the book, the axioms for the order characteristics are modified so that the lines of the projective plane or those of the hyperbolic plane are specialized order characteristics. To do this the geometry is developed in a set homeomorphic either to the projective plane or to a disk within it. The fundamental number k of the system becomes 2 and the assumption is made that there is exactly one order characteristic which contains two arbitrary distinct points of G. Convexity is studied in a region obtained by removing an order characteristic from G, special attention being given to the differentiability properties of the boundary of convex regions. Curves of (point) order 3 are considered without the assumption that they have continuously moving tangents. The results are obtained by use of the underlying ideas of the contraction and expansion theorems. Another of the topics discussed in this section is that of curves of maximal class index. These are continuously differentiable curves in the projective plane for which a number k exists so that each point of the projective plane is within at least k-2 tangents but within at most k tangents. All such curves are constructed. Two Plücker-type formulas are given for these curves which connect a topologically defined genus, the number of their cusps, inflexion points and double tangents. The induction proofs depend on continuous deformation of the curves. Also in this section is the senior author's generalization of the Möbius theorem that a single closed differentiable curve, without cusps or double points, of odd order has at least 3 inflexion points. A final section on the order geometry of the plane in which the fundamental number k of the order characteristics is subject only to the condition $k \geq 2$ deals mostly with the 4-vertex and related theorems.

The remaining third of the book deals with higher dimensional geometry. In most cases the basic space is the n-dimensional real projective space P_n and the order characteristics are the hyperplanes. Among the topics discussed are differentiability of curves of finite order, completion of an arc of order n to a closed curve of order n, the theorem that a curve of order $n + 1$ is the union of a finite number of arcs of order n and monotone sequences. In the closing pages order characteristics are defined for a compact metric space and generalizations of some of the earlier results are given.

A supplement contains an account of the work on the subject of a number of authors without proofs which is not covered in the previous parts of the book.

Nearly all the proofs are within the framework of set theoretic topology. Anyone with a working knowledge of this subject would be able to read the book. As the only complete work on order goemetry which is presently available, it will be an indispensable reference for those interested in the subject.

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