VARIETIES AND D.G. NEAR-RINGS

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In this note we show that the variety of near-rings generated by d.g. near-rings is the class of all near-rings R satisfying

$$0x = x0 = 0 \text{ for all } x \in R. \qquad (1)$$

This extends a result of J. J. Malone (3) on the embedding of near-rings.

A near-ring R is a set with two binary operations, addition and multiplication such that R is a group with respect to addition, a semi-group with respect to multiplication and $x(y+z) = xy+xz$ for all x, y, $z \in R$. If $x \in R$ satisfies $(y+z)x = yx+zx$ for all y, $z \in R$, we say that x is distributive. A distributively generated (d.g.) near-ring is one which is generated as an additive group by its distributive elements.

A variety of near-rings is the class of all near-rings satisfying a given set of laws. Let $\boldsymbol{X}$ be a class of near-rings. Then let $\mathrm{v}\boldsymbol{X}$ be the smallest variety containing $\boldsymbol{X}$, $\mathrm{s}\boldsymbol{X}$ the class of all sub near-rings of $\boldsymbol{X}$ near-rings, $\mathrm{Q}\boldsymbol{X}$ the class of all homomorphic images of $\boldsymbol{X}$ near-rings and $\mathrm{R}\boldsymbol{X}$ the class of all residually $\boldsymbol{X}$ near-rings. For further explanation of these ideas see P. M. Cohn (1).

Let G be a group written additively (G is not necessarily abelian). Then we can define $T(G)$, the near-ring of all mappings from G to G where addition in $T(G)$ is given by $g(x+y) = gx+gy$, $g \in G$, x, $y \in T(G)$, and multiplication in $T(G)$ is the usual product of maps. $T(G)$ contains the following sub near-rings, each containing the next:

$T_0(G)$, the set of all mappings preserving the identity of G;

$E(G)$ the d.g. near-ring generated by all the endomorphisms of G;

$I(G)$ the d.g. near-ring generated by all the inner automorphisms of G.

Denote by $\boldsymbol{O}$ the variety of near-rings satisfying (1) and let

$$\boldsymbol{T} = \{R;\ R \cong T_0(G) \text{ for some group } G\},$$

$$\boldsymbol{I} = \{R;\ R \cong I(G) \text{ for some group } G\},$$

$$\boldsymbol{F} = \{R;\ |R| < \aleph_0\}.$$

We will show that $\boldsymbol{O} = \mathrm{v}\boldsymbol{I}$, by showing that $\boldsymbol{O} \leqq \mathrm{SQR}\boldsymbol{I}$, from which we can deduce that $\boldsymbol{O} = \mathrm{SQR}\boldsymbol{I} = \mathrm{v}\boldsymbol{I}$, using results from P. M. Cohn (1). In (3) J. J. Malone proved that $\boldsymbol{O} \cap \boldsymbol{F} = \mathrm{s}(\boldsymbol{I} \cap \boldsymbol{F})$.

Lemma 1. $\boldsymbol{O} = \mathrm{s}\boldsymbol{T}$.

Proof. $sT \leqq O$ is immediate. $O \leqq sT$ follows immediately from Theorem 1 of J. J. Malone and H. E. Heatherly (4), which states that if $R \in O$, then R can be embedded in $T_0(G)$ for any group G such that G contains properly the additive group of R.

By using this result, we can restrict our attention to $T_0(G)$ where G belongs to a class of groups such that any group can be embedded in a group of this class. By Theorem 11.5.4 of W. R. Scott (5), such a class is that of the simple (non-abelian) groups. So let G be a simple non-abelian group. Let $\{\delta_\lambda;\ \lambda \in \Lambda\}$ be the set of all finite subsets of G, $R = T_0(G)$, $S = I(G)$. Then A. Fröhlich (2) has established

Lemma 2. *Let $r \in R$ and let δ_λ be a finite subset of G. Then there is an element $s_\lambda \in S$ such that $gr = gs_\lambda$ for all $g \in \delta_\lambda$.*

This is essentially result 5.2 of (2). If G is finite, this shows that $R = S$.

For every $\lambda \in \Lambda$, let $S_\lambda \cong S$. Denote by T the direct product $\prod_{\lambda \in \Lambda} S_\lambda$. Then we will show that R is isomorphic to the homomorphic image of a sub near-ring U of T, whose projection into each factor is onto. U is then a subdirect product of the S_λ and hence lies in $\mathrm{R}I$ since $S_\lambda \in I$ (see P. Cohn (1) for a proof that $R \in \mathrm{R}X$ if and only if R is a subdirect product of X near-rings).

We define a partial ordering on Λ by $\lambda \leqq \mu$ if and only if $\delta_\lambda \leqq \delta_\mu$. Then given λ_1, λ_2, there is a μ satisfying $\mu \geqq \lambda_i$, $i = 1, 2$. Just take $\delta_\mu = \delta_{\lambda_1} \cup \delta_{\lambda_2}$. Let $t \in T$. Call t eventually constant if given $g \in G$, $\lambda \in \Lambda$, there is a $\mu \geqq \lambda$ such that for all $\eta \geqq \mu$, $gt(\eta) = h$ is independent of η. We are here considering $t(\eta)$ as lying in S for all η. Let $U = \{t;\ t \in T \text{ and } t \text{ is eventually constant}\}$.

Lemma 3. *U is a subdirect product of the S_λ, $\lambda \in \Lambda$. That is, $U \in \mathrm{R}I$.*

Proof. We only need to show that U is a sub near-ring, since the constant elements $t \in T$ satisfying $t(\lambda) = t(\mu)$ for all λ, μ in Λ lie in U and so the projection of U into each factor will be onto. Let $u, v \in U$, and let $g \in G$, $\lambda \in \Lambda$ be given. Then we have $\mu_1 \geqq \lambda$, $\mu_2 \geqq \lambda$ such that if $\eta \geqq \mu_1$, $gu(\eta) = h$, h independent of η, if $\eta \geqq \mu_2$, $gv(\eta) = k$, k independent of η. Let $\mu \geqq \mu_i$, $i = 1, 2$. Then for $\eta \geqq \mu$, $gu(\eta) = h$, $gv(\eta) = k$ and so $g(u+v)(\eta) = h+k$. Hence $u+v \in U$. Also we have $\mu_3 \geqq \lambda$ such that if $\eta \geqq \mu_3$, $hv(\eta) = l$, l independent of η. Let $\mu' \geqq \mu_i$, $i = 1, 3$. Then for $\eta \geqq \mu'$, $g(uv)(\eta) = (gu(\eta))v(\eta) = hv(\eta) = l$, l independent of η. Hence $uv \in U$. Finally it is obvious that $u \in U$ gives $-u \in U$ and that the additive and multiplicative identities are in U.

We are interested in what happens "eventually". So we define an equivalence relationship on the elements of U as follows. Let $u, v \in U$. Then $u \sim v$ if given $g \in G$, $\lambda \in \Lambda$, there is a $\mu \geqq \lambda$ such that for all $\eta \geqq \mu$, $gu(\eta) = gv(\eta)$. The fact that $\sim$ is symmetric and reflexive is immediate. If $u \sim v$, $v \sim w$, then given $g \in G$, $\lambda \in \Lambda$, there is a $\mu_1 \geqq \lambda$ such that for all $\eta \geqq \mu_1$, $gu(\eta) = gv(\eta)$, and a $\mu_2 \geqq \lambda$ such that for all $\eta \geqq \mu_2$, $gv(\eta) = gw(\eta)$. So if $\mu \geqq \mu_i$, $i = 1, 2$, then for all $\eta \geqq \mu$, $gu(\eta) = gv(\eta) = gw(\eta)$. Hence $u \sim w$. We have

Lemma 4. *$\sim$ is an equivalence relationship on U.*

We next show that $\sim$ is compatible with multiplication and addition in U.

Lemma 5. *Let 0 be the additive identity in U, and let*

$$N = \{u;\ u \in U,\ u \sim 0\}.$$

Then N is an ideal in U and $\sim$ is compatible with multiplication and addition in U.

Proof. Let v, $w \in U$, $u \in N$. Let $g \in G$, $\lambda \in \Lambda$. Then as in the proof of Lemma 3, there is a $\mu \geqq \lambda$ such that $gw(\eta)$, $gv(\eta)$ and $gu(\eta)$ are constant and $gu(\eta) = 0$ for all $\eta \geqq \mu$. So $g(-v+u+v)(\eta) = -gv(\eta)+gu(\eta)+gv(\eta) = 0$, i.e. $-v+u+v \in N$. We now need to show that $uw \sim 0$ and $(w+u)v-wv \sim 0$. If $\eta \geqq \mu$, then $guw(\eta) = gu(\eta)w(\eta) = 0w(\eta) = 0$. So $uw \sim 0$. If $gw(\eta) = k$ for $\eta \geqq \mu$, there is a $\mu_1 \geqq \lambda$ such that $kv(\eta)$ is constant for $\eta \geqq \mu_1$. Then if $\mu_2 \geqq \mu_1$, $\mu_2 \geqq \mu$, we have

$$g(w+u)v(\eta) = (gw(\eta)+gu(\eta))v(\eta) = kv(\eta) = gwv(\eta) \text{ as } gu(\eta) = 0,$$

for $\eta \geqq \mu_2$. So $(w+u)v-wv \sim 0$ and N is an ideal in U. Finally if $u \sim v$, then by the definition of $\sim$ it is immediate that $u-v \sim 0$. Hence $\sim$ is compatible with multiplication and addition in U.

We can now prove our result.

Theorem 6. $\boldsymbol{O} = \mathrm{SQR}\boldsymbol{I}$.

Proof. Let θ: $R \to U/N$, which is a near-ring by Lemma 5, be given by $r\theta = u+N$ where $u(\lambda)$ is given by $gr = gu(\lambda)$ for all $g \in \delta_\lambda$. This is possible by Lemma 2. Given $g \in G$, $\lambda \in \Lambda$, let μ be such that $g \in \delta_\mu$ and $\mu \geqq \lambda$, e.g. $\delta_\mu = \delta_\lambda \cup \{g\}$. Then if $\eta \geqq \mu$, $gu(\eta) = gr$ is constant. So $u \in U$. But $u(\eta)$ is not uniquely defined. Suppose that $u'(\eta)$ also satisfies $gr = gu'(\eta)$ if $g \in \delta_\eta$. Then $g(u-u')(\eta) = 0$ if $g \in \delta_\eta$ and hence for all $\eta \geqq \mu$ as defined above So $u+N = u'+N$ and $r\theta$ is uniquely defined.

Let r, $s \in R$, $r\theta = u+N$, $s\theta = v+N$. Given $g \in G$, $\lambda \in \Lambda$, let $\delta_\mu = \delta_\lambda \cup \{g\}$. Then $gu(\eta) = gr$ and $gv(\eta) = gs$ for all $\eta \geqq \mu$. So

$$g(u+v)(\eta) = gu(\eta)+gv(\eta) = gr+gs = g(r+s).$$

Hence if $(r+s)\theta = w$, then $w+N = u+v+N$.

Now let $g \in G$, $gr = h$, $\lambda \in \Lambda$. Define μ by $\delta_\mu = \delta_\lambda \cup \{g, h\}$. Then

$$gu(\eta) = gr,\ gv(\eta) = gs \text{ and } g(uv)(\eta) = (gu(\eta))v(\eta) = hv(\eta) = hs = grs$$

for all $\eta \geqq \mu$. So if $(rs)\theta = w$, then $w+N = uv+N$. This shows that θ is a homomorphism.

Suppose that $r\theta = N+u$ and $u \in N$. Then given $g \in G$, $\lambda \in \Lambda$, let μ_1 be defined by $\delta_{\mu_1} = \delta_\lambda \cup \{g\}$. We have $gu(\eta) = gr$ for all $\eta \geqq \mu_1$. But $u \sim 0$. So given $g \in G$, $\mu_2 \in \Lambda$, there is a $\mu_2 \geqq \mu_1$ such that $gu(\eta) = 0$ for all $\eta \geqq \mu_2$.

But $\eta \geqq \mu_2$ gives us $gr = gu(\eta) = 0$. This is true for all $g \in G$. So $r = 0$, and θ is an isomorphism.

Finally if $u \in U$, given $g \in G$, $\lambda \in \Lambda$, there is a $\mu \geqq \lambda$ such that $gu(\eta) = h$ is constant for $\eta \geqq \mu$. In particular if μ_1 is defined by $\delta_{\mu_1} = \delta_\mu \cup \{g\}$, we have $gu(\eta) = h$ for $\eta \geqq \mu_1$. Define $r \in R$ by $gr = h$, and let $v+N = r\theta$. Then for $\eta \geqq \mu_1$, $gv(\eta) = gr = h = gu(\eta)$. So $v+N = u+N$. Hence θ is onto and $R \cong U/N$.

By Lemma 3, $U \in$ R$\boldsymbol{I}$ and so $U/N \in$ QR$\boldsymbol{I}$. Hence $R \in$ QR$\boldsymbol{I}$ and by the remarks made after Lemma 1, $\boldsymbol{T} \leqq$ QR$\boldsymbol{I}$. By the remarks made just before Lemma 1 and Lemma 1 itself, $\boldsymbol{O} =$ SQR$\boldsymbol{I}$, thus finishing the proof of the theorem.

This extends J. J. Malone's result (3) that $\boldsymbol{O} \cap \boldsymbol{F} =$ S$(\boldsymbol{I} \cap \boldsymbol{F})$ in the sense that we remove the $\boldsymbol{F}$, but have to replace $\boldsymbol{I}$ by QR$\boldsymbol{I}$. It would be interesting to know if we can get rid of QR or replace it by something simpler.

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REFERENCES

(1) P. M. Cohn, *Universal Algebra* (Harper and Row, 1965).

(2) A. Fröhlich, The near-ring generated by the inner automorphisms of a finite simple group, *J. London Math. Soc.* 33 (1958), 95-107.

(3) J. J. Malone, A near-ring analogue of a ring embedding theorem, *J. Algebra* **16** (1970), 237-238.

(4) J. J. Malone and H. E. Heatherly, On some near-ring embeddings, *Quart. J. Math. Oxford Ser.* (2), **20** (1969), 81-85.

(5) W. R. Scott, *Group Theory* (Prentice Hall, 1964).

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