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STARLIKE INTEGRAL OPERATORS

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We study integral transforms of functions belonging to the Jakubowski class $S(m,M)$ and determine the range of values of the exponent for which the integral is a convex or a close to convex function.

1. Introduction

Let S denote the family of functions $f(z) = z + a_2z^2 + \ldots$, which are regular and univalent in the open unit disc $E = \{z \, , \, |z| < 1\}$ and let K , S^* , and C respectively denote the subclasses of S which are convex, starlike with respect to the origin and close to convex in E . Jakubowski [4] has defined a subclass $S(m,M)$ of S^* and a subclass $K(m,M)$ of K in the following manner. Let m and M be positive real numbers satisfying the condition: $|m - 1| < M \le m$. A function f of S belongs to the class $S(m,M)$ if

$$\left| \frac{z\,f'(z)}{f(z)} - m \right| < M \, , \quad z \text{ in } E \, . \tag{1}$$

A function g of S belongs to the class $K(m,M)$ if

$$\left| 1 + \frac{zg''(z)}{g'(z)} - m \right| < M \, , \quad z \text{ in } E \, . \tag{2}$$

From (1) we get

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$$\left|\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} - m\right| < M , \quad z \text{ in } E ,$$

or

$$m - M < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < m + M , \quad z \text{ in } E , \tag{3}$$

also we have

$$\left|\operatorname{Im}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < M , \quad z \text{ in } E . \tag{4}$$

It is obvious from (3) that $S(m,M) \subset S^*(m-M)$. If $g \in K(m,M)$, we get bounds identical to (3) and (4) for $\operatorname{Re}\{1+zg''(z)/g'(z)\}$ and $\operatorname{Im}\{1+zg''(z)/g'(z)\}$. The fact that these bounds are constant can be used to investigate the univalence of integrals involving members of the classes $S(m,M)$ or $K(m,M)$.

2. An integral operator that maps $S(m,M)$ to C

In the following all powers are principal ones. Let $f \in S$ and define the function F by the integral

$$F(z) = \int_0^z (f(t)/t)^\gamma dt , \quad z \in E . \tag{5}$$

For real γ the above integral has been extensively studied when f belongs to K , S^* or C [2,6,7]. When γ is complex and $f \in K$ or S^* , then F is univalent only in a disc which is contained in E [1]. Merkes and Wright [6] have shown that if $f \in S^*$ then F is close to convex if $-\frac{1}{2} \leq \gamma \leq \frac{3}{2}$. The corresponding result for the class $S(m,M)$ is contained in Theorem 1:

THEOREM 1. *Let* γ *be real and* $f \in S(m,M)$ *, then* $F \in C$ *if*

$$-3/2(1+m+M) \leq \gamma \leq 3/2(1-m+M) .$$

Proof. Kaplan [5] has shown that a function F belongs to C if, and only if,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + re^{i\theta}\frac{F''(re^{i\theta})}{F'(re^{i\theta})}\right] d\theta > -\pi , \tag{6}$$

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for $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$. From (5) we get

$$\begin{aligned} \operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right] &= 1-\gamma+\gamma\left[\operatorname{Re}\frac{zf'(z)}{f(z)}\right] \\ &> 1-|\gamma|-|\gamma|(m+M) \\ &= 1-|\gamma|(1+m+M)\ , \end{aligned} \tag{7}$$

where we have used (3). Hence

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1+re^{i\theta}\,\frac{F''(re^{i\theta})}{F'(re^{i\theta})}\right] d\theta > (\theta_2-\theta_1)[1-|\gamma|(1+m+M)]\ .$$

The last quantity is not less than $-\pi$ provided

$$1-|\gamma|(1+\ +\ ) \geq -\frac{1}{2}\ ,$$

or

$$|\gamma| \leq \frac{3}{2(1+m+M)}\ . \tag{8}$$

If $\gamma > 0$, (7) can be replaced by

$$\operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right] > 1-\gamma+\gamma(m-M)\ ,$$

and this leads to the result: $F \in C$ if

$$0 \leq \gamma \leq \frac{3}{2(1-m+M)}\ . \tag{9}$$

When we combine (8) and (9), we get the conclusion of Theorem 1. Note that when $f \in S(m,M)$, then F is univalent over a larger range of the exponent γ as compared to when $f \in S^*$. We now generalise the result of Theorem 1 to the case when γ is complex. Let F be defined by (5). We have

THEOREM 2. *Let $f \in S(m,M)$ and γ be a complex number,* $\operatorname{Re}\gamma \geq 0$ *; then $F \in C$ if*

$$|\gamma| \leq \frac{3}{2\sqrt{M^2+(1-m+M)^2}}\ .$$

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Proof. Let $\gamma = p + iq$ and $zf'(z)/f(z) = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. A simple calculation gives

$$\mathrm{Re}\left[1 + \frac{zF''(z)}{F'(z)}\right] = 1 - p + pu(r,\theta) - qv(r,\theta)$$
$$> 1 - p + p(m-M) - |q|M \ ,$$

where we have used (3) and (4). Proceeding as in the proof of Theorem 1, we get: $F \in C$ if

$$\frac{3}{2} - p(1-m+M) - |q|M \geq 0 \ . \tag{10}$$

We can write $p = |\gamma| \cos \alpha$ and $|q| = |\gamma| \sin \alpha$, where $0 \leq \alpha \leq \frac{\pi}{2}$. Denote $\tan \frac{\alpha}{2}$ by t , then $\cos \alpha = (1-t^2)/(1+t^2)$ and $\sin \alpha = 2t/(1+t^2)$ and (10) becomes

$$[\tfrac{3}{2} - |\gamma|(1-m+M)] - 2|\gamma|Mt + [\tfrac{3}{2} + |\gamma|(1-m+M)]t^2 \geq 0 \ . \tag{11}$$

If (11) is to hold for $0 \leq t \leq 1$, then

$$\frac{3}{2} - |\gamma|(1-m+M) \geq 0 \ ,$$

and

$$|\gamma|^2 M^2 - [\tfrac{9}{4} - |\gamma|^2 (1-m+M)^2] \leq 0 \ ,$$

or

$$|\gamma| \leq \frac{3}{2\sqrt{M^2 + (1-m+M)^2}} \ ,$$

and the proof of Theorem 2 is complete.

Let $f \in K(m,M)$ and define the function G by

$$G(z) = \int_0^z [g'(t)]^\gamma \, dt \ , \quad z \in E \ . \tag{12}$$

Since $g \in K(m,M)$ if and only if $zg' \in S(m,M)$, we can apply the results of Theorems 1 and 2 to the function zg' to get

COROLLARY. (a) *Let* γ *be a real constant and* $g \in K(m,M)$; *then* $G \in C$ *if*

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$$-3/2(1+m+M) \leq \gamma \leq 3/2(1-m+M) .$$

(b) *Let* γ *be complex,* Re $\gamma \geq 0$ *and* $g \in K(m,M)$ *; then* $G \in C$ *if*

$$|\gamma| \leq \frac{3}{2\sqrt{M^2+(1-m+M)^2}} .$$

3. Transforms of products of functions

Let $f \in K$, $g \in S(m,M)$ and define the functions H and J by the following integrals

$$H(z) = \int_0^z [f'(t)]^\alpha [g(t)/t]^\beta dt , \quad z \in E , \tag{13}$$

$$J(z) = \int_0^z [f(t)/t]^\alpha [g(t)/t]^\beta dt , \quad z \in E , \tag{14}$$

where α is a nonnegative real number and β is complex. For real β and the pair of functions f and g both belongong to K (or C) , the transform (13) has been studied by Causey and Reade [3] who have shown that $H \in K$ (or C) , if α,β lie in a closed convex region in the α-β plane. For complex β and $g \in S(m,M)$, Pandey and Bhargava [8] have shown recently that if $f \in K(\mu)$ then $H \in K(\mu)$ provided $0 \leq |\beta| \leq (1-\alpha)/2M$. We improve upon this result in the following Theorem 3.

THEOREM 3. *Let* $f \in K(\mu)$, $0 \leq \mu < 1$, $g \in S(m,M)$, $\alpha \geq 0$, Re $\beta \geq 0$, *then* $H \in K(\eta)$, $0 \leq \eta < 1$, *provided*

$$\alpha \leq \frac{1-\eta}{1-\mu} , \quad |\beta| \leq \frac{1-\eta-(1-\mu)\alpha}{\sqrt{M^2+(1-m+M)^2}} .$$

Proof. From (13) we get

$$1+\frac{zH''(z)}{H'(z)} = \alpha\left[1+\frac{zf''(z)}{f'(z)}\right] - \alpha + \beta\,\frac{zg'(z)}{g(z)} + 1 - \beta .$$

Let $\beta = p+iq$ and $zg'(z)/g(z) = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. We get

$$\operatorname{Re}\left[1+\frac{zH''(z)}{H'(z)}\right] = \alpha \operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] - \alpha + pu(r,\theta) - qv(r,\theta) + 1 - p ,$$

$$> \alpha\mu - \alpha + p(m-M) - |q|M + 1 - p ,$$

where we have used (3) and the fact that $f \in K(\mu)$. Thus $H \in K(\eta)$ if

$$1 - (1-\mu)\alpha - (1-m+M)p - |q|M \geq \eta ,$$

or

$$1 - \eta - (1-\mu)\alpha - (1-m+M)p - |q|M \geq 0 . \tag{15}$$

As before, we write $p = |\beta|(1-t^2)/(1+t^2)$ and $q = 2|\beta|t/(1+t^2)$, $0 \leq t \leq 1$, and (15) becomes

$$[1-\eta-(1-\mu)\alpha-|\beta|(1-m+M)] - 2M|\beta|t + [1-\eta-(1-\mu)\alpha+|\beta|(1-m+M)]t^2 . \tag{16}$$

If (16) is to hold for all t, $0 \leq t \leq 1$, then

$$1 - \eta - (1-\mu)\alpha - |\beta|(1-m+M) \leq 0 , \tag{17}$$

<u>and</u> the discriminant of (16) must be nonpositive, which gives

$$|\beta|^2M^2 \leq [1-\eta-(1-\mu)\alpha]^2 - |\beta|^2(1-m+M)^2. \tag{18}$$

Since $1 - m + M > 0$, (17) implies $1 - \eta - (1-\mu)\alpha \geq 0$, or

$$\alpha \leq \frac{1-\eta}{1-\mu} \tag{19}$$

and (18) gives

$$|\beta|^2 \leq \frac{1-\eta-(1-\mu)\alpha}{\sqrt{M^2+(1-m+M)^2}} .$$

This completes the proof of Theorem 3. Notice that the special case $\beta = 0$ gives a result of Patil and Thakare [9] which states that $H \in K(\eta)$ provided $\alpha \leq (1-\eta)/(1-\mu)$ which is precisely condition (19) above. Now we consider the function J defined by (14) above. We have

THEOREM 4. *Let* $f \in K$, $g \in S(m,M)$, α *be a nonnegative real number,* $\operatorname{Re}\beta \geq 0$, *then* J *is convex in* E *provided* $0 \leq \alpha \leq 2$ *and*

$$|\beta| \leq \frac{1-\alpha/2}{\sqrt{M^2+(1-m+M)^2}} .$$

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The proof is similar to that of Theorem 3. We make use of the well known result that if $f \in K$, then for $|z| < 1$,

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] \geq \frac{1}{2} .$$

4. Concluding remarks

We have considered transforms involving functions of the class $S(m,M)$. The existence of constant upper and lower bounds for the functions of this class helps in determining the range of the exponent, real or complex, for which the transform belongs to a subclass of S . Since members of the class $K(m,M)$ also have identical bounds, one can find results analogous to Theorems 3 and 4 when, in (13) and (14), $g(t)/t$ is replaced by $g'(t)$, $g \in K(m,M)$

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