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HOLOMORPHIC FLOWS WITH PERIODIC ORBITS ON STEIN SURFACES

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Abstract

In this paper we study the classification of holomorphic flows on Stein spaces of dimension two. We assume that the flow has periodic orbits, not necessarily with a same period. Then we prove a linearization result for the flow, under some natural conditions on the surface.

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1. Introduction

In this paper we address the problem of classification of holomorphic flows on Stein spaces of dimension two. Though the study of holomorphic actions of the complex multiplicative group $\mathbb{C}^*$ appears to be well developed, with an extensive list of linearization and classification results for the action as well as for the ambient manifold [8, 9, 17], the study of holomorphic flows (that is, holomorphic $\mathbb{C}$-actions) seems to be a harder topic. This is in part due to the fact that the group $\mathbb{C}^*$ is reductive, which is not the case for the additive group $\mathbb{C}$. In addition, each $\mathbb{C}^*$-action generates a flow in an obvious manner, but the converse is not true and, indeed, there is a wide class of examples of holomorphic flows which do not come from $\mathbb{C}^*$-actions. At this level of generality there is slim hope of getting extensive classification results as indicated in some previous work [7]. Therefore we make a hypothesis on the orbits of the flow that may allow a classification. In this work we consider the case where the flow has periodic orbits. This means that the nonsingular orbits of the flow are periodic but not necessarily with a common period. Since by Suzuki [18] periodic orbits of holomorphic flows on Stein surfaces only accumulate at the singular set, our hypothesis can be seen as a dynamical hypothesis.

Let us now recall some of the motivation for our current approach based also on holomorphic foliations theory. A fundamental contribution to the study of holomorphic actions on Stein surfaces was made by Suzuki who introduced into this subject the techniques of theory of foliations and potential theory (see [18, 19]). In our current framework Suzuki's main result is probably that *any analytic* $\mathbb{C}^*$*-action on* $\mathbb{C}^2$ *is analytically linearizable*. The classification of holomorphic $\mathbb{C}$-actions with proper orbits on $\mathbb{C}^2$ was also obtained by Suzuki [19, Theorem 4], and not all of them are linearizable. In connection with this, in [2] the authors address the classification of a Stein analytic space N of dimension two, with a normal singularity $p \in N$, endowed with a $\mathbb{C}^*$-action φ having a *dicritical singularity* at p, meaning that p is a fixed point such that every nonsingular orbit of φ close enough to p accumulates at and only at p. These are called *quasi-homogeneous singularities* in the framework of analytic and algebraic geometry [15]. This study continues in [5] where the authors study the nondicritical case. Let us now state our main result. Given a holomorphic flow ψ on a Stein manifold, a nonsingular orbit is biholomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$ if it has a nontrivial isotropy group. In the last case the orbit is called *periodic*. We prove the following linearization result.

THEOREM 1.1. *Let ψ be a holomorphic flow with periodic orbits* (*not necessarily with a common period*) *and isolated singularities on a connected Stein surface N with $H^2(N, \mathbb{Z}) = 0$ and $H_1(N, \mathbb{C}) = 0$. Then, up to a reparametrization, the flow is induced by a holomorphic $\mathbb{C}^*$-action and we have the following possibilities.*

(i) *ψ has a dicritical singularity and is globally linearizable.*
(ii) *ψ exhibits no dicritical singularity but has a nondicritical analytically linearizable singularity, and the corresponding foliation is the pull-back of a linear foliation on $\mathbb{C}^2$.*

We shall see that the flow is periodic in case (i) and induces a $\mathbb{C}^*$-action. In case (ii) the corresponding holomorphic vector field is a multiple of a complete vector field with periodic flow. It will follow from the proof we give that the result holds partially for holomorphic flows with isolated singularities and *generic orbit* (generic in the sense of Theorem 2.2) periodic, that is, diffeomorphic to $\mathbb{C}^*$. Indeed, for item (ii), we can prove that the tangent vector field $Z_\psi := (\partial\psi_t/\partial t)|_{t=0}$ can be multiplied by a suitable *meromorphic* function τ (the period map) so that it becomes periodic with constant period and therefore associated with a $\mathbb{C}^*$-action (see the proof of Lemma 2.9). If all orbits are periodic the map τ is holomorphic without zeros.

The result is sharp. Indeed, letting $N = \mathbb{C}^* \times \mathbb{C}$, then N is Stein and equipped with a periodic (nonsingular) horizontal flow. Nevertheless, N is not biholomorphic to $\mathbb{C}^2$. This shows that the hypothesis $H_1(N, \mathbb{C}) = 0$ cannot be dropped in Theorem 1.1. For (ii), we consider a linear vector field $Z_0 = nx(\partial/\partial x) - my(\partial/\partial y)$ with $n, m \in \mathbb{N}$ in affine coordinates $(x, y) \in \mathbb{C}^2$. Then we consider the transverse section $\Sigma \subset \mathbb{C}^2$ given by $\{x = 1\} \cap \{|y| < 1\}$. Let N be the union of the y-axis with the *saturation* $\mathrm{Sat}_{Z_0}(\Sigma) \subset \mathbb{C}^2$ of the transverse section Σ by the orbits of Z_0. Then N gives an open subset of $\mathbb{C}^2$ invariant by the flow of Z_0, which is periodic, and which exhibits a nondicritical singularity at the origin $0 \in N$. The foliation $\mathcal{F}_{Z_0}$ induced by Z_0 on N is obviously the pull-back of a linear foliation on $\mathbb{C}^2$, the ambient surface is Stein, satisfies $H_1(N, \mathbb{C}) = 0$ and $H^2(N, \mathbb{Z}) = 0$. On the other hand, N is not biholomorphic to $\mathbb{C}^2$ and this shows that the conclusion in (ii) cannot be improved.

From the main result in [19] we promptly conclude the following corollary.

COROLLARY 1.2. *A holomorphic flow with isolated singularities and periodic orbits on* $\mathbb{C}^2$ *is analytically linearizable of the form* $t \circ (x, y) = (e^{nt}x, e^{mt}y)$ *where* $t \in \mathbb{C}$, $(x, y) \in \mathbb{C}^2$, *and* $m, n \in \mathbb{Z}$.

2. Resolution of singularities

Throughout this section N denotes a nonsingular Stein analytic space of dimension two endowed with a holomorphic flow $\psi : \mathbb{C} \times N \to N$ having isolated singularities. We denote by $\mathcal{F}_\psi$ the foliation induced by ψ and study the behavior of the flow in a neighborhood of a nondicritical singularity (fixed point).

PROPOSITION 2.1. *Let* N *be a nonsingular Stein analytic space of dimension two. Let* ψ *be a holomorphic flow with periodic orbits on* N*, with isolated fixed points, and let* $p \in N$ *be a nondicritical singularity of the corresponding foliation* $\mathcal{F}_\psi$*. Then* $\mathcal{F}_\psi$ *is analytically linearizable in a neighborhood of* p.

Let us recall some of Suzuki's results.

THEOREM 2.2 [18]. *Given a* $\mathbb{C}$*-action* ψ *on a normal Stein analytic space* N *of dimension two, the following statements hold.*

(i) *There is a subset* $e \subset N$ *of logarithmic capacity zero such that* $\psi_t(e) = e$*, for any* $t \in \mathbb{C}$*, and all orbits of* ψ *in* $N \setminus e$ *are biholomorphic.*
(ii) *Any leaf of* $\mathcal{F}_\psi$ *containing an orbit of* ψ *isomorphic to* $\mathbb{C}^*$ *is closed in* $N \setminus \operatorname{sing}(\mathcal{F}_\psi)$.
(iii) *If the leaves of* $\mathcal{F}_\psi$ *are properly embedded in* $N \setminus \operatorname{sing}(\mathcal{F}_\psi)$ *then there is a meromorphic first integral of* $\mathcal{F}_\psi$ *on* N*, not constant, and one can find a Riemann surface* S *and a surjective holomorphic map* $p : N \setminus \operatorname{sing}(\mathcal{F}_\psi) \to S$*, such that:*
    (iii.1) *the irreducible components of the fibers* $\{p^{-1}(w); w \in S\}$ *of* p *are the leaves of* $\mathcal{F}_\psi$*;*
    (iii.2) *the subset* $\mathcal{E} \subset N$ *defined as the union of all the nonirreducible levels* $p^{-1}(w)$*,* $w \in S$*, has zero logarithmic capacity.*
(iv) *If the generic orbit is biholomorphic to* $\mathbb{C}^*$*, then each leaf of* $\mathcal{F}_\psi$ *is closed in* $N \setminus \operatorname{sing}(\mathcal{F}_\psi)$ *and therefore there is a meromorphic first integral as in (iii).*

The resolution theorem of Seidenberg [16] asserts that there is a proper holomorphic map which is a finite sequence of quadratic blow-ups $\rho : \tilde{N} \to N$ such that the following statements hold.

(1) The *exceptional divisor* $D = \rho^{-1}(p) = \bigcup_{j=1}^m \sigma_j \subset \tilde{N}$ is a union of projective lines, copies of the Riemann sphere, with normal crossings, no cycles and no triple points.
(2) The pull-back foliation $\tilde{\mathcal{F}}_\psi := \rho^*(\mathcal{F}_\psi)$ on $\tilde{N}$ is such that any singularity $\tilde{p} \in \operatorname{sing}(\tilde{\mathcal{F}}_\psi) \cap D$ is isolated and *reduced* of one of the following types in local coordinates:

(a) $x\,dy - \lambda y\,dx + \text{h. o. t.} = 0,\ \lambda \in \mathbb{C} \setminus \mathbb{Q}_+$ (*nondegenerate*);
(b) $x^{k+1}\,dy - [y(1 + \lambda x^k) + \text{h. o. t.}]\,dx = 0,\ k \in \mathbb{N},\ \lambda \in \mathbb{C}$ (*saddle-node*).

Here, by h. o. t. we mean 'higher-order terms'.

Conditions (1) and (2) above hold for any isolated singularity p of a holomorphic foliation in dimension two. Using now the flow ψ that generates $\mathcal{F}_\psi$ and the fact that the orbits are periodic, as well as the fact that N is Stein, we obtain the following result.

PROPOSITION 2.3. *Let N be a nonsingular Stein analytic space of dimension two. Let ψ be a holomorphic flow on N with periodic orbits and isolated fixed points, and let $p \in N$ be a nondicritical singularity of $\mathcal{F}_\psi$. There is a resolution $\rho : \widetilde{N} \to N$ of the singularity p of $\mathcal{F}_\psi$ such that:*

(i) *each singular point of $\widetilde{\mathcal{F}}_\psi$ has local form $d(x^\ell y^k) = 0$, for $k, \ell \in \mathbb{N}$, in suitable local coordinates;*

(ii) *there is a holomorphic flow $\tilde{\psi}$ on $\tilde{N}$ induced by ψ in the sense that*

$$\tilde{\psi}(s, \tilde{p}) = \psi(s, \rho(\tilde{p})), \quad \forall s \in \mathbb{C}^*,\ \tilde{p} \in \tilde{N};$$

(iii) *the exceptional divisor is a linear chain formed by invariant Riemann spheres σ_j, $j \geq 1$, with two separatrices $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ intersecting D outside the corners.*

We recall that a compact analytic divisor D of dimension one on a complex surface N is a *linear chain* if it is a union of compact Riemann surfaces, elements of the divisor D, say $D_1, \ldots, D_n$, such that $D_i \cap D_j$ is nonempty with $i < j$ if and only if $i = j - 1$, and in this case it is a point, for $j = 2, \ldots, n$. Each intersection $D_i \cap D_j$ is called a *corner*.

In the first part of Proposition 2.3 we use the same argument as in the proof of Proposition 1 in [5] and, for sake of clarity, we briefly repeat it here. The first part of the argument is the following extension lemma from [5] (see Lemma 1).

LEMMA 2.4. *Let $\tilde{Z}$ be a meromorphic vector field defined in a complex manifold $\tilde{N}$, denote by $\mathcal{F}_{\tilde{Z}}$ the corresponding foliation in $\tilde{N}$ and assume that for some invariant codimension-one compact analytic divisor $D = \bigcup_{j=1}^r D_j \subset \tilde{N}$, the following statements hold.*

(i) *$\tilde{Z}$ is holomorphic with isolated singularities and complete in $\tilde{V} \setminus D$.*

(ii) *The divisor D is connected, has normal crossings, no cycles and no triple points, and each singularity of the foliation $\mathcal{F}_{\tilde{Z}}$ in D is reduced of the local form $d(x^k y^\ell) = 0$, for $k, \ell \in \mathbb{N}$.*

(iii) *There is a separatrix $\tilde{\Gamma}$ transverse to D with $\tilde{\Gamma} \cap D = \tilde{\Gamma} \cap D_1 = \{p\}$ and $\tilde{Z}|_{\tilde{\Gamma}}$ has an isolated singularity at p.*

Then $\tilde{Z}$ is holomorphic, complete and with isolated singularities in D.

PROOF OF PROPOSITION 2.3. By hypothesis, the nonsingular orbits of ψ are diffeomorphic to $\mathbb{C}^*$. Since N is Stein of dimension two and the generic orbit of ψ

is $\mathbb{C}^*$ then by the main theorem of [18] the foliation $\mathcal{F}_\psi$ admits a meromorphic first integral in N. By composition with the resolution map $\rho : \tilde{N} \to N$ we have a meromorphic first integral for $\tilde{\mathcal{F}}_\psi$ on $\tilde{N}$. Thus, by the local form of the reduced singularities [10–12] we conclude that all singularities arising in the exceptional (resolution) divisor are linearizable with local holomorphic first integral, that is, of the local form $d(x^\ell y^k) = 0$, $k, \ell \in \mathbb{N}$, in suitable coordinates. The natural lift $\tilde{\psi}$ of the flow ψ to $\tilde{V} \setminus D$ must extend to a flow on D: indeed, $\tilde{\psi}$ induces a holomorphic vector field $\tilde{Z}$ on $\tilde{V} \setminus D$, and the vector field $\tilde{Z}$ has no essential singularity on the divisor D so that we can extend it to D as a meromorphic (possibly holomorphic) vector field which we shall also denote by $\tilde{Z}$. Let us now prove that $\tilde{Z}$ has no poles in D and has isolated singularities only. By [3] we can find a separatrix Γ of $\mathcal{F}_\psi$ through p. This gives a smooth curve $\tilde{\Gamma}$ on the resolution which meets the exceptional divisor D at a singular point $\tilde{p}$ of local type $\tilde{\mathcal{F}}_\psi : kx\, dy + \ell y\, dx = 0$, $k, \ell \in \mathbb{N}$. Then by Lemma 2.4 the vector field $\tilde{Z}$ extends holomorphically to D, with isolated singularities. This extension is complete because D is compact. Therefore $\tilde{\psi}$ extends to $\tilde{N}$, satisfying the relation $\tilde{\psi}_t(\tilde{p}) = \psi_t(\rho(\tilde{p}))$, for all $t \in \mathbb{C}$, for all $\tilde{p} \in \tilde{N}$. We know that $\tilde{Z}$ is a complete holomorphic vector field with isolated singularities tangent to each component D_j of D, which is a copy of the Riemann sphere $\overline{\mathbb{C}}$. Thus, the restriction $\tilde{Z}_j := \tilde{Z}|_{D_j}$ is linear and may be written in some suitable affine coordinate $z : \mathbb{C} \to D_j \cong \overline{\mathbb{C}}$ as $\tilde{Z}_j(z) = n_j z(\partial/\partial z)$ for some $n_j \in \mathbb{N}$. In particular, $\tilde{Z}_j$ has two singularities in D_j. Hence, each component of the resolution divisor exhibits exactly two singularities of $\tilde{Z}$ and, since there are no cycles in the resolution divisor, the exceptional divisor is a linear chain, with two separatrices $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ intersecting D outside the corners. □

We now prove Proposition 2.1. We denote by $\Gamma_1, \Gamma_2 \subset N$ the separatrices obtained by blowing down $\tilde{\Gamma}_1, \tilde{\Gamma}_2$, respectively. Using the fact that the orbits are biholomorphic to $\mathbb{C}^*$ and that a Stein manifold contains no compact curves we obtain, as in [5, Lemmas 2 and 3], the following lemma.

Lemma 2.5. *In the above situation:*

(1) *each separatrix Γ_j is contained in a curve $C_j \subset N$ which is smooth, diffeomorphic to $\mathbb{C}$ and such that $C_1 \cap C_2 = \{p\}$;*
(2) $C_j \cap \operatorname{sing}(\mathcal{F}_{\tilde{\psi}}) = \{p\}$.

We now apply [19] and consider a primitive meromorphic first integral $F : N \dashrightarrow \mathbb{C}P(1)$ for $\mathcal{F}_\psi$. According to [2], $\mathcal{F}_\psi$ has no dicritical singularity (indeed, a dicritical singularity must be unique as a singularity of the foliation). Therefore F has no indeterminacy point and we have a holomorphic function $F : N \to \mathbb{C}P(1)$. We denote by R the image set $R = F(N) \subset \mathbb{C}P(1)$. The map F is *algebroid* in the sense of [19], that is, (i) the levels $F^{-1}(z)$, $z \in R$ are isomorphic to compact Riemann surfaces minus a finite set of points, and (ii) for any $z \in R$ there is a neighborhood $\delta = \delta(z)$ such that $F^{-1}(\delta)$ is Stein. In our case the orbits of ψ are isomorphic to $\mathbb{C}^*$ and we can take N as a neighborhood of its levels. Also F is of *finite type* as a consequence of [19, Proposition 3, p. 534] and of our hypothesis $H^2(V, \mathbb{Z}) = 0$.

Since $F: N \to R$ is primitive and surjective, (R, F) is already the *base space* of $F: N \to R$. Thus, according to the 'Remarque' in [19, p. 531], the fact that $H_1(N, \mathbb{C}) = 0$ implies that $H_1(R, \mathbb{C}) = 0$, that is, R is simply-connected. Moreover, according to the same remark, since F is free of indeterminacy points, R cannot be compact. Therefore R is isomorphic to $\mathbb{C}$ or the unit disc $\mathbb{D}$. We can assume that $F(p) = 0$ so that $C_j \subset F^{-1}(0)$, $j = 1, 2$.

LEMMA 2.6. $F^{-1}(0) = C_1 \cup C_2$.

PROOF. Since $F: N \to R$ is algebroid and primitive and $N^* = N - F^{-1}(0)$ is Stein we can apply [19, Lemma 7, p. 535] to conclude that there is an exact sequence

$$H_1(\mathbb{C}^*; \mathbb{Q}) \to H_1(N^*; \mathbb{Q}) \to H_1(R^*; \mathbb{Q}) \to 0$$

where $R^* = R - \{0\}$.

Thus rank $H_1(N^*) \leq$ rank $H_1(\mathbb{C}^*)$ + rank $H_1(R^*)$. Since $R^* = \mathbb{C} - \{0\}$ or $\mathbb{D} - \{0\}$ we obtain rank $H_1(N^*) \leq 2$. This implies that rank $H_1(N^*) = 2$, because if $f_j \in \mathcal{O}(N)$ is a reduced equation for C_j, then the 1-forms $df_j/f_j = \theta_j$, $j = 1, 2$, are holomorphic in N^* and independent in the cohomology. The existence of these reduced equations is a consequence of the fact that N is Stein with $H^2(N, \mathbb{Z}) = 0$. The above shows that $F^{-1}(0)$ has exactly two irreducible components, that is, $F^{-1}(0) = C_1 \cup C_2$. □

Using again the hypothesis that N is Stein with $H^2(N, \mathbb{Z}) = 0$ we can write $F = \prod_{j=1}^{r} F_j^{n_j}$ with $F_j \in \mathcal{O}(N)$, $n_j \in \mathbb{Z}$. Since F is holomorphic, $R \subset \mathbb{C}$ and $F^{-1}(0) = C_1 \cup C_2$, we have indeed, up to notation, $F = f_1^{n_1} f_2^{n_2} h$ where $h \in \mathcal{O}(N)^*$ is a unit. Since $H_1(N, \mathbb{C}) = 0$ we can write $h = h_1^{n_1}$ for some unit h_1 on N. Replacing f_1 by $f_1 h_1$ we can write $F = f_1^{n_1} f_2^{n_2}$ obtaining a first integral $F: N \to R (R = \mathbb{D}$ or $R = \mathbb{C})$ of the form $F = f_1^{n_1} f_2^{n_2}$ where $(f_j = 0)$ is a reduced equation for C_j. Finally, since Γ_1 and Γ_2 are transverse at p this implies that the map F takes these two transverse curves into the coordinate axes of the xy-plane. This and the following two lemmas will imply that $\Phi = (f_1, f_2): N \to \mathbb{C}^2$ is biholomorphic at the point p.

LEMMA 2.7. *If $z \in R \setminus \{0\}$ then the fiber $F^{-1}(z)$ is irreducible and therefore isomorphic to $\mathbb{C}^*$.*

PROOF. Since the map F is primitive it is enough to show that $0 \in R$ is the unique critical value of F. Indeed, if $\lambda \in R - \{0\}$ is another critical value of F then $F^{-1}(\lambda)$ contains a singularity $q \in V$, $q \neq p$, of $\mathcal{F}_\psi$ which is not of dicritical type. The same conclusions that hold for p are valid for q. In particular, $F^{-1}(\lambda)$ is the union of two irreducible curves C_1', C_2' and $C_1' \cap C_2' = \{q\}$. Also $C_i \cap C_j' = \emptyset$. Applying then the same arguments as in the proof of Lemma 2.6 we obtain

$$\text{rank } H_1(N - (F^1(0) \cup F^{-1}(\lambda))) \leq \text{rank } H_1(\mathbb{C}^*) + \text{rank } H_1(R - \{0, \lambda\}).$$

Thus

$$\text{rank } H_1(N - C_1 \cup C_2 \cup C_1' \cup C_2') \leq 1 + 2 = 3.$$

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On the other hand, as in the proof of Lemma 2.6, we can conclude that rank $H_1(N - C_1 \cup C_2 \cup C_1' \cup C_2') \geq 4$, a contradiction. This implies that $F|_{N \setminus F^{-1}(0)} : N \setminus F^{-1}(0) \to R \setminus \{0\}$ is a nonsingular primitive map and therefore has irreducible fibers. □

LEMMA 2.8. *The resolution of the singularity $p \in \operatorname{sing}(\mathcal{F}_\psi)$ exhibits a single projective line.*

PROOF. Take a singularity $\tilde{q} \in D$ in the exceptional divisor of the resolution. In a neighborhood of $\tilde{q}$ the lifted foliation $\tilde{\mathcal{F}}_\psi$ admits a holomorphic first integral of a local form as $x^n y^m$, $n, m \in \mathbb{N}$, in suitable coordinates $(x, y) \in \mathbb{C}^2$ centered at $\tilde{q}$. There is at least one component D_j of D which contains $\tilde{q}$ and we may assume that $(y = 0) \subset D_j$. Thus we can define a local 'vertical' projection $\sigma : L_0 \to D_j$ of a given (nonseparatrix) leaf L_0 passing close enough to $\tilde{q}$ onto the component D_j of D that contains $(y = 0)$ as the restriction of the map $\pi_2(x, y) = y$. If L_0 is given locally around $\tilde{q}$ by $x^n y^m = c \in \mathbb{C}^*$ then σ defines a finite-to-one map. Let us now consider for each singularity $\tilde{q}_\nu$ of $\operatorname{sing}(\tilde{\mathcal{F}}_\psi)$ on the projective line D_j a small disk $D_\nu \subset D_j$ centered at $\tilde{q}_\nu$ and its inverse image $\mathcal{D}_\nu \subset L_0$ by σ on the leaf L_0. There are two possibilities for $\mathcal{D}_\nu$: either it is a disk on the cylinder $L_0 \equiv \mathbb{C}^* \simeq S^1 \times [0, 1]$ or it corresponds to an end of L_0. On the other hand, the projection map σ induces a finite-to-one holomorphic covering $L_0 \setminus \bigcup_\nu \mathcal{D}_\nu \to D_j \setminus \bigcup_\nu D_\nu$. By the study of the resolution of p we know that D is a linear chain, so D_j contains exactly two singularities, and therefore we have exactly two disks $D_\nu \subset D_j$. This implies that $D_j \setminus \bigcup_\nu D_\nu$ is conformally equivalent to the Riemann sphere $\mathbb{C}P(1)$ minus two disjoint disks, that is, it is biholomorphic to the cylinder $\mathbb{C}^*$. This already implies that the holomorphic universal covering of $L_0 \setminus \bigcup_\nu \mathcal{D}_\nu$ is the complex plane $\mathbb{C}$ and therefore necessarily $L_0 \setminus \bigcup_\nu \mathcal{D}_\nu$ is biholomorphic to $\mathbb{C}^*$. Therefore we must have that the lifted disks $\mathcal{D}_\nu$ correspond to ends of L_0 and not to disks on L_0. Hence the transverse separatrices of the foliation $\tilde{\mathcal{F}}_\psi$ through the singular points $D_\nu \cap D_j$ correspond to the separatrices Γ_1 and Γ_2, not to other components of the divisor D. This proves that D contains a single projective line. □

PROOF OF PROPOSITION 2.1. Since the fibers of $F = f_1^{n_1} f_2^{n_2}$ are irreducible and D contains a single projective line, we conclude that $p \in \operatorname{sing}(\mathcal{F}_\psi)$ is already reduced, analytically linearizable and, furthermore, the map $\Phi = (f_1, f_2)$ has a nonsingular derivative at p. □

LEMMA 2.9. *Let Z be a complete holomorphic vector field with isolated singularities, all of nondicritical type, and periodic orbits on a Stein surface N. Then the period of the orbits defines a holomorphic function $\tau : N \to \mathbb{C}$. In particular, the foliation $\mathcal{F}_Z$, induced by Z, is also induced by a $\mathbb{C}^*$-action on N.*

PROOF. We first define the map τ on $N \setminus \operatorname{sing}(Z)$ as follows: given a point $p \in N$ which is not a singular point of Z, we consider $\tau(p) \in \mathbb{C}$ as the period of the orbit $\mathcal{O}_p$

of Z through p. It is now not difficult to see that this is a *meromorphic* function constant along the orbits of Z. This is in fact already in the work of Suzuki [18] and mentioned in [1, Section 5]: the main reason is that any tube covering U_T of $\mathcal{F}$ is biholomorphic to $U\mathbb{C}$ and, in the case where U_T is Stein, this follows from the results of Nishino [13] and Yamaguchi [20, 21]. The fact that U_T is Stein when N is Stein and Z periodic is proved by Suzuki in [18]. In our case, the orbits of the flow are periodic so that τ has no poles (the poles correspond to the orbits of 'infinite period'). Thus τ is holomorphic on $N \setminus \text{sing}(Z)$ and by the classical Hartogs extension theorem the map τ extends to a *holomorphic* map $\tau : N \to \mathbb{C}$. Also notice that, since all orbits are periodic, τ has no zeros on N, that is, $1/\tau$ is also holomorphic on N. Now define the vector field $\hat{Z} := \tau \cdot Z$ on N. Since Z is complete and τ is constant along the orbits of Z then $\hat{Z}$ is also complete. The flow maps $\hat{Z}_t$ and Z_t of $\hat{Z}$ and Z, respectively, are related by $\hat{Z}_t(p) = Z_{\tau(p)t}(p)$, for all $p \in N$, for all $t \in \mathbb{C}$. Thus $\hat{Z}_1 := Z_\tau \equiv \text{Id}$. This shows that Z is a multiple of a periodic complete holomorphic vector field and concludes the proof of the lemma. □

3. Proof of the theorem

Denote by $\mathcal{F}_\psi$ the one-dimensional holomorphic foliation induced by ψ on N. The leaves of $\mathcal{F}_\psi$ are the nonsingular orbits of ψ and, since the singularities of ψ are isolated, the singular set of $\mathcal{F}_\psi$ is $\text{sing}(\mathcal{F}_\psi) = \text{Fix}(\psi)$, the set of fixed points of ψ. By hypothesis, the flow ψ has periodic orbits, thus since N is Stein, by [18] the foliation $\mathcal{F}_\psi$ admits a meromorphic first integral $f : N \dashrightarrow \mathbb{C}P(1)$ for $\mathcal{F}_\psi$. This already implies that *the nonsingular orbits of the flow only accumulate at the singular set*. Also, according to Stein factorization theorem we can assume that the first integral is primitive and is onto an open Riemann surface $R \in \{\mathbb{C}, \mathbb{D}\}$ as explained in the paragraphs before Lemma 2.6. If ψ has no fixed point on N then f defines a holomorphic fibration over R, which is not possible thanks to the hypothesis that $H_1(N, \mathbb{C}) = 0$ (recall that the fibers are $\mathbb{C}^*$ which is topologically a cylinder). Therefore ψ must have some fixed point, that is, some singularity $p \in N$ for $\mathcal{F}_\psi$. Suppose that p is dicritical. Then the resolution of singularities of p by the blow-up method exhibits some projective line E which is completely transverse to the pull-back foliation [4, 16]. For each point $q \in E$ on this projective line there is a single orbit $\mathcal{O}_q$ of the lifted flow accumulating at, and only at, this point (indeed, otherwise we would have $\overline{\mathcal{O}_q} \setminus \mathcal{O}_q$ containing more than one point, what is not possible since $\mathcal{O}_q$ is biholomorphic to $\mathbb{C}^*$). This defines a holomorphic function τ on E which associates with each $q \in E$ the period of the corresponding orbit $\mathcal{O}_q$. Since E is compact the period function τ must be constant and therefore there is an open subset of N (similar to a 'sector' with vertex at the point p) where the flow ψ has a constant period. By the identity principle the flow ψ has a constant period $\tau \in \mathbb{C}$ and therefore corresponds to a holomorphic action φ of the multiplicative group $\mathbb{C}^*$. The point $p \in \text{sing}(\mathcal{F}_\psi)$ is a singularity (fixed point) of the $\mathbb{C}^*$-action φ. Since p is a dicritical singularity then (the action φ and therefore) ψ is globally linearizable

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by [2] or by [14]: indeed, a holomorphic $\mathbb{C}^*$-action is analytically linearizable in a neighborhood of an isolated fixed point, and then, since we are in the case where p is a dicritical singularity, we can apply the main result of [14] to conclude that the action is globally linearizable. Assume now that $p \in \text{sing}(\mathcal{F}_\psi)$ is nondicritical, more specifically that $\mathcal{F}_\psi$ exhibits no dicritical singularity (otherwise we can apply the case above to conclude the linearization). Then $\mathcal{F}_\psi$ admits a holomorphic first integral $f : N \to \mathbb{C}P(1)$ (indeed, a meromorphic first integral is holomorphic, taking values on $\mathbb{C}P(1)$ in the neighborhood of a nondicritical singularity). In particular, $\mathcal{F}_\psi$ admits a holomorphic first integral in a neighborhood of p (given by f or $1/f$) and by Proposition 2.1 the foliation is analytically linearizable in a neighborhood of the singularity p. From Lemma 2.9 we know that ψ can be replaced by a holomorphic action $\hat{\psi}$ with constant period and therefore associated with a $\mathbb{C}^*$-action $\hat{\varphi}$. The vector field $Z_{\hat{\psi}}$ corresponding to the action $\hat{\psi}$ is therefore analytically linearizable in a neighborhood of p (it is a multiple $\hat{Z} = h \cdot Z_0$ of a linearizable vector field of the local form $Z_0 = nx(\partial/\partial x) - my(\partial/\partial y)$, $n, m \in \mathbb{N}$, by a holomorphic first integral and, since $\hat{Z}$ also has constant period, h is also constant so that $\hat{Z}$ is also analytically linearizable at p). On the other hand, evoking Theorem 1 in [5], we conclude that $\mathcal{F}_{\hat{\varphi}}$ and therefore $\mathcal{F}_{\hat{\psi}} = \mathcal{F}_\psi$ is the pull-back of a linear foliation with holomorphic first integral.

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