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ON FUNCTIONS WHICH SATISFY SOME DIFFERENTIAL INEQUALITIES ON RIEMANNIAN MANIFOLDS

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Introduction

Most of the problems in differential geometry can be reduced to problems in differential equations and differential inequalities on Riemannian manifolds. Our main purpose of this paper is to study such differential inequalities on complete Riemannian manifolds. In [5], H. Omori proved a very important theorem. S. Y. Cheng and S. T. Yau gave a simplification and a generalization of it which was called the generalized maximum principle in [2] and [7], and many interesting applications in differential geometry in [2], [3], [7], and [8].

THE GENERALIZED MAXIMUM PRINCIPLE: Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M. Then, for all $\varepsilon > 0$, there exists a point x in M such that at x

$$\sup f - \varepsilon < f(x) ,$$
$$|\nabla f| < \varepsilon ,$$

and

$$\Delta f < \varepsilon .$$

In particular, $\lim_{\varepsilon \to 0} f(x) = \sup f$.

As the maximum principle plays an important role in geometry on compact manifolds, so the generalized maximum principle does on noncompact manifolds. Roughly speaking, if we want to compute the maximal value of some function on noncompact manifold, we should compute its asymptotic maximal value. If we have differential inequalities on Riemannian manifold which are closely related to the Laplacian Δ, then we can

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apply the generalized maximum principle. By the generalized maximum principle, we shall give certain theorems on functions which satisfy differential inequalities on complete Riemannian manifolds.

In § 1, we give the notation of differential geometric foundations on complete Riemannian manifolds which will be used later. In § 2, we prove a Liouville theorem for functions which satisfy the differential inequality

$$\Delta f \geqq \varphi(f, |\nabla f|) .$$

In § 3, we prove the following theorem:
Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant K. Let f be a bounded C^3-function such that, for some constant $0 < C < 1$,

$$\Delta f \geqq 0 ,$$

and

$$|\sum_j f_j(\Delta f)_j| \leqq C \sum_{i,j} f_{ij}^2 .$$

Then

$$|\nabla f| \leqq 2\sqrt{\sup f - \inf f} \cdot \sqrt{|K|}\ (\sup f - f)^{1/2} .$$

In § 4, we give the boundedness of functions which satisfy the differential inequality

$$\Delta f \geqq \varphi(f, |\nabla f|) ,$$

and apply it to a generalized Schwarz lemma for harmonic maps of dilatation bounded by K.

§ 1. Complete Riemannian manifolds

In this paper the differentiability of Riemannian manifolds always means the differentiability of class C^∞.

Let M be a complete Riemannian manifold of dimension n. Let f be a C^3-function on M. Let x be any point of M and let $X_1, X_2, \cdots, X_n$ be an orthonormal frame field in an open neighborhood U of x.

Let $f_i (i = 1, 2, \cdots, n)$ denote the covariant differentiation of f with respect to X_i, i.e., $f_i = \nabla_{X_i} f$. Then, the gradient of f which will be denoted by ∇f is the vector field on M which is given on U by

$$\nabla f = \sum_{i=1}^{n} f_i X_i . \tag{1.1}$$

Hence, we have

$$|\nabla f| = \left(\sum_{i=1}^{n} f_i^2\right)^{1/2} . \tag{1.2}$$

We put $f_{ij} = \nabla_{X_j}\nabla_{X_i} f$. The Hessian of f is by definition the second covariant differential $\nabla^2 f$ of f, i.e.,

$$\mathrm{Hess}\, f(X, Y) = \nabla_X \nabla_Y f = X(Yf) - (\nabla_X Y) f ,$$

for all vector fields X, Y on M. The Laplacian Δf of f is by definition trace (Hess f). In other words, then

$$\begin{aligned} \Delta f(x) &= \sum_i \mathrm{Hess}\, f(X_i, X_i)_x \\ &= \sum_i f_{ii}(x) . \end{aligned} \tag{1.3}$$

Then, the norm of Hess f is given by

$$|\mathrm{Hess}\, f| = \left(\sum_{i,j} f_{ij}^2\right)^{1/2} . \tag{1.4}$$

Let R denote the Ricci curvature tensor of M. We put $R_{ij} = R(X_i, X_j)$. Let v be a tangent vector at x. Then we denote by Ric (v) the Ricci curvature in the direction v, i.e., Ric $(v) = R(v, v)$.

Now, we consider an orthonormal frame field $\{X_i\}$ such that at x, $\nabla_{X_i} X_j = 0$ $(i, j = 1, 2, \cdots, n)$. We put $f_{ijk} = \nabla_{X_k}\nabla_{X_j}\nabla_{X_i} f$. By tensor calculation (cf. Proof of formula of Bochner-Lichnerowicz in [1]), we have, at x

$$\begin{aligned} \sum_{i,j} f_j f_{jii} &= \sum_{i,j} f_j f_{iji} \\ &= \sum_{i,j} f_j f_{iij} + \sum_{i,j} R_{ij} f_i f_j \\ &= \sum_j f_j (\Delta f)_j + \mathrm{Ric}\,(\nabla f) . \end{aligned} \tag{1.5}$$

We shall fix a point $x_0 \in M$ and use $r(x)$ to denote the distance function from x_0. Let σ be any geodesic parametrized by arclength joining a point x to x_0. Let $\sigma'(t)$ be the tangent vector of σ. Define

$$K_\sigma(x) = \min_{0 \leqq k \leqq r(x)} \left[\frac{n-1}{r(x) - k} - \frac{1}{(r(x) - k)^2} \int_k^{r(x)} (t - k)^2 \,\mathrm{Ric}\,(\sigma'(t)) dt\right] .$$

If x is not on the cut locus of x_0, then we can take σ to be unique minimizing from x_0 to x and define $K(x) = K_\sigma(x)$. Otherwise, we define $K(x) = \min_\sigma K_\sigma(x)$, where σ ranges over all the minimal geodesic from x_0 to x.

Then, if x is not on the cut locus of x_0, we have

(1.6) $$\Delta r(x) \leqq K(x)$$

(cf. Lemma 1 in [7]).

§ 2. The generalized maximum principle and a Liouville theorem

We shall state the generalized maximum principle proved in [2] and give a remark on it that will be needed in § 3.

THEOREM 1 (The generalized maximum principle). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M.*

Then for all $\varepsilon > 0$, there exists a point x in M such that at x

(1) $\sup f - \varepsilon < f(x)$,

(2) $|\nabla f| < \varepsilon$,

(3) $\nabla f < \varepsilon$.

Furthermore, if f has no maximum, then there exists a sequence of positive numbers $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ $(k \to \infty)$, and for all k, (1) may be replaced by

(1′) $\sup f - \varepsilon_k < f(x) < \sup f - \dfrac{\varepsilon_k}{2}$.

Proof. By the proof of Theorem 3 in [2], it is sufficient to show the following easy lemma.

LEMMA 1. *Let a be a real number and let $\{a_n\}$ $(n = 1, 2, \cdots)$ be a sequence of real numbers such that, for all positive integer n, $a_n < a$ and* $\lim_{n\to\infty} a_n = a$. *Then there exists a sequence of positive numbers $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ $(k \to \infty)$, and for all k, there exists a positive integer n such that*

$$a - \varepsilon_k < a_n < a - \frac{\varepsilon_k}{2}.$$

Now, we shall prove a Liouville theorem. Let M be a complete Riemannian manifold. We shall use the notation and the definition in § 1. For some constant $0 < p < 1$, the following condition will be called condition $(C[p])$: For all sequence of points $x_k \in M$ $(k = 1, 2, \cdots)$ such that $r(x_k) \to \infty$,

$$\limsup_{k\to\infty} \frac{K(x_k)}{r(x_k)^{1-p}[\log(r(x_k)^2 + 2)]^{1-p}} < \infty .$$

If M is a complete Riemannian manifold with Ricci curvature bounded from below, then $K(x)$ is bounded from above by some constant when $r(x) \geqq 1$ and hence M satisfies the condition $(C[p])$.

THEOREM 2. *Let M be a connected complete Riemannian manifold satisfying the condition $(C[p])$. Let f be a C^2-function bounded from above on M such that*

$$\Delta f \geqq \varphi(f, |\nabla f|)\,,$$

where $\varphi(x, y)$ is a continuous nonnegative function defined for all x and $y \geqq 0$ such that

(1) $\varphi(x, 0) = 0$,

(2) *for all sequences $\{x_k\}$ and $\{y_k\}$ $(k = 1, 2, \cdots)$ such that*

$$x_k \longrightarrow \sup f,\ y_k > 0 \text{ and } y_k \to 0,$$

$$\liminf_{t\to\infty} \frac{\varphi(x_k, y_k)}{y_k^p} > 0\,.$$

Then, f is a constant.

COROLLARY 1. *Let M be a connected complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M such that, for some constants $0 < p < 1$ and $C > 0$,*

$$\Delta f \geqq C|\nabla f|^p\,.$$

Then, f is constant.

Remark 1 (Corollary 1 in [2]). Let M be a connected complete two-dimensional Riemannian manifold with nonnegative Ricci curvature. Then, subharmonic functions bounded from above are constants.

Remark 2. Let $M = \boldsymbol{R}^n$ be the Euclidean space of dimension n with standard Euclidean metric. Let $x = (x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^n$ and we put

$$r(x) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}\,.$$

For a constant $p > 1$, we choose the positive integer k, n such that

$$p > \frac{2k+4}{2k+1}\,, \qquad n > 2k+2\,.$$

Then, the function on $\boldsymbol{R}^n$ defined by

$$f(x) = -\frac{1}{(r(x)^2+1)^k}$$

satisfies the following differential inequality:

$$\Delta f \geqq \frac{1}{(2k)^{p-1}} |\Delta f|^p .$$

Proof of Theorem 2. We shall fix a point $x_0 \in M$ and use $r(x)$ to denote the distance function from x_0. Then clearly we can assume $f(x_0) > 0$ and $\sup f > 0$. For all $k > 0$, let

$$g(x) = \frac{f(x)}{[\log (r(x)^2 + 2)]^k} .$$

Then since

$$g(x_0) > 0 \text{ and } \lim_{r(x)\to\infty} g(x) \leqq 0 ,$$

we see that g must attain its maximum at some point x_k. If x_k is not on the cut locus of x_0, then we can differentiate at x_k to obtain

$$\nabla g(x_k) = 0 , \tag{2.1}$$

$$\Delta g(x_k) \leqq 0 . \tag{2.2}$$

By the proof of Theorem 3 in [2], we have

$$f(x_k) \longrightarrow \sup f \ (k \longrightarrow 0) . \tag{2.3}$$

By hypothesis on f, f is a subharmonic function on M. If f is not a constant, it follows from the maximum principle of E. Hopf (Theorem 2.1 in [6]) that f has no maximum. Hence, we have

$$r(x_k) \longrightarrow \infty \ (k \longrightarrow 0) . \tag{2.4}$$

Direct computation shows that at x_k,

$$\nabla g = \frac{\nabla f}{[\log (r^2 + 2)]^k} - \frac{2krf\nabla r}{[\log (r^2 + 2)]^{k+1}(r^2 + 2)} , \tag{2.5}$$

$$\begin{aligned} \Delta g = & \frac{\Delta f}{[\log (r^2 + 2)]^k} - \frac{4kr\nabla f \cdot \nabla r}{[\log (r^2 + 2)]^{k+1}(r^2 + 2)} \\ & - \frac{2kf}{[\log (r^2 + 2)]^{k+1}(r^2 + 2)} \\ & - \frac{2kfr\Delta r}{[\log (r^2 + 2)]^{k+1}(r^2 + 2)} + \frac{4k(k+1)fr^2}{[\log (r^2 + 2)]^{k+2}(r^2 + 2)^2} \end{aligned} \tag{2.6}$$

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$$+\frac{4kfr^2}{[\log(r^2+2)]^{k+1}(r^2+2)^2}\leqq 0\,.$$

Hence, by (2.6), we have

$$\begin{aligned}\Delta f \leqq & \frac{4kr\nabla f\cdot\nabla r}{[\log(r^2+2)](r^2+2)}+\frac{2kf}{[\log(r^2+2)](r^2+2)}\\ &+\frac{2kfr\Delta r}{[\log(r^2+2)](r^2+2)}-\frac{4k(k+1)fr^2}{[\log(r^2+2)]^2(r^2+2)^2}\\ &-\frac{4kfr^2}{[\log(r^2+2)](r^2+2)^2}\,.\end{aligned}\tag{2.7}$$

Putting (1.6), (2.5) and hypothesis into (2.7), we obtain

$$\begin{aligned}&\varphi\left(f,\ \frac{2kr|f|}{[\log(r^2+2)](r^2+2)}\right)\\ &\leqq \frac{8k^2r^2f}{[\log(r^2+2)]^2(r^2+2)^2}+\frac{2kf}{[\log(r^2+2)](r^2+2)}\\ &\quad-\frac{4k(k+1)fr^2}{[\log(r^2+2)]^2(r^2+2)^2}-\frac{4kfr^2}{[\log(r^2+2)](r^2+2)^2}\\ &\quad+\frac{2kfr\,K(x_k)}{[\log(r^2+2)](r^2+2)}\,.\end{aligned}\tag{2.8}$$

Now consider the case where x_k is on the cut locus of x_0. By a method of Calabi (cf. Proof of Theorem 3 in [2]), we have the inequality (2.8).

Multiplying (2.8) by $([\log(r^2+2)]^p(r^2+2)^p)/kr^p$, we have

$$\begin{aligned}&2^pk^{p-1}|f|^p\varphi\left(f,\ \frac{2kr|f|}{[\log(r^2+2)](r^2+2)}\right)\left(\frac{2kr|f|}{[\log(r^2+2)](r^2+2)}\right)^{-p}\\ &\leqq \frac{8kr^{2-p}f}{[\log(r^2+2)]^{2-p}(r^2+2)^{2-p}}+\frac{2fr^{-p}}{[\log(r^2+2)]^{1-p}(r^2+2)^{1-p}}\\ &\quad-\frac{4(k+1)fr^{2-p}}{[\log(r^2+2)]^{2-p}(r^2+2)^{2-p}}-\frac{4fr^{2-p}}{[\log(r^2+2)]^{1-p}(r^2+2)^{2-p}}\\ &\quad+2f\left(\frac{r^2}{r^2+2}\right)^{1-p}\cdot\frac{K(x_k)}{r^{1-p}[\log(r^2+2)]^{1-p}}\,.\end{aligned}\tag{2.9}$$

When $k\to 0$, by the condition $(C[p])$, (2.3) and (2.4), (2.9) is a contradiction.

Theorem 2 is thereby proved.

§ 3. Gradient estimate of functions

Gradient estimates of partial differential equations on a complete Riemannian manifold were given in [7].

We shall prove the following:

THEOREM 3. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant K. Let f be a bounded C^3-function such that*

(C.1) $\Delta f \geqq 0$, *i.e., f is subharmonic,*

(C.2) *For some constant* $0 < C < 1$,

$$|\sum_i f_j(\Delta f)_j| \leqq C \sum_{i,j} f_{ij}^2 .$$

Then

$$|\nabla f| \leqq 2\sqrt{\sup f - \inf f} \cdot \sqrt{|K|}\ (\sup f - f)^{1/2} .$$

COROLLARY 1. *Let M be a connected complete Riemannian manifold with nonnegative Ricci curvature. Let f be a bounded C^2-function which satisfies the conditions* (C. 1) *and* (C. 2). *Then, f is a constant.*

Proof of Theorem 3. We put $\dim M = n$. We shall use the notation given in § 1. For any constants, $a > \sup f$, $b > 0$ and $1 > p > 1/2$, let

$$g(x) = \frac{a - f(x)}{(|\nabla f|^2(x) + b)^p} .$$

If g has no minimum, then, by Theorem 1, for any positive number ε_k (we shall denote ε_k by ε), there is a point $y \in M$ such that at y

$$\inf g + \frac{\varepsilon}{2} < g < \inf g + \varepsilon,\ |\nabla g| < \varepsilon, \text{ and } \Delta g > -\varepsilon. \tag{3.1}$$

Then direct computation shows

$$g_i = -\frac{f_i}{(|\nabla f|^2 + b)^p} - \frac{2p(a-f)\sum_j f_j f_{ji}}{(|\nabla f|^2 + b)^{p+1}}, \tag{3.2}$$

$$\begin{aligned} g_{ii} = &-\frac{f_{ii}}{(|\nabla f|^2 + b)^p} + \frac{4pf_i \sum_j f_j f_{ji}}{(|\nabla f|^2 + b)^{p+1}} \\ &- \frac{2p(a-f)\sum_j f_{ji}^2}{(|\nabla f|^2 + b)^{p+1}} - \frac{2p(a-f)\sum_j f_j f_{jii}}{(|\nabla f|^2 + b)^{p+1}} \end{aligned} \tag{3.3}$$

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$$+\frac{4p(p+1)(a-f)(\sum_j f_j f_{ji})^2}{(|\nabla f|^2+b)^{p+2}}.$$

For any real numbers α and β, we have

$$k\alpha\beta + m\alpha^2 \leqq (\alpha+\beta)^2,$$

provided

(*) $$k>0,\ m>0 \quad \text{and } (2-k)^2-4(1-m)\leqq 0.$$

By (3.1), we have

$$\left|\frac{f_i}{(|\nabla f^2+b)^p}+\frac{2p(a-f)\sum_j f_j f_{ji}}{(|\nabla f|^2+b)^{p+1}}\right|<\varepsilon.$$

From the above inequality, we have

$$\frac{2pk(a-f)f_i\sum_j f_j f_{ji}}{(|\nabla f|^2+b)^{2p+1}}+\frac{mf_i^2}{(|\nabla f|^2+b)^{2p}}<\varepsilon^2.$$

Hence,

(3.4) $$\frac{4p\sum_{i,j} f_i f_j f_{ji}}{(|\nabla f|^2+b)^{p+1}}<\frac{2(|\nabla f|^2+b)^p}{k(a-f)}\left(n\varepsilon^2-\frac{m|\nabla f|^2}{(|\nabla f|^2+b)^{2p}}\right).$$

For any real numbers α and β, we have

$$-\lambda\alpha^2+\mu\beta^2\leqq(\alpha+\beta)^2,$$

provided

(**) $$\lambda>0,\ \mu>0 \quad \text{and } (1+\lambda)(1-\mu)\geqq 1.$$

By (3.1) and the above inequality, we have

$$\frac{4p^2\mu(a-f)^2\sum_i(\sum_j f_j f_{ji})^2}{(|\nabla f|^2+b)^{2p+2}}\leqq n\varepsilon^2+\frac{\lambda|\nabla f|^2}{(|\nabla f|^2+b)^{2p}}.$$

Hence, for some constant $0<r<1$,

(3.5) $$\frac{4p(p+r)(a-f)\sum_i(\sum_j f_j f_{ji})^2}{(|\nabla f|^2+b)^{p+2}} \leqq \frac{p+r}{p}\cdot\frac{1}{\mu}\cdot\frac{(|\nabla f|^2+b)^p}{(a-f)}\left(n\varepsilon^2+\frac{\lambda|\nabla f|^2}{(|\nabla f|^2+b)^{2p}}\right).$$

By tensor calculation in § 1 and hypothesis on M, we have

$$\begin{aligned}\sum_{i,j} f_j f_{jii} &= \sum_{i,j} f_j f_{iji}\\ &= \sum_{i,j} f_j f_{iij} + \sum_{i,j} R_{ij} f_i f_j\\ &\geqq \sum_j f_j(\Delta f)_j + K|\Delta f|^2 .\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{2p(a-f)\sum_{i,j} f_j f_{jii}}{(|\nabla f|^2+b)^{p+1}}\\ (3.6)\qquad &\leqq -\frac{2p(a-f)K|\nabla f|^2}{(|\nabla f|^2+b)^{p+1}} - \frac{2p(a-f)\sum_j f_j(\Delta f)_j}{(|\nabla f|^2+b)^{p+1}}\\ &\leqq \frac{2p(a-f)K|\nabla f|^2}{(|\nabla f|^2+b)^{p+1}} + \frac{2p(a-f)|\sum_j f_j(\Delta f)_j|}{(|\nabla f|^2+b)^{p+1}} .\end{aligned}$$

Putting (3.2), (3.4), (3.5) and (3.6) into (3.1), we have

$$\begin{aligned}-\varepsilon < &-\frac{\Delta f}{(|\nabla f|^2+b)^p} + \frac{2(|\nabla f|^2+b)^p}{k(a-f)}\left(n\varepsilon^2 - \frac{m|\nabla f|^2}{(|\nabla f|^2+b)^{2p}}\right)\\ &+\frac{2p|K|(a-f)|\nabla f|^2}{(|\nabla f|^2+b)^{p+1}} + \frac{2p(a-f)|\sum_j f_j(\Delta f)_j}{(|\nabla f|^2+b)^{p+1}}\\ &+\frac{p+r}{p}\cdot\frac{1}{\mu}\cdot\frac{(|\nabla f^2+b)^p}{(a-f)}\left(n\varepsilon^2 + \frac{\lambda|\nabla f|^2}{(|\nabla f|^2+b)^{2p}}\right)\\ &+\frac{4p(1-r)(a-f)\sum_i(\sum_j f_j f_{ji})}{(|\nabla f|^2+b)^{p+2}} - \frac{2p(a-f)\sum_{i,j}(f_{ij})^2}{(|\nabla f|^2+b)^{p+1}} .\end{aligned}$$

By the fact that f satisfies (C.1) and (C.2) and the Schwarz inequality, we obtain

$$\begin{aligned}-\varepsilon < &\frac{1}{g}\left(\frac{2n}{k} + \frac{p+r}{p}\cdot\frac{n}{\mu}\right)\varepsilon^2\\ &+\frac{2p|K|(a-f)|\nabla f|^2}{(|\nabla f|^2+b)^{p+1}} - \frac{|\nabla f|^2}{(|\nabla f|^2+b)^p(a-f)}\left(\frac{2m}{k} - \frac{p+r}{p}\cdot\frac{\lambda}{\mu}\right)\\ &-2p(1-C-2(1-r))\frac{(a-f)\sum_{i,j}(f_{ij})^2}{(|\nabla f|^2+b)^{p+1}} .\end{aligned}$$

We can choose a number $1 > r > 0$ such that $1 - C - 2(1-r) > 0$. Therefore, we obtain

$$\begin{aligned}(3.7)\qquad -\varepsilon < &\frac{1}{g}\left(\frac{2n}{k} + \frac{p+r}{p}\cdot\frac{n}{\mu}\right)\varepsilon^2 + g\cdot\frac{2p|K||\nabla f|^2}{(|\nabla f|^2+b)}\\ &-\frac{1}{g^{(p-1)/p}}\cdot\frac{|\nabla f|^2}{(a-f)^{(2p-1)/p}(|\nabla f|^2+b)}\cdot D\end{aligned}$$

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where $D = 2m/k - (p + r)/p \cdot \lambda/\mu$.

Let us now prove

$$\inf g > 0 . \tag{3.8}$$

In fact, if $\inf g = 0$, by (3.1), we have

$$\frac{\varepsilon}{2} < g(y) < \varepsilon .$$

When $\varepsilon \to 0$, $|\nabla f|(y) \to \infty$ and (3.7) is a contradiction.

Hence, we have

$$\sup |\nabla f|^2 < \infty . \tag{3.9}$$

As before, we claim that $\lim_{\varepsilon \to 0} \inf |\nabla f|^2(y) \neq 0$ when

$$b < \frac{\sup|\nabla f|^2(a - \sup f)^{1/p}}{(a - \inf f)^{1/p} - (a - \sup f)^{1/p}} .$$

In fact, if this were not true, then we could find y such that

$$\frac{a - \inf f}{(\sup |\nabla f|^2 + b)^p} \geqq \frac{a - f(y)}{(|\nabla f|^2(y) + b)^p} .$$

By (3.1) and the assumption, we have

$$\frac{a - \inf f}{(\sup |\nabla f|^2 + b)^p} \geqq \frac{a - \sup f}{b^p} .$$

This contradicts our assumption.

Let

$$B = \lim_{\varepsilon \to 0} \inf |\nabla f|^2(y) .$$

Therefore, when $\varepsilon \to 0$ in (3.7), we have

$$0 \leqq \inf g \cdot 2p|K| - \frac{1}{(\inf g)^{(1-p)/p}} \cdot \frac{1}{(a - \inf f)^{(2p-1)/p}} \cdot \frac{B}{B + b} \cdot D . \tag{3.10}$$

If g has a minimum at some point x_0, then, by Theorem 1, for any $\varepsilon > 0$, there is a point y such that at y

$$g < \inf g + \varepsilon,\ |\nabla g| < \varepsilon,\ \Delta g > -\varepsilon , \qquad \text{and } y \longrightarrow x_0\ (\varepsilon \longrightarrow 0) . \tag{3.11}$$

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Then, the same argument as above implies (3.10).

By (3.10), we have

$$\frac{1}{(\inf g)^{1/p}} \leqq 2p|K|(a - \inf f)^{(2p-1)p} \cdot \frac{B+b}{B} \cdot \frac{1}{D} .$$

Therefore, we obtain

$$\frac{|\nabla f|^2 + b}{(a-f)^{1/p}} \leqq 2p|K|(a - \inf f)^{(2p-1)/p} \cdot \frac{B+b}{B} \cdot \frac{1}{D} . \tag{3.12}$$

When $b \to 0$, we have

$$|\nabla f|^2 \leqq \frac{2p}{D} \cdot |K|(a - \inf f)^{(2p-1)/p}(a-f)^{1/p} . \tag{3.13}$$

By letting $p \to 1$ and $a \to \sup f$ in (3.13), we obtain

$$|\nabla f|^2 \leqq \frac{2}{D} \cdot |K|(\sup f - \inf f)(\sup f - f) . \tag{3.14}$$

Finally, we shall determine the constant D. By conditions (*) and (**), we can choose a number D such that $D = 1/2$.

Therefore, we obtain

$$|\nabla f| \leqq 2\sqrt{\sup f - \inf f} \cdot \sqrt{|K|}\, (\sup f - f)^{1/2} \tag{3.15}$$

This completes the proof of Theorem 3.

We can now use the same method as in the proof of Theorem 3 and show the following:

THEOREM 4. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant K. Let f be a C^3-function which satisfies the following conditions*:

(C.1) $0 < \inf f$, $\sup f < \infty$,

(C.2) $\Delta(f^2) \geqq 0$,

(C.3) *for some constant* $0 < C < 1$,

$$|\sum_j f_j(\Delta f)_j| \leqq C \sum_{i,j} f_{ij}^2 .$$

Then

$$|\nabla f| \leqq \sqrt{\frac{|K|}{2 \inf (f^2)}} (\sup (f^2) - f^2) .$$

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§ 4. Boundedness of functions

In [8], S. T. Yau considered boundedness of certain function which satisfy a differential inequality and proved a generalized Schwarz lemma for Kähler manifolds.

We shall give a generalization of it.

THEOREM 5. *Let M be a complete Riemannian manifolds with Ricci curvature bounded from below. Let f be a C^2-function bounded from below on M such that*

$$\Delta f \geqq \varphi(f, |\nabla f|),$$

where $\varphi(x, y)$ is a function defined for $x \geqq a$ ($a = \inf f$) and $y \geqq 0$ such that, for some constant $p > 0$,

(1) *$\varphi(x, y)$ is continuous in x and differentiable of C^2 in y,*

(2) *$\varphi(x, 0) = g(x)$,*

(3) $$\liminf_{x \to 0} \frac{(\partial\varphi/\partial y)(x, 0)}{x^{p/2}} > -\infty,$$

(4) $$\frac{\partial^2 \varphi}{\partial y^2} \geqq 0,$$

where $g(x)$ is a continuous function defined for $x \geqq a$ such that

(i) *There exists a number $b > a$ such that*

$$g(x) > 0 \quad (x > b),$$

(ii) $$\liminf_{x \to \infty} \frac{g(x)}{x^{1+p}} > 0.$$

Then

$$f(x) \leqq b \qquad (x \in M).$$

Remark 1. Let $g(x) = -K_1 x + K_2 x$ ($K_1 \geqq 0, K_2 > 0$) and $\varphi(x, y) = g(x)$. This is a typical example of φ and was considered in [8].

THEOREM 6 ([4]). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant $-A$. Let N be a Riemannian manifold with sectional curvature bounded from above by a constant $-B$ ($B > 0$).*

Let f be a harmonic map $M \to N$ of dilatation bounded by K. Then

$$f^* g_N \leqq \frac{k^2 K^2}{2} \cdot \frac{A}{B} g_M,$$

where $k = \min(\dim M, \dim N)$ *and* g_M, g_N *are Riemannian metric of* M *and* N *respectively.*

In particular, if $A = 0$, *then* f *is a constant.*

Proof. Let $m = \dim M$ and $n = \dim N$. Let $x \in M$ and $\theta_i (i = 1, 2, \cdots, m)$ be an orthonormal coframe field in an open neighborhood of x. Then, we have

$$g_M = \sum_{i=1}^{m} \theta_i^2 .$$

Let $\omega_\alpha (\alpha = 1, 2, \cdots, n)$ be an orthonormal coframe field in an open neighborhood of $f(x) \in N$.

Then we can define

$$f^* \omega_\alpha = \sum_{i=1}^{m} a_{\alpha i} \theta_i ,$$

and

$$u = \sum_{\alpha, i} a_{\alpha i}^2 .$$

Clearly, we have

$$f^* g_N \leqq u g_M . \tag{4.1}$$

By formula of the Laplacian of u (cf. [4]) and a simple calculation, we obtain

$$\Delta u \geqq -2Au + \frac{4B}{k^2 K^2} u^2 . \tag{4.2}$$

By Theorem 5 and Remark 1, we have

$$u(x) \leqq \frac{k^2 K^2}{2} \cdot \frac{A}{B} .$$

Hence, (4.1) and the last inequality prove Theorem 6.

Proof of Theorem 5. Let c be a constant such that $a + c > 0$.
Let

$$g(x) = \frac{1}{(f(x) + c)^{p/2}} .$$

Let $\varepsilon > 0$ be any number. Then, by Theorem 1, there exists a point y such that at y,

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$$\frac{1}{(f+c)^{p/2}} < \inf \frac{1}{(f+c)^{p/2}} + \varepsilon\,, \tag{4.3}$$

$$\left|\nabla\left(\frac{1}{(f+c)^{p/2}}\right)\right| < \varepsilon\,, \tag{4.4}$$

$$\Delta\left(\frac{1}{(f+c)^{p/2}}\right) > -\varepsilon\,. \tag{4.5}$$

Then direct computation shows

$$\left|\nabla\left(\frac{1}{(f+c)^{p/2}}\right)\right|^2 = \frac{p^2}{4}\cdot\frac{|\nabla f|^2}{(f+c)^{p+2}}\,, \tag{4.6}$$

$$\Delta\left(\frac{1}{(f+c)^{p/2}}\right) = -\frac{p}{2}\cdot\frac{\Delta f}{(f+c)^{p/2+1}} + \frac{p}{2}\left(\frac{p}{2}+1\right)\frac{|\nabla f|^2}{(f+c)^{p/2+2}}\,. \tag{4.7}$$

By (4.5) and (4.7), we have

$$\frac{p}{2}\cdot\frac{\Delta f}{(f+c)^{p/2+1}} < \varepsilon + \frac{p}{2}\left(\frac{p}{2}+1\right)\frac{|\nabla f|^2}{(f+c)^{p/2+2}}\,. \tag{4.8}$$

Dividing (4.8) by $p/2(f+c)^{p/2}$, we have

$$\frac{\Delta f}{(f+c)^{p+1}} < \frac{2}{p}\cdot\frac{1}{(f+c)^{p/2}}\cdot\varepsilon + \left(\frac{p}{2}+1\right)\frac{|\nabla f|^2}{(f+c)^{p+2}}\,. \tag{4.9}$$

By (4.4), (4.6) and the hypothesis, we obtain

$$\frac{\varphi(f,|\nabla f|)}{(f+c)^{p+1}} < \frac{2}{p}\cdot\frac{1}{(f+c)^{p/2}}\,\varepsilon + \left(\frac{p}{2}+1\right)\frac{4}{p^2}\varepsilon^2\,. \tag{4.10}$$

Using Taylor's formula on φ, we have

$$\varphi(x,y) = \varphi(x,0) + \frac{\partial\varphi}{\partial y}(x,0)y + \frac{1}{2}\frac{\partial^2\varphi}{\partial y^2}(x,\theta y)y^2\,, \tag{4.11}$$

where θ is some number with $0<\theta<1$.

Putting (2), (3) and (4.11) into (4.10), we obtain

$$\frac{g(f)}{(f+c)^{p+1}} + \frac{(\partial\varphi/\partial y)(x,0)}{(f+c)^{p+1}}\cdot|\nabla f| < \frac{2}{p}\cdot\frac{1}{(f+c)^{p/2}}\cdot\varepsilon + \left(\frac{p}{2}+1\right)\frac{4}{p^2}\varepsilon^2\,. \tag{4.12}$$

By (4.2), we have

(4.13) $$\frac{(\partial\varphi/\partial y)(x, 0)}{(f + c)^{p+1}}\cdot|\nabla f| \geqq \min\left(0, \frac{2}{p}\cdot\frac{(\partial\varphi/\partial y)(x, 0)}{(f + c)^{p/2}}\cdot\varepsilon\right).$$

Let us now prove

(4.14) $$\sup f < \infty .$$

In fact, if $\sup f = \infty$, by (4.3) we have

$$f(y) \longrightarrow \infty \quad (\varepsilon \longrightarrow 0) .$$

When $\varepsilon \to 0$, by (3), (ii) and (4.13), (4.12) is a contradiction.

Let $m = \sup f$. When $\varepsilon \to 0$, by (4.12), we obtain

(4.15) $$g(m) \leqq 0 .$$

Therefore, by the condition (i), we obtain

$$m \leqq b .$$

This completes the proof of Theorem 5.

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