NEAR-RINGS OF QUOTIENTS OF ENDOMORPHISM NEAR-RINGS

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0. Introduction

Let $\mathscr{C}$ be a category with finite products and a final object and let X be any group object in $\mathscr{C}$. The set of $\mathscr{C}$-morphisms, $\mathrm{Mor}_{\mathscr{C}}(X, X)$ is, in a natural way, a near-ring which we call the *endomorphism near-ring of X in $\mathscr{C}$*. Such near-rings have previously been studied in the case where $\mathscr{C}$ is the category of pointed sets and mappings, (**6**). Generally speaking, if Γ is an additive group and S is a semigroup of endomorphisms of Γ then a near-ring can be generated naturally by taking all zero preserving mappings of Γ into itself which commute with S (see **1**). This type of near-ring is again an endomorphism near-ring, only the category $\mathscr{C}$ is the category of S-acts and S-morphisms (see (**4**) for definition of S-act, etc.).

The question answered in this paper is the following. Under what conditions do endomorphism near-rings of this type have near-rings of quotients which are 2-primitive with d.c.c. on right ideals and an identity? The conditions obtained are described in terms of conditions on the semigroup S and the group Γ, and are formalised by introducing the concept of a $\mathscr{Q}$-system. 2-primitive near-rings with d.c.c. on right ideals and an identity have been classified in terms of endomorphism near-rings by Wielandt (**6**) (see also Ramakotaiah (**5**) and Holcombe (**3**)).

1. Terminology

A *near-ring* is a set N with two binary operations, addition (+) and multiplication (.), such that

(i) N is a group with respect to addition.

(ii) N is a semigroup with respect to multiplication.

(iii) For any $n, n_1, n_2 \in N$, $n.(n_1+n_2) = n.n_1+n.n_2$.

(iv) If 0 is the additive identity of N, then $0.n = n.0 = 0$ for all $n \in N$.

A subnear-ring S of a near-ring N is a subset S of N, which is a near-ring under the induced binary operations.

A mapping f: $N \to N_1$ of two near-rings N, N_1 is a near-ring homomorphism if $(n+n')f = nf+n'f$; $(n.n')f = (nf).(n'f)$ for any $n, n' \in N$.

If N is a near-ring, then an additive group M is an *N-module*, if there exists a mapping $(m, n) \rightarrow m.n$ of $M \times N$ into M, such that

(i) $m.(n+n_1) = m.n+m.n_1$ for all $m \in M$; $n, n_1 \in N$.

(ii) $m.(n.n_1) = (m.n).n_1$ for all $m \in M$; $n, n_1 \in N$.

A mapping ψ: $M \rightarrow M_1$ (where M and M_1 are N-modules) is called an *N-homomorphism* if

$$(m+m')\psi = m\psi + m'\psi \quad \text{for any } m, m' \in M$$

$$(mn)\psi = (m\psi)n \quad \text{for any } m \in M;\ n \in N.$$

An *N-submodule* M' of an N-module M is the kernel of an N-homomorphism from M, so that $M' \lhd M$ and $(m+m').n - m.n \in M'$ for all $m \in M, m' \in M', n \in N$.

An *N-subgroup* M'' of an N-module M is an additive subgroup M'' of M such that $m''.n \in M''$ for all $m'' \in M''$, $n \in N$.

A near-ring N is clearly an N-module.

A *right ideal of* N is an N-submodule of the N-module N.

A *right N-subgroup of* N^+ is an N-subgroup of the N-module N.

An N-module M is *of type* 2 if M possesses no proper, non-trivial N-subgroups and $MN = \{mn \mid m \in M;\ n \in N\} \neq \{0\}$.

Let S be a subset of an N-module M, we define

$$(S)_r = \{n \in N \mid sn = 0 \quad \text{for all } s \in S\}$$

and call this the *right annihilator of S* (*in N*). Clearly this is a right ideal of N.

A near-ring N is *2-primitive* if there exists an N-module M of type 2 such that $(M)_r = (0)$.

For any near-ring N, define $J_2(N)$ to be the intersection of the right annihilators of all N-modules of type 2, with the convention that $J_2(N) = N$ if N possesses no N-modules of type 2.

A near-ring N is *2-prime* if $aNb = (0) \Rightarrow a = 0$ or $b = 0$ $(a, b \in N)$.

A near-ring satisfies the *descending chain condition* (*d.c.c.*) *on right ideals* if, given a chain of right ideals in the near-ring,

$$R_1 \supsetneqq R_2 \supsetneqq \dots \supsetneqq R_p \supsetneqq \dots$$

then there exists an integer q with $R_q = R_{q+1} = \dots$

A near-ring satisfies the *ascending chain condition* (*a.c.c.*) on right annihilators if given a chain of right ideals of the near-ring, which are right annihilators,

$$(Z_1)_r \subsetneqq (Z_2)_r \subsetneqq \dots \subsetneqq (Z_p)_r \subsetneqq \dots$$

then there exists an integer q with $(Z_q)_r = (Z_{q+1})_r \dots$.

An element n of a near-ring N is a *regular element* if

$$n_1 n = n_2 n \Rightarrow n_1 = n_2 \quad (n_1, n_2 \in N)$$

and

$$nn_3 = nn_4 \Rightarrow n_3 = n_4 \quad (n_3, n_4 \in N).$$

A near-ring N has a *near-ring of right quotients* Q if

(i) N can be embedded (as a near-ring) in Q, and Q has an identity;

(ii) x is a regular element of N then $\exists y \in Q$ such that $x.y = y.x = 1_Q$ (we write $y = x^{-1}$);

(iii) $q \in Q$ then $q = nx^{-1}$ where $n \in N$ and x is a regular element of N.

Let Γ be an additive group and S a semigroup of endomorphisms of Γ. We define $\mathrm{Map}_S(\Gamma)$ to be the set of mappings $\{n \mid n\colon \Gamma \to \Gamma;\ (0_\Gamma).n = 0_\Gamma;\ \gamma sn = \gamma ns, \forall \gamma \in \Gamma, \forall s \in S\}$. Then $\mathrm{Map}_S(\Gamma)$ is a near-ring (with a multiplicative identity), called the *endomorphism near-ring of* Γ *in the category of S-acts and S-maps*.

2. $\mathcal{Q}$-Systems

Theorem 2.1. *Suppose that* Γ *is an additive group, and* S *is a multiplicative semigroup of endomorphisms of* $(\Gamma, +)$, *which includes the identity endomorphism, but not the zero endomorphism. Suppose that* S *is left and right cancellative, left and right reversible and for all* $s \in S$, $\gamma s = 0 \Rightarrow \gamma = 0$, $(\gamma \in \Gamma)$. *Then* S *has a group* G *of left quotients, and* G *acts as a group of automorphisms on an additive group* Δ.

Proof. The existence of G is a standard result (**2**).

Consider $\Gamma \times S$. Let (γ, s), $(\gamma_1, s_1) \in \Gamma \times S$, define the relation $\sim$ by $(\gamma, s) \sim (\gamma_1, s_1) \Leftrightarrow \exists a, b, \in S$ such that $sb = s_1 a$ and $\gamma b = \gamma_1 a$.

It is a fairly mechanical procedure to verify that $\sim$ is well defined and an equivalence relation (see (**6**)). Partitioning $\Gamma \times S$ into equivalence classes we write γ/s to represent the equivalence class containing (γ, s). Let

$$\Delta = \{\gamma/s \mid \gamma \in \Gamma, s \in S\}.$$

We show that Δ is an additive group and G is a group of automorphisms of Δ.

Let γ/s, $\gamma'/s' \in \Delta$ and define $\gamma/s + \gamma'/s' = (\gamma a + \gamma' b)/m$ where $a, b \in S$, such that $s'b = sa = m$. This operation is well defined. For suppose $(\gamma_1, s_1) \sim (\gamma, s)$ and $(\gamma_1', s_1') \sim (\gamma', s')$; we show that $\gamma_1/s_1 + \gamma_1'/s_1' = \gamma/s + \gamma'/s'$. Let $\alpha, \beta \in S$ such that $s\alpha = s_1\beta$ and $\gamma\alpha = \gamma_1\beta$. Also let $\lambda, \mu \in S$ such that $s'\lambda = s_1'\mu$ and $\gamma'\lambda = \gamma_1'\mu$. Then $\gamma_1/s_1 + \gamma_1'/s_1' = (\gamma_1 x + \gamma_1' y)/s_1 x$, where $x, y \in S$ such that $s_1' x = s_1' y$. Choose $e, f \in S$ with $s_1 xf = sae$. Then $sae = s'be$, $s_1 xf = s_1' yf$. Therefore $s'be = s_1' yf$. Now there exist $k, h \in S$ such that $aek = \alpha h$. Then $saek = s\alpha h = s_1\beta h = s_1 xfk$; thus $\beta h = xfk$. $\gamma aek = \gamma\alpha h = \gamma_1 xfk$. Then $(\gamma ae - \gamma_1 xf)k = 0 \Rightarrow \gamma ae = \gamma_1 xf$. Also $\gamma' bel = \gamma'\lambda m'$, where $l, m' \in S$ such that $bel = \lambda m'$. Now $s'bel = sael = s_1' yfl = s'\lambda m'$. Then $s_1' yfl = s_1'\mu m' \Rightarrow yfl = \mu m'$ and $\gamma' bel = \gamma'\lambda m' = \gamma_1'\mu m' = \gamma_1' yfl \Rightarrow \gamma' be = \gamma_1' yf$. Since e and f are endomorphisms of Γ, $(\gamma a + \gamma' b)e = \gamma ae + \gamma' be = \gamma_1 xf + \gamma_1' yf = (\gamma_1 x + \gamma_1' y)f$: but $sae = s_1 xf$; hence $(\gamma a + \gamma' b, sa) \sim (\gamma_1 x + \gamma_1' y, s_1 x)$. Thus addition is well defined, and the group axioms are satisfied. If $\gamma/s \in \Delta$ and $g \in G$, with $g = r/s_1$ $(r, s_1 \in S)$

define $(\gamma/s).g = \gamma u_1/(s_1 b_1)$, where $g = r/s_1$ and $u_1, b_1 \in S$ such that $su_1 = rb_1$. It may be checked that this is a well-defined operation and g is an automorphism of $(\Delta, +)$, and G acts faithfully on Δ. For example, let $\gamma/s, \gamma'/s', \in \Delta$ and $g = r/s_1 \in G$. Then $(\gamma/s+\gamma'/s')(r/s_1) = ((\gamma a+\gamma' b)/sa)(r/s_1)$, where $sa = s'b$ so

$$(\gamma/s+\gamma'/s')(r/s_1) = (\gamma a+\gamma' b)(u_1/(s_1 b_1)) = (\gamma a u_1+\gamma' b u_1)/(s_1 b_1),$$

where $sau_1 = rb_1$. Now $(\gamma/s)(r/s_1) = (\gamma u_2)/(s_1 b_2)$, where $su_2 = rb_2$, and $(\gamma'/s')(r/s_1) = (\gamma' u_3)/(s_1 b_3)$, where $s'u_3 = rb_3$. Then

$$(\gamma/s)(r/s_1)+(\gamma'/s')(r/s_1) = (\gamma u_2 c+\gamma' u_3 d)/(s_1 b_2 c),$$

where $s_1 b_2 c = s_1 b_3 d$, i.e. $b_2 c = b_3 d$. Choose $x, y \in S$ such that $s_1 b_2 cx = s_1 b_1 y$. Then $b_2 cx = b_1 y = b_3 dx$ and $su_2 cx = rb_2 cx = rb_3 dx = rb_1 y = sau_1 y$; hence $u_2 cx = au_1 y \Rightarrow \gamma u_2 cx = \gamma a u_1 y$. Also $s'u_3 dx = rb_3 dx = s'bu_1 y$; so $u_3 dx = bu_1 y$ and $\gamma' u_3 dx = \gamma' b u_1 y$. Then $(\gamma a u_1+\gamma' b u_1)y = \gamma a u_1 y+\gamma' b u_1 y = (\gamma u_2 c+\gamma' u_3 d)x$, i.e. $(\gamma a u_1+\gamma' b u_1,\ s_1 b_1) \sim (\gamma u_2 c+\gamma' u_3 d,\ s_1 b_2 c)$. In this way we see that G is a group of endomorphisms of Δ, and is in fact easily seen to be a group of automorphisms of Δ.

Lemma 2.2. *In the terminology of* 2.1, *G is a group of regular automorphisms of Δ, if and only if, for every* $0 \neq \gamma \in \Gamma$, $\gamma s_1 = \gamma s_2 \Rightarrow s_1 = s_2$, $(s_1, s_2 \in S)$.

Proof. (i) Suppose that G is a group of regular automorphisms. Let $\gamma \in \Gamma$; $s_1 \in S, s_2 \in S$ with $\gamma s_1 = \gamma s_2$ and $s_1 \neq s_2$. Then $(\gamma/1)(s_1/s_2) = (\gamma.a)/(s_2 b)$, where $1.a = s_1 b$. Hence $(\gamma/1)(s_1/s_2) = (\gamma s_1 b)/(s_2 b) = (\gamma s_2 b)/(s_2 b) = \gamma$. Since $g = s_1/s_2 \in G$, we have found an element $\gamma \in \Gamma$ and $g \in G$ such that $\gamma g = \gamma$. But g is regular; so $\gamma = 0$.

(ii) Suppose that $\gamma/s \in \Delta$, $r/s_1 \in G$ and $\gamma \neq 0$, and $(\gamma/s)(r/s_1) = \gamma/s$. Then $\gamma/s = \gamma u_1/(s_1 b_1)$, where $rb_1 = su_1$. There exist $u_2, b_2 \in S$ such that $\gamma u_2 = \gamma u_1 b_2$ and $su_2 = s_1 b_1 b_2$. Now $\gamma \neq 0 \Rightarrow u_2 = u_1 b_2$ and thus

$$s_1 b_1 b_2 = su_2 = su_1 b_2 \Rightarrow s_1 b_1 = su_1 = rb_1 \Rightarrow s_1 = r.$$

Thus r/s_1 is the identity automorphism of Δ, and G is a group of regular automorphisms of Δ.

Lemma 2.3. *In the terminology of Theorem* 2.1 *if* $\Gamma = \{0\} \cup \left\{\bigcup_{i=1}^{p} \gamma_i S\right\}$ *for suitable* $\gamma_1, \gamma_2, \ldots, \gamma_p \in \Gamma$, *such that* $\gamma_1 S \cap \gamma_j S = \varnothing$ *for* $i \neq j$, *then Δ has p orbits under the action of G.*

Proof. Let $\delta \in \Delta$, $\delta \neq 0$. Then $\delta = \gamma/s$ for some $\gamma \in \Gamma$, $s \in S$, and $\gamma = \gamma_i s_i$ for some $s_i \in S$, and some $i \in \{1, 2, \ldots, p\}$. Then $\delta = (\gamma_i s_i)/s = \gamma_i(s_i/s) \in \bigcup_{i=1}^{p} \gamma_i G$. Thus $\Delta = \left\{\bigcup_{i=1}^{p} \gamma_i G\right\} \cup \{0\}$. Now suppose that $\delta' \neq 0$ and $\delta' \in \gamma_i G \cap \gamma_j G$ for $i \neq j$. Let $\delta' = \gamma_i(r/s) = \gamma_j(y/z)$, where $r, s, y, z \in S$. There exist $\alpha, \beta \in S$ such that $s\alpha = z\beta = m$ (say); so we have $\delta' = (\gamma_i r)/s = (\gamma_j y)/z$ and

$\delta'm = (\gamma_i r/s)(m/1) = \gamma_i r u_1/b_1$, where $u_1, b_1 \in S$ and $mb_1 = su_1$, i.e. $s\alpha b_1 = su_1$, i.e. $\alpha b_1 = u_1$. Thus $\delta'm = (\gamma_i r\alpha b_1)/b_1 = \gamma_i r\alpha$. Also $\delta'm = \gamma_j y u_2/b_2$, where $u_2, b_2 \in S$ and $mb_2 = zu_2$, i.e. $\beta b_2 = u_2$. Thus $\delta'm = \gamma_j y\beta = \gamma_i r\alpha$; which implies that $\gamma_j S \cap \gamma_i S \neq \varnothing$, a contradiction.

Definition. A system (Γ, S) satisfying the hypothesis of Theorem 2.1 will be called a *$\mathcal{Q}$-system*, if and only if for every $0 \neq \gamma \in \Gamma$, $\gamma s_1 = \gamma s_2$, $(s_1, s_2 \in S)$.

Theorem 2.4. *If* (Γ, S) *and* (Γ_1, S_1) *are $\mathcal{Q}$-systems, then* $(\Gamma \oplus \Gamma_1, S \times S_1)$ *is a $\mathcal{Q}$-system.*

Proof. $S \times S_1$ is a semigroup of monomorphisms of $\Gamma \oplus \Gamma_1$ in the natural way. Clearly $S \times S_1$ is left and right cancellative, left and right reversible. Let $(\gamma, \gamma_1) \in \Gamma \oplus \Gamma_1$ with $(\gamma, \gamma_1) \neq (0, 0_1)$ and $(\gamma, \gamma_1)(s, s_1) = (\gamma, \gamma_1)(s', s_1')$ for $s, s' \in S$; $s_1, s_1' \in S_1$. Then $\gamma s = \gamma s' \Rightarrow s = s'$ and $\gamma_1 s_1 = \gamma_1 s_1' \Rightarrow s_1 = s_1'$. Finally, if $\Gamma = \gamma_1 S \cup \gamma_2 S \cup \ldots \cup \gamma_p S \cup \{0\}$ and $\Gamma_1 = \gamma_{11} S_1 \cup \gamma_{12} S_1 \cup \ldots \cup \gamma_{1q} S_1 \cup \{0_1\}$ then $\Gamma \oplus \Gamma_1 = \bigcup_{t=1}^{m} \delta_t (S \times S_1) \cup \{(0, 0_1)\}$, where $m = pq + p + q$ and the δ_t are elements of the form $(\gamma_i, 0_1)$, $(0, \gamma_{1j})$, (γ_i, γ_{1j}). Let

$$(\gamma, \gamma') \in (\lambda_i, \lambda_{1j})(S \times S_1) \cap (\lambda_l, \lambda_{1m})(S \times S_1)$$

for some $\lambda_i, \lambda_l \in \{0, \gamma_1, \gamma_2, \ldots, \gamma_p\}$ and $\lambda_{1j}, \lambda_{1m} \in \{0_1, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1q}\}$. Suppose $(\gamma, \gamma') = (\lambda_i s, \lambda_{1j} s_1) = (\lambda_l s', \lambda_{1m} s_1)$ then $\lambda_i s = \lambda_l s'$ and $\lambda_{1j} s_1 = \lambda_{1m} s_1'$. Clearly $\lambda_i = \lambda_l$ and $\lambda_{1j} = \lambda_{1m}$. This proves the result.

Examples. Let Γ be the additive group of integers and S the semigroup of positive integers under multiplication. Then (Γ, S) is a $\mathcal{Q}$-system if we define $(\gamma)s = \gamma s$ for all $\gamma \in \Gamma$, $s \in S$. Also let Γ be any finite group and $S = \{$identity automorphism$\}$ then (Γ, S) is a $\mathcal{Q}$-system.

3. The near-rings associated with $\mathcal{Q}$-systems

To each $\mathcal{Q}$-system (Γ, S), we have associated an additive group Δ and a group G of regular automorphisms of Δ admitting only finitely many orbits on Δ. We now define two near-rings in a natural way. Let $N = \mathrm{Map}_S(\Gamma)$ and $Q = \mathrm{Map}_G(\Delta)$. We can imbed N in Q in a natural way. Let $n \in N$; define $\bar{n}$: $\Delta \to \Delta$ by $0 . \bar{n} = 0$ and $[\gamma_i(r/s)]\bar{n} = (\gamma_i n)(r/s)$, where $\gamma_i(r/s)$ is a typical non-zero element of Δ, and $\Delta = \{0\} \cup \left(\bigcup_{i=1}^{p} \gamma_i G \right)$. It is not difficult to see that $\bar{n} \in Q$. Define a map ξ: $N \to Q$ by $n\xi = \bar{n}$, $\forall n \in N$. Let $n, n_1 \in N$; then $(n + n_1)\xi = \overline{n + n_1}$. For any $\delta \in \Delta$, $\delta \neq 0$,

$$\delta(\overline{n + n_1}) = \gamma_i(a/b)(\overline{n + n_1}) = \gamma_i(n + n_1) . (a/b) = (\gamma_i n + \gamma_i n_1) . (a/b)$$

$$= \gamma_i(n . a/b + n_1 . a/b) = \delta n + \delta n_1.$$

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Also $\delta(\overline{n.n_1}) = (\gamma_i(a/b))(\overline{nn_1}) = (\gamma_i nn_1).(a/b)$. Let $\gamma_i nn_1 = \gamma_j r_1$, for some $j \in \{1, \ldots, p\}$ and $r_1 \in S$, so that $\delta(\overline{nn_1}) = \gamma_j r_1(a/b)$. Now

$$\delta(\bar{n}.\bar{n}_1) = \gamma_1 n.(a/b)\bar{n}_1 = \gamma_k r_2(a/b)\bar{n}_1,$$

where $\gamma_i n = \gamma_k r_2$ for some $k \in \{1, \ldots, p\}$ and $r_2 \in S$. Then

$$\delta(\bar{n}.\bar{n}_1) = \gamma_k.n_1(r_2.a/b).$$

Let $\gamma_k n_1 = \gamma_l r_3$ for some $l \in \{1, \ldots, p\}$ and $r_3 \in S$, then

$$\gamma_i nn_1 = \gamma_j r_1 = (\gamma_k r_2)n_1 = \gamma_l r_3 r_2.$$

Hence $\delta(\bar{n}.\bar{n}_1) = \gamma_l r_3 r_2.(a/b) = \gamma_j r_1.(a/b) = \delta(\bar{n}\bar{n}_1)$. Thus ξ is a near-ring homomorphism. If $n \in \ker \xi$ then $\bar{n} = 0$, and $\Delta\bar{n} = 0$. Thus if $0 \neq \delta \in \Delta$, and $\delta = \gamma_i(a/b)$, then $[\gamma_i(a/b)]\bar{n} = \gamma_i n(a/b) = 0$. In particular, $\gamma_i n = 0$ and this is clearly true for $i = 1, \ldots, p$. Hence n is the zero mapping and ξ is bijective. We have

Theorem 3.1. *If* (Γ, S) *is a* $\mathcal{Q}$*-system, and* Δ *and* G *the groups constructed by Theorem* 2.1, *then* $\mathrm{Map}_S(\Gamma)$ *may be imbedded in* $\mathrm{Map}_G(\Delta)$.

Remark. The near-ring $Q = \mathrm{Map}_G(\Delta)$ is a 2-primitive near-ring with identity and descending chain condition on right ideals (see (3)). The remainder of this section will show that Q is a near-ring of right quotients of N.

Let $I = \{1, 2, \ldots, p\}$. Suppose $n \in N$ and for any $k \in I$ define

$$I_k(n) = \{i \in I \mid \gamma_i n \in \gamma_k S\}.$$

It is clear that $I_k(n)$ may be empty and that if $l \in I$ with $k \neq l$, then

$$I_k(n) \cap I_l(n) = \emptyset.$$

Lemma 3.2. *If* n *is a regular element of* N, *then each* $I_k(n)$ *contains one element for any* $k \in I$.

Proof. If $I_k(n) = \emptyset$ for some $k \in I$, define n_1 so that

$$\gamma_i s n_1 = \gamma_i s \text{ for all } i \in I \text{ with } i \neq k \text{ and any } s \in S;$$
$$\gamma_k s n_1 = 0 \text{ for all } s \in S \text{ and } 0n_1 = 0.$$

Then $n_1 \in N$, $\gamma_i n.1 = \gamma_i nn_1$ $(i \neq k)$ and $\gamma_k n.1 = \gamma_j s = \gamma_j s n_1 = \gamma_k nn_1$, where $\gamma_k n = \gamma_j s$ for some $j \in I\backslash\{k\}$ and $s \in S$. Thus $n.1 = n.n_1 \Rightarrow n_1 = 1$ which contradicts $\gamma_k n_1 = 0$. Thus $I_k(n) \neq \emptyset$ for any $k \in I$.

Theorem 3.3. *If* n *is a regular element of* N, *then there exists* $q \in Q$ *such that* $\bar{n}.q = q.\bar{n} = 1_Q$.

Proof. Let $\gamma_i n = \gamma_{j_i}.s_1$ where $j_i \in I$; $s_i \in S$; $i = 1, \ldots, p$.

The integers $j_1, \ldots, j_p$ are a permutation of $1, \ldots, p$ by Lemma 3.2; let this permutation be denoted by π. Thus $j_i = \pi(i)$, $i \in I$. Let s_i^{-1} be the inverse of s_i in G $(i \in I)$. Define $q\colon \Delta \to \Delta$ by $0q = 0$ and $(\gamma_{\pi(i)}g_i)q = \gamma_i s_i^{-1} g_i$ for

$i \in I$, $g_i \in G$. Let $\delta \in \Delta$, $g \in G$ and suppose that $\delta = \gamma_{\pi(l)} \cdot g_1$ for some $l \in I$, $g_1 \in G$. Then

$$(\delta g)q = (\gamma_{\pi(l)} \cdot g_1 g)q = \gamma_l s_l^{-1}(g_1 \cdot g) = (\gamma_l s_l^{-1} g_1)g = (\gamma_{\pi(l)} g_1)qg = \delta qg.$$

Hence $q \in Q$.

Now let $\delta' \in \Delta$ with $\delta' \neq 0$, and $\delta' = \gamma_i g_i = \gamma_i(r_i/t_i)$, where $i \in I$, r_i, $t_i \in S$. Then $\delta' \bar{n} q = (\gamma_i(r_i/t_i)\bar{n}q = (\gamma_i n)(r_i/t_i)q = (\gamma_{\pi(i)} s_i)(r_i/t_i)q = \gamma_i s_i^{-1} s_i(r_i/t_i) = \delta'$. Hence $\bar{n}q = 1_Q$. Also $\delta' q \bar{n} = (\gamma_i(r_i/t_i))q\bar{n} = \gamma_{\pi(k)}(r_i/t_i)q\bar{n}$, where $\pi(k) = i$. Then

$$\delta' q \bar{n} = \gamma_k s_k^{-1}(r_i/t_i)\bar{n} = \gamma_k n s_k^{-1}(r_i/t_i).$$

Now $\gamma_k n = \gamma_{\pi(k)} s_k$ and so $\delta' q \bar{n} = \delta'$. Thus $\bar{n}q = q\bar{n} = 1_Q$. Therefore we can invert the imbedded regular elements of N in Q.

Theorem 3.4. *If x is an arbitrary non-zero element of Q, then there exist θ, $n_1 \in N$, with θ regular in N such that $x = \bar{n}_1 \theta^{-1}$, where θ^{-1} is the inverse in Q of the element θ.*

We first need the following lemma, which is proved by a standard induction argument.

Lemma 3.5. *Let $r_1, \ldots, r_\sigma, t_1, \ldots, t_\sigma \in S$, then there exist $m \in S$ and $h_1, \ldots, h_\sigma \in S$ such that $mr_i = h_i t_i$ for $i = 1, \ldots, \sigma$.*

Proof of Theorem 3.4. Let $x \in Q$, with $x \neq 0$. Put $X = \{\alpha \in I \mid \gamma_\alpha x = 0\}$. Suppose that x: $\gamma_i \to \gamma_{j_i} g_i$, $i \in I \backslash X$, $j_i \in I$, $g_i \in G$. We have that x: $\gamma_\alpha \to 0$ for $\alpha \in X$. For any $k \in I$, put $I_k^*(x) = \{i \in I \mid \gamma_i x \in \gamma_k G\}$.

Some of these $I_k^*(x)$ may be empty. If $v \in I_k^*(x)$ for some $k \in I$ then $\gamma_v x = \gamma_k g_{kv}$ say, $(g_{kv} \in G)$ and so $\gamma_v x = \gamma_k(r_{kv}/t_{kv})$ for suitable r_{kv}, $t_{kv} \in S$. From Lemma 3.5, there exist $m_k \in S$ and $h_{kv} \in S$, $(v \in I_k^*(x))$ such that $m_k r_{kv} = h_{kv} t_{kv}$ for all $v \in I_k^*(x)$, and each $k \in I$.

Now we define a mapping n_1: $\Gamma \to \Gamma$ by putting

$$(\gamma_v \cdot s)n_1 = \gamma_k(h_{kv})s \quad \text{for all } v \in I_k^*(x),\ k \in I,$$
$$0 \cdot n_1 = 0,\ (\gamma_\alpha s)n_1 = 0 \quad \text{for } \alpha \in X.$$

Clearly $n_1 \in N$.

Let $I' = \{j \in I \mid \gamma_i x \in \gamma_j G$ for some $i \in I\}$. (Thus I' is the set of indices whose associated orbits appear in the image of x in Δ.)

Define θ: $\Gamma \to \Gamma$ by

$$\gamma_t s\theta = \gamma_t m_t s \quad \text{for all } t \in I',$$
$$\gamma_i s\theta = \gamma_i s \quad \text{for all } i \in I \backslash I',$$
$$0 \cdot \theta = 0.$$

Then θ is a regular element of N and the element θ^{-1}: $\Delta \to \Delta$ is given by

$$\gamma_t g \to \gamma_t m_t^{-1} g \quad \text{for all } t \in I',$$
$$\gamma_i g \to \gamma_i g \quad \text{for all } i \in I \backslash I'.$$

Put $y = n_1\theta^{-1}$, then for $i \in I_k^*(x)$, $\gamma_i y = \gamma_i n_1 \theta^{-1} = (\gamma_k h_{ki})\theta^{-1} = \gamma_k m_k^{-1} h_{ki}$ as $k \in I'$. Now $m_k r_{ki} = h_{ki} t_{ki}$ and so in G, $r_{ki} t^{-1} = m_k^{-1} h_{ki}$. Thus $\gamma_i y = \gamma_k r_{ki} t_{ki}^{-1} = \gamma_i x$. If $j \in X$, then $\gamma_j y = \gamma_j n_1 \theta^{-1} = 0$ as $\gamma_j n_1 = 0$. Hence $y = x = n_1\theta^{-1}$. Thus Q is a near-ring of right quotients of N.

Theorem 3.6. *If* (Γ, S) *is a* $\mathscr{Q}$*-system, then* $N = \mathrm{Map}_S(\Gamma)$ *is a* 2*-prime near-ring.*

Proof. Assume that $n, n' \in N$ with $nNn' = 0$ but $n \neq 0$ and $n' \neq 0$. Let $I = \{1, 2, \ldots, p\}$. Since $n \neq 0$, then $\gamma_i n = \gamma_j s$ for some $i, j \in I$ and $s \in S$; also $\gamma_t n' = \gamma_r s'$ for some $t, r \in I$ and $s' \in S$. Define $n_1 : \Gamma \to \Gamma$ by

$$\gamma_j s_1 n_1 = \gamma_t s_1 \quad \text{for any } s_1 \in S,$$

$$\gamma_l s_1 n_1 = 0 \quad \text{for any } s_1 \in S \text{ and } l \in I,\ l \neq j.$$

Then $n_1 \in N$ and $\gamma_i n n_1 n' = \gamma_j s n_1 n' = \gamma_t s n' = \gamma_r s' s \neq 0$. But $n n_1 n' \in nNn' = 0$; thus we have a contradiction.

The theorems of this section show that if (Γ, S) is a $\mathscr{Q}$-system then the near-ring $\mathrm{Map}_S(\Gamma)$ will have a near-ring of right quotients which is in fact 2-primitive and artinian and is of the form $\mathrm{Map}_G(\Delta)$ for suitable groups Δ, G. Notice that the near-ring of quotients, although an endomorphism near-ring is associated with the category of G-acts and G-morphisms rather than with the related category of S-acts and S-morphisms.

Some of the theorems of ring theory which describe the connection between a ring and its ring of right quotients (if it has one) can be generalised easily to the near-ring case. Using results of Betsch (**1**) concerning near-rings with d.c.c. on right ideals which have a zero J_2-radical, it is possible to prove that $N = \mathrm{Map}_S(\Gamma)$ has a.c.c. on right annihilators and possesses no infinite direct sums of right ideals (see (**3**)).

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