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ON SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS OF BRIOT–BOUQUET TYPE

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Abstract

We study systems of partial differential equations of Briot–Bouquet type. The existence of holomorphic solutions to such systems largely depends on the eigenvalues of an associated matrix. For the noninteger case, we generalise the well-known result of Gérard and Tahara ['Holomorphic and singular solutions of nonlinear singular first order partial differential equations', *Publ. Res. Inst. Math. Sci.* **26** (1990), 979–1000] for Briot–Bouquet type equations to Briot–Bouquet type systems. For the integer case, we introduce a sequence of blow-up like changes of variables and give necessary and sufficient conditions for the existence of holomorphic solutions. We also give some examples to illustrate our results.

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1. Introduction

A Briot–Bouquet system usually refers to the following ordinary differential system

$$tU' = F(t, U),$$

where $U = (u_1, \dots, u_m) \in \mathbf{C}^m$ and $F = (f_1, \dots, f_m)$ is holomorphic satisfying $F(0, 0) = 0$. Since the work of Briot and Bouquet [1], many authors have worked on Briot–Bouquet systems (see, for example, [2, 3, 8, 10, 11] and [6] for an extensive bibliography).

A nonlinear first-order partial differential system is said to be of Briot–Bouquet type, if it takes the form

$$tU_t = F(t, x, U, U_x), \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, $U_x = (u_{1,x_1}, \dots, u_{m,x_n}) \in \mathbf{C}^{mn}$, $F(t, x, U, V)$ is holomorphic in a polydisc Δ centred at the origin of $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^m \times \mathbf{C}^{mn}$, $F(0, x, 0, 0) = 0$ and

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$\partial f_i/\partial u_{j,x_k}(0,x,0,0) = 0$ for all i, j, k. Under these assumptions, we can rewrite (1.1) as

$$tU_t = A(x)t + \Lambda(x)U + G(t,x,U,U_x), \tag{1.2}$$

where $A(x) = (a_1(x), \ldots, a_m(x))$, $\Lambda(x) = [b_{ij}(x)]_{i,j=1}^m$ is an $m \times m$ matrix and

$$G(t,x,U,V) = \sum_{k+|\alpha|+|\beta|\geq 2} c_{k\alpha\beta}(x)t^k U^\alpha V^\beta.$$

Let λ_i, $1 \leq i \leq m$, be the eigenvalues of $\Lambda(0)$.

There have been some studies of Briot–Bouquet type partial differential equations, that is, the case $m = 1$. In [4], Gérard and Tahara studied holomorphic and singular solutions to such partial differential equations. In particular, they proved the following result.

Theorem 1.1. *If $\lambda_1 \notin \mathbf{Z}^+$, then a Briot–Bouquet type partial differential equation has a unique solution $u_1(t,x)$ holomorphic near $t = 0$ satisfying $u_1(0,x) = 0$.*

Our first result is a generalisation to systems of partial differential equations of Briot–Bouquet type. More precisely, we prove the following result.

Theorem 1.2. *If $\lambda_i \notin \mathbf{Z}^+$ for $1 \leq i \leq m$, then a Briot–Bouquet type partial differential system has a unique solution $U(t,x)$ holomorphic near $t = 0$ satisfying $U(0,x) = 0$.*

When $\lambda_i \in \mathbf{Z}^+$ for some i, we also provide necessary and sufficient conditions for the existence of holomorphic solutions of (1.2). For simplicity, we only carry out the details for $m = 1$ and $m = 2$. Here, we state a typical result for $m = 1$. (Yamane [10] also studied the integer case for $m = 1$, however his focus was quite different.)

When $m = 1$, we rewrite (1.2) as

$$tu_t = a(x)t + \lambda(x)u + G(t,x,u,u_x). \tag{1.3}$$

For notational purpose, write $a_1(x) = a(x)$. Suppose $\lambda(0) = q \in \mathbf{Z}^+$. We will perform some blow-up like changes of variable to put (1.3) into the following prepared form

$$t\bar{u}_t = a_q(x)t + (\lambda(x) - q + 1)\bar{u} + \bar{G}(t,x,\bar{u},\bar{u}_x),$$

where $a_q(x)$ will be inductively obtained from the expression of $G(t,x,u,u_x)$. We then obtain the following result.

Theorem 1.3. *Assume $\lambda(0) = q \in \mathbf{Z}^+$ in (1.3).*

(1) *If $\lambda(x) \not\equiv q$, then (1.3) has a unique solution $u(t,x)$ holomorphic near $t = 0$ satisfying $u(0,x) = 0$ if $a_q(x)/(\lambda(x) - q + 1)$ is holomorphic, and otherwise there are no solutions.*

(2) *If $\lambda(x) \equiv q$, then (1.3) has infinitely many solutions $u(t,x)$ holomorphic near $t = 0$ satisfying $u(0,x) = 0$ if $a_q(x) \equiv 0$, and otherwise there are no solutions.*

In Section 2, we prove Theorem 1.2. In Section 3, we first prove Theorem 1.3, and then we prove similar results for $m = 2$. Using exactly the same approach, similar results for arbitrary m can be proved, although the results will be lengthy to state.

2. The noninteger case

We prove Theorem 1.2 in this section. The method of proof is along the same lines as in [4], although with many changed details. For completeness and in preparation for the integer case, we carry out the necessary details below.

Assume $\lambda_i \notin \mathbf{Z}^+$ for $1 \le i \le m$. Suppose that the series

$$U = \sum_{j=1}^{\infty} U_j t^j, \quad U_j = (U_{j,1}, \ldots, U_{j,m}) \in \mathbf{C}^m,$$

solves (1.2) formally. Then

$$\sum_{j=1}^{\infty} jU_j t^j = A(x)t + \sum_{j=1}^{\infty} \Lambda(x)U_j t^j + \sum_{k+|\alpha|+|\beta|\ge 2} c_{k\alpha\beta}(x) g_{\alpha\beta}(U, U_x) t^{k+|\alpha|+|\beta|},$$

where $g_{\alpha\beta}(U, V)$ is determined by $U^\alpha V^\beta$, but does not contain t. Comparing the coefficients of t^j gives

$$(I - \Lambda(x))U_1 = A(x), \tag{2.1}$$

and

$$(jI - \Lambda(x))U_j = \sum_{k+|\alpha|+|\beta|=j} c_{k\alpha\beta}(x) g_{\alpha\beta}(U, U_x) \quad \text{for } j \ge 2. \tag{2.2}$$

Since $|\alpha| + |\beta| \le j$, the coefficient $g_{\alpha\beta}(U, U_x)$ only contains U_i with $i < j$. Since none of λ_i is a positive integer, the matrix $(jI - \Lambda(x))$ is invertible in a small neighbourhood of the origin. Hence we can solve for U_j recursively using (2.1) and (2.2).

Write $D_a = \{x \in \mathbf{C}^n : |x_i| < a, \ 1 \le i \le n\}$. For δ small enough,

$$|c_{k\alpha\beta}(x)| < C_{k\alpha\beta} \quad \text{for } x \in D_\delta, \tag{2.3}$$

and $\sum_{k+|\alpha|+|\beta|\ge 2} C_{k\alpha\beta} t^k U^\alpha V^\beta$ is convergent near the origin of $\mathbf{C} \times \mathbf{C}^m \times \mathbf{C}^{mn}$. Moreover,

$$|A(x)| < A \quad \text{for } x \in D_\delta. \tag{2.4}$$

By [6, Section 4, Proposition 1.1.1], there exists an $\epsilon > 0$ such that for all $j \ge 1$,

$$|jI - \Lambda(x)| > \epsilon j \quad \text{for } x \in D_\delta. \tag{2.5}$$

We also need the following lemma (see [5, Lemma 5.1.3]).

Lemma 2.1. *Suppose $u(x)$ is a holomorphic function on D_δ and $r \in (0, \delta)$. If*

$$|u(x)| \le \frac{C}{(\delta - r)^p} \quad \text{on } D_r,$$

then

$$\left|\frac{\partial u(x)}{\partial x_i}\right| \le \frac{Ce(p+1)}{(\delta - r)^{p+1}} \quad \text{on } D_r \text{ for } 1 \le i \le n.$$

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Set $S = (Y, \dots, Y) \in \mathbf{C}^m$ and $T = (eY, \dots, eY) \in \mathbf{C}^{mn}$. Consider the analytic equation

$$\epsilon Y = At + \frac{1}{\delta - r} \sum_{k+|\alpha|+|\beta|\geq 2} \frac{C_{k\alpha\beta}}{(\delta - r)^{k+|\alpha|+|\beta|-2}} t^k S^\alpha T^\beta. \tag{2.6}$$

By the implicit function theorem, (2.6) has a unique holomorphic solution of the form

$$Y = \sum_{j=1}^{\infty} Y_j t^j,$$

where

$$Y_1 = \epsilon^{-1} A, \tag{2.7}$$

and

$$Y_j = \epsilon^{-1} \frac{1}{\delta - r} \sum_{k+|\alpha|+|\beta|=j} \frac{C_{k\alpha\beta}}{(\delta - r)^{k+|\alpha|+|\beta|-2}} g_{\alpha\beta}(S, T) \quad \text{for } j \geq 2. \tag{2.8}$$

By induction on j, we can show that Y_j is of the form

$$Y_j = \frac{C_j}{(\delta - r)^{j-1}}.$$

Comparing (2.1) and (2.2) with (2.7) and (2.8), and using (2.3), (2.4), (2.5) and Lemma 2.1, we see inductively (see [4, Section 1]) that

$$|U_j| \leq \frac{Y_j}{j}, \quad \left|\frac{\partial U_j}{\partial x_k}\right| \leq eY_j \quad \text{for } j \geq 1.$$

Here $|U_j| = \max\{|U_{j,1}|, \dots, |U_{j,m}|\}$ and $|\partial U_j/\partial x_k| = \max\{|\partial U_{j,1}/\partial x_1|, \dots, |\partial U_{j,m}/\partial x_n|\}$. This shows that $U = \sum_{j=1}^{\infty} U_j t^j$ is convergent and completes the proof of Theorem 1.2.

3. The integer case

In this section, we study the case when some of the $\lambda_i \in \mathbf{Z}^+$ (see [7]). For simplicity, we only write out the details in dimensions 1 and 2, although it will be clear that similar results hold in higher dimensions.

3.1. The case $m = 1$. By assumption, $\lambda_1 = q \in \mathbf{Z}^+$. Write $\lambda(x) = q - \rho(x)$, with $\rho(0) = 0$. Rewrite (1.3) as

$$tu_t = a(x)t + (q - \rho(x))u + G(t, x, u, u_x), \tag{3.1}$$

where

$$G(t, x, u, u_x) = \sum_{k+\alpha+|\beta|\geq 2} c_{k\alpha\beta}(x) t^k u^\alpha u_x^\beta,$$

with $\beta = (\beta_1, \dots, \beta_n)$ and $u_x^\beta = u_{x_1}^{\beta_1} \cdots u_{x_n}^{\beta_n}$. As in the previous section, we first try to find a formal solution $u = \sum_{j=1}^{\infty} u_j t^j$ to (3.1). The recursive formulas are exactly as in (2.1) and (2.2).

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If $q = 1$, then (2.1) gives

$$\rho(x)u_1 = a(x). \tag{3.2}$$

If $\rho(x) \not\equiv 0$, then (3.2) has a unique holomorphic solution if and only if $a(x)/\rho(x)$ is holomorphic. If $\rho(x) \equiv 0$, then u_1 is solvable if and only if $a(x) \equiv 0$, in which case u_1 is arbitrary. Since $j - 1 + \rho(0) \neq 0$ for $j \geq 2$, the u_j's are determined by (2.2) as before. From the proof of Theorem 1.2, we also see that every formal solution is convergent.

If $q > 1$, then we make the blow-up like change of variable

$$u = t\Big(\tilde{u} + \frac{a(x)}{\rho(x) - q + 1}\Big).$$

Note that $a(x)/(\rho(x) - q + 1)$ is the leading coefficient u_1. Therefore, there exists a solution $u(t, x)$ holomorphic near $t = 0$ satisfying $u(0, x) = 0$ for the original equation if and only if there exists a solution $\tilde{u}(t, x)$ holomorphic near $t = 0$ satisfying $\tilde{u}(0, x) = 0$ for the new equation, which will still be of Briot–Bouquet type. One readily checks that the equation for $\tilde{u}$ is of the form (after cancelling t on both sides)

$$t\tilde{u}_t = a_2(x)t + (q - 1 - \rho(x))\tilde{u} + \tilde{G}(t, x, \tilde{u}, \tilde{u}_x),$$

where

$$a_2(x) = \sum_{k+\alpha+|\beta|=2} c_{k\alpha\beta}(x)\varphi(x)^{\alpha}\varphi_x(x)^{\beta}, \quad \varphi(x) = \frac{a(x)}{\rho(x) - q + 1}. \tag{3.3}$$

After $q - 1$ such steps, (3.1) takes the following prepared form

$$t\bar{u}_t = a_q(x)t + (1 - \rho(x))\bar{u} + \bar{G}(t, x, \bar{u}, \bar{u}_x),$$

where $a_q(x)$ can be obtained inductively as in (3.3). As above, if $\rho(x) \not\equiv 0$, then (3.1) has a unique holomorphic solution if and only if $a_q(x)/\rho(x)$ is holomorphic. And if $\rho(x) \equiv 0$, then (3.1) has infinitely many holomorphic solutions if and only if $a_q(x) \equiv 0$.

This completes the proof of Theorem 1.3.

3.2. The case $m = 2$. First suppose that only $\lambda_1 = q \in \mathbf{Z}^+$. After a suitable linear change of variables we can assume that $\Lambda(x)$ is diagonal in a neighbourhood of $x = 0$. Write $\Lambda(x) = \mathrm{diag}(q - \rho(x), \eta(x))$, with $\rho(0) = 0$. Then (1.2) can be written as

$$\begin{cases} tu_t = a(x)t + (q - \rho(x))u + f(t, x, u, v, u_x, v_x), \\ tv_t = b(x)t + \eta(x)v + g(t, x, u, v, u_x, v_x), \end{cases} \tag{3.4}$$

where

$$f(t, x, u, v, u_x, v_x) = \sum_{k+\alpha+|\beta|+\gamma+|\delta|\geq 2} c_{k\alpha\beta\gamma\delta}(x)t^k u^{\alpha} u_x^{\beta} v^{\gamma} v_x^{\delta}$$

and

$$g(t, x, u, v, u_x, v_x) = \sum_{k+\alpha+|\beta|+\gamma+|\delta|\geq 2} d_{k\alpha\beta\gamma\delta}(x)t^k u^{\alpha} u_x^{\beta} v^{\gamma} v_x^{\delta},$$

with $\beta = (\beta_1, \ldots, \beta_n)$, $u_x^{\beta} = u_{x_1}^{\beta_1} \cdots u_{x_n}^{\beta_n}$, $\delta = (\delta_1, \ldots, \delta_n)$ and $v_x^{\delta} = v_{x_1}^{\delta_1} \cdots v_{x_n}^{\delta_n}$. Write $a_1(x) = a(x)$ and $b_1(x) = b(x)$.

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Again, we try to find a formal solution using (2.1) and (2.2), which will be convergent if it exists.

If $q > 1$, then we consider the change of variables:

$$u = t\Big(\tilde{u} + \frac{a(x)}{\rho(x) - q + 1}\Big), \quad v = t\Big(\tilde{v} + \frac{b(x)}{1 - \eta(x)}\Big).$$

Similar to the $m = 1$ case, the system for $(\tilde{u}, \tilde{v})$ takes the form

$$\begin{cases} t\tilde{u}_t = a_2(x)t + (q - 1 - \rho(x))\tilde{u} + \tilde{f}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \\ t\tilde{v}_t = b_2(x)t + (\eta(x) - 1)\tilde{v} + \tilde{g}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \end{cases}$$

where

$$a_2(x) = \sum_{k+\alpha+|\beta|+\gamma+|\delta|\geq 2} c_{k\alpha\beta\gamma\delta}(x)\varphi(x)^{\alpha}\varphi_x(x)^{\beta}\psi(x)^{\gamma}\psi_x(x)^{\delta} \tag{3.5}$$

and

$$b_2(x) = \sum_{k+\alpha+|\beta|+\gamma+|\delta|\geq 2} d_{k\alpha\beta\gamma\delta}(x)\varphi(x)^{\alpha}\varphi_x(x)^{\beta}\psi(x)^{\gamma}\psi_x(x)^{\delta}, \tag{3.6}$$

with

$$\varphi(x) = \frac{a(x)}{\rho(x) - q + 1}, \quad \psi(x) = \frac{b(x)}{1 - \eta(x)}.$$

After $q - 1$ steps, (3.4) takes the prepared form

$$\begin{cases} t\bar{u}_t = a_q(x)t + (1 - \rho(x))\bar{u} + \bar{f}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \\ t\bar{v}_t = b_q(x)t + (\eta(x) - q + 1)\bar{v} + \bar{g}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \end{cases} \tag{3.7}$$

where $a_q(x)$ and $b_q(x)$ are obtained inductively as in (3.5) and (3.6). Then arguing as before, we obtain the following result.

THEOREM 3.1. *Assume that $\Lambda(0)$ has one positive integer eigenvalue q. Let $\rho(x)$ and $a_q(x)$ be as in (3.7).*

(1) *If $\rho(x) \not\equiv 0$, then (3.4) has a unique holomorphic solution $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x)/\rho(x)$ is holomorphic, and otherwise there are no solutions.*

(2) *If $\rho(x) \equiv 0$, then (3.4) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x) \equiv 0$, and otherwise there are no solutions.*

Suppose now that both $\lambda_1 = q \in \mathbf{Z}^+$ and $\lambda_2 = p \in \mathbf{Z}^+$, with $q \leq p$. First suppose that $q < p$. Then $\Lambda(x)$ is diagonalisable in a neighbourhood of $x = 0$ and we can write $\Lambda(x) = \operatorname{diag}(q - \rho(x), p - \eta(x))$, with $\rho(0) = \eta(0) = 0$. Then (1.2) can be written as

$$\begin{cases} tu_t = a(x)t + (q - \rho(x))u + f(t, x, u, v, u_x, v_x), \\ tv_t = b(x)t + (p - \eta(x))v + g(t, x, u, v, u_x, v_x). \end{cases} \tag{3.8}$$

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If $q > 1$, we consider the change of variables

$$u = t\left(\tilde{u} + \frac{a(x)}{\rho(x) - q + 1}\right), \quad v = t\left(\tilde{v} + \frac{b(x)}{\eta(x) - p + 1}\right).$$

Then the system for $(\tilde{u}, \tilde{v})$ takes the form

$$\begin{cases} t\tilde{u}_t = a_2(x)t + (q - 1 - \rho(x))\tilde{u} + \tilde{f}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \\ t\tilde{v}_t = b_2(x)t + (p - 1 - \eta(x))\tilde{v} + \tilde{g}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \end{cases}$$

where $a_2(x)$ and $b_2(x)$ are as in (3.5) and (3.6), but with $\psi(x) = b(x)/(\eta(x) - p + 1)$.

After $q - 1$ such steps, (3.8) takes the prepared form

$$\begin{cases} t\bar{u}_t = a_q(x)t + (1 - \rho(x))\bar{u} + \bar{f}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \\ t\bar{v}_t = b_q(x)t + (p - q + 1 - \eta(x))\bar{v} + \bar{g}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x). \end{cases} \tag{3.9}$$

If $a_q(x)/\rho(x)$ is holomorphic, then we take $p - q$ more steps and (3.8) takes the prepared form

$$\begin{cases} t\hat{u}_t = a_p(x)t + (q - p + 1 - \rho(x))\hat{u} + \hat{f}(t, x, \hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x), \\ t\hat{v}_t = b_p(x)t + (1 - \eta(x))\hat{v} + \hat{g}(t, x, \hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x). \end{cases} \tag{3.10}$$

Then arguing as before, we obtain the following result.

THEOREM 3.2. *Assume that $\Lambda(0)$ has two positive integer eigenvalues q and p, with $q < p$. Let $\rho(x)$ and $a_q(x)$ be as in (3.9) and $\eta(x)$ and $b_p(x)$ be as in (3.10).*

(1) *If $\rho(x) \not\equiv 0$ and $\eta(x) \not\equiv 0$, then (3.8) has a unique holomorphic solution $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if both $a_q(x)/\rho(x)$ and $b_p(x)/\eta(x)$ are holomorphic, and otherwise there are no solutions.*

(2) *If $\rho(x) \equiv 0$ and $\eta(x) \not\equiv 0$, then (3.8) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x) \equiv 0$ and $b_p(x)/\eta(x)$ is holomorphic, and otherwise there are no solutions.*

(3) *If $\rho(x) \not\equiv 0$ and $\eta(x) \equiv 0$, then (3.8) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x)/\rho(x)$ is holomorphic and $b_p(x) \equiv 0$, and otherwise there are no solutions.*

(4) *If $\rho(x) \equiv 0$ and $\eta(x) \equiv 0$, then (3.8) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x) \equiv 0$ and $b_p(x) \equiv 0$, and otherwise there are no solutions.*

Now we assume that $\lambda_1 = \lambda_2 = q \in \mathbf{Z}^+$. There are two issues we need to address. The first is that even if $\Lambda(0)$ is diagonalisable, $\Lambda(x)$ is not necessarily diagonalisable in a neighbourhood of $x = 0$. The second is that the eigenvalue functions of $\Lambda(x)$ are not necessarily univalent in a neighbourhood of $x = 0$.

First we assume that $\Lambda(x)$ is diagonalisable in a neighbourhood of $x = 0$. Then (1.2) can be written as

$$\begin{cases} tu_t = a(x)t + (q - \rho(x))u + f(t, x, u, v, u_x, v_x), \\ tv_t = b(x)t + (q - \eta(x))v + g(t, x, u, v, u_x, v_x). \end{cases} \tag{3.11}$$

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If $q > 1$, then we consider the change of variables

$$u = t\Big(\tilde{u} + \frac{a(x)}{\rho(x) - q + 1}\Big), \qquad v = t\Big(\tilde{v} + \frac{b(x)}{\eta(x) - q + 1}\Big).$$

Then the system for $(\tilde{u}, \tilde{v})$ takes the form

$$\begin{cases} t\tilde{u}_t = a_2(x)t + (q - 1 - \rho(x))\tilde{u} + \tilde{f}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \\ t\tilde{v}_t = b_2(x)t + (q - 1 - \eta(x))\tilde{v} + \tilde{g}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \end{cases}$$

where $a_2(x)$ and $b_2(x)$ are as in (3.5) and (3.6), but with $\psi(x) = b(x)/(\eta(x) - q + 1)$.

After $q - 1$ such steps, (3.11) takes the prepared form

$$\begin{cases} t\bar{u}_t = a_q(x)t + (1 - \rho(x))\bar{u} + \bar{f}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \\ t\bar{v}_t = b_q(x)t + (1 - \eta(x))\bar{v} + \bar{g}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x). \end{cases} \tag{3.12}$$

Then arguing as before, gives the following result.

Theorem 3.3. *Assume that $\Lambda(0)$ has two equal positive integer eigenvalues q and $\Lambda(x)$ is diagonalisable in a neighbourhood of $x = 0$. Let $\rho(x)$, $\eta(x)$, $a_q(x)$ and $b_q(x)$ be as in (3.12).*

(1) *If $\rho(x) \not\equiv 0$ and $\eta(x) \not\equiv 0$, then (3.11) has a unique holomorphic solution $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if both $a_q(x)/\rho(x)$ and $b_q(x)/\eta(x)$ are holomorphic, and otherwise there are no solutions.*

(2) *If $\rho(x) \equiv 0$ and $\eta(x) \not\equiv 0$, then (3.11) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x) \equiv 0$ and $b_q(x)/\eta(x)$ is holomorphic, and otherwise there are no solutions.*

(3) *If $\rho(x) \not\equiv 0$ and $\eta(x) \equiv 0$, then (3.11) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x)/\rho(x)$ is holomorphic and $b_q(x) \equiv 0$, and otherwise there are no solutions.*

(4) *If $\rho(x) \equiv 0$ and $\eta(x) \equiv 0$, then (3.11) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $a_q(x) \equiv 0$ and $b_q(x) \equiv 0$, and otherwise there are no solutions.*

Next assume that $\Lambda(x)$ is not necessarily diagonalisable, but has univalent eigenvalue functions, in a neighbourhood of $x = 0$. Then after a suitable linear change of variables (1.2) can be written as

$$\begin{cases} tu_t = a(x)t + (q - \rho(x))u + \epsilon(x)v + f(t, x, u, v, u_x, v_x), \\ tv_t = b(x)t + (q - \eta(x))v + g(t, x, u, v, u_x, v_x). \end{cases} \tag{3.13}$$

If $q > 1$, then we consider the change of variables

$$u = t\Big(\tilde{u} + \frac{a(x) + \epsilon(x)b(x)/(\eta(x) - q + 1)}{\rho(x) - q + 1}\Big), \qquad v = t\Big(\tilde{v} + \frac{b(x)}{\eta(x) - q + 1}\Big).$$

Then the system for $(\tilde{u}, \tilde{v})$ takes the form

$$\begin{cases} t\tilde{u}_t = a_2(x)t + (q - 1 - \rho(x))\tilde{u} + \epsilon(x)\tilde{v} + \tilde{f}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \\ t\tilde{v}_t = b_2(x)t + (q - 1 - \eta(x))\tilde{v} + \tilde{g}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \end{cases}$$

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where $a_2(x)$ and $b_2(x)$ are as in (3.5) and (3.6), but with

$$\varphi(x) = \frac{a(x) + \epsilon(x)b(x)/(\eta(x) - q + 1)}{\rho(x) - q + 1}, \quad \psi(x) = \frac{b(x)}{\eta(x) - q + 1}.$$

After $q - 1$ such steps, (3.13) takes the prepared form

$$\begin{cases} t\bar{u}_t = a_q(x)t + (1 - \rho(x))\bar{u} + \epsilon(x)\bar{v} + \bar{f}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \\ t\bar{v}_t = b_q(x)t + (1 - \eta(x))\bar{v} + \bar{g}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x). \end{cases} \tag{3.14}$$

For a formal solution $\bar{u} = \sum_{j=1}^{\infty} \bar{u}_j(x)t^j$, $\bar{v} = \sum_{j=1}^{\infty} \bar{v}_j(x)t^j$, equation (2.1) gives

$$\begin{cases} \rho(x)\bar{u}_1(x) = a_q(x) + \epsilon(x)\bar{v}_1(x), \\ \eta(x)\bar{v}_1(x) = b_q(x). \end{cases}$$

A similar argument to those above yields the following result.

THEOREM 3.4. *Assume that $\Lambda(0)$ has two equal positive integer eigenvalues q and the eigenvalue functions of $\Lambda(x)$ are univalent in a neighbourhood of $x = 0$. Let $\rho(x)$, $\eta(x)$, $\epsilon(x)$, $a_q(x)$ and $b_q(x)$ be as in (3.14).*

(1) *If $\rho(x) \not\equiv 0$ and $\eta(x) \not\equiv 0$, then (3.13) has a unique holomorphic solution $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if both $b_q(x)/\eta(x)$ and $(a_q(x) + \epsilon(x)b_q(x)/\eta(x))/\rho(x)$ are holomorphic, and otherwise there are no solutions.*

(2) *If $\rho(x) \equiv 0$ and $\eta(x) \not\equiv 0$, then (3.13) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $b_q(x)/\eta(x)$ is holomorphic and $a_q(x) + \epsilon(x)b_q(x)/\eta(x) \equiv 0$, and otherwise there are no solutions.*

(3) *If $\eta(x) \equiv 0$, then (3.13) has infinitely many holomorphic solutions $(u(t, x), v(t, x))$ near $(t, x) = (0, 0)$ satisfying $(u(0, x), v(0, x)) = (0, 0)$ if $b_q(x) \equiv 0$, and otherwise there are no solutions.*

Finally assume that $\Lambda(x)$ does not have univalent eigenvalue functions in a neighbourhood of $x = 0$. After a suitable linear change of variables, (1.2) can be written as

$$\begin{cases} tu_t = a(x)t + (q - \rho(x))u + \epsilon(x)v + f(t, x, u, v, u_x, v_x), \\ tv_t = b(x)t + \delta(x)u + (q - \eta(x))v + g(t, x, u, v, u_x, v_x), \end{cases} \tag{3.15}$$

where $\eta(x)\rho(x) - \delta(x)\epsilon(x)$ is not identically zero.

If $q > 1$, then we consider the change of variables

$$u = t(\tilde{u} + \varphi(x)), \quad v = t(\tilde{v} + \psi(x)),$$

where $\varphi(x)$ and $\psi(x)$ satisfy

$$\begin{cases} (q - 1 - \rho(x))\varphi(x) + \epsilon(x)\psi(x) = -a(x), \\ \delta(x)\varphi(x) + (q - 1 - \eta(x))\psi(x) = -b(x). \end{cases} \tag{3.16}$$

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Note that this system is always solvable under our assumptions. The system for $(\tilde{u}, \tilde{v})$ then takes the form

$$\begin{cases} t\tilde{u}_t = a_2(x)t + (q-1-\rho(x))\tilde{u} + \epsilon(x)\tilde{v} + \tilde{f}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \\ t\tilde{v}_t = b_2(x)t + \delta(x)\delta u + (q-1-\eta(x))\tilde{v} + \tilde{g}(t, x, \tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x), \end{cases}$$

where $a_2(x)$ and $b_2(x)$ are as in (3.5) and (3.6), but with $\varphi(x)$ and $\psi(x)$ as in (3.16).

After $q-1$ such changes, (3.15) takes the prepared form

$$\begin{cases} t\bar{u}_t = a_q(x)t + (1-\rho(x))\bar{u} + \epsilon(x)\bar{v} + \bar{f}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x), \\ t\bar{v}_t = b_q(x)t + \delta(x)\bar{u} + (1-\eta(x))\bar{v} + \bar{g}(t, x, \bar{u}, \bar{v}, \bar{u}_x, \bar{v}_x). \end{cases} \tag{3.17}$$

For a formal solution $\bar{u} = \sum_{j=1}^{\infty} \bar{u}_j(x)t^j$, $\bar{v} = \sum_{j=1}^{\infty} \bar{v}_j(x)t^j$, equation (2.1) gives

$$\begin{cases} \rho(x)\bar{u}_1(x) - \epsilon(x)\bar{v}_1(x) = a_q(x), \\ -\delta(x)\bar{u}_1(x) + \eta(x)\bar{v}_1(x) = b_q(x). \end{cases}$$

Arguing as before leads to the following result.

THEOREM 3.5. *Assume that* $\Lambda(0)$ *has two equal positive integer eigenvalues* q *and the eigenvalue functions of* $\Lambda(x)$ *are not univalent in a neighbourhood of* $x = 0$. *Let* $\rho(x)$, $\eta(x)$, $\epsilon(x)$, $\delta(x)$, $a_q(x)$ *and* $b_q(x)$ *be as in (3.17). Then (3.15) has a unique holomorphic solution* $(u(t, x), v(t, x))$ *near* $(t, x) = (0, 0)$ *satisfying* $(u(0, x), v(0, x)) = (0, 0)$ *if both*

$$\frac{\eta(x)a_q(x) + \epsilon(x)b_q(x)}{\eta(x)\rho(x) - \epsilon(x)\delta(x)} \quad \text{and} \quad \frac{\delta(x)a_q(x) + \rho(x)b_q(x)}{\eta(x)\rho(x) - \epsilon(x)\delta(x)}$$

are holomorphic, and otherwise there are no solutions.

This concludes our discussion of the case $m = 2$. As we can see from both the proofs and the statements of the above theorems, similar results will hold in arbitrary dimensions.

REMARK 3.6. In [9], Tahara obtained a similar result for the case $m = 1$. Our approach is very different to that of [9]. The main novelty is the introduction of a series of blow-up like transformations which pinpoint the obstacles to the existence of holomorphic solutions of systems of partial differential equations of Briot–Bouquet type.

Finally, we give some examples to illustrate our results. For simplicity, we focus on Theorem 3.1.

Consider (3.4) with $a(x) = b(x) = \rho(x) = \eta(x) = x$, $q = 2$, $f(t, x, u, v, u_x, v_x) = tu$ and $g(t, x, u, v, u_x, v_x) = tv$. This gives the system

$$\begin{cases} tu_t = xt + (2-x)u + tu, \\ tv_t = xt + xv + tv. \end{cases} \tag{3.18}$$

With the change of variables

$$u = t\Big(\tilde{u} + \frac{x}{x-1}\Big), \quad v = t\Big(\tilde{v} + \frac{x}{1-x}\Big),$$

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the system (3.18) becomes

$$\begin{cases} t\tilde{u}_t = \dfrac{x}{x-1}t + (1-x)\tilde{u} + t\tilde{u}, \\ t\tilde{v}_t = \dfrac{x}{1-x}t + (x-1)\tilde{v} + t\tilde{v}. \end{cases}$$

We can then conclude using Theorem 3.1(1) that (3.18) has a unique holomorphic solution $(u(t,x), v(t,x))$ near $(t,x) = (0,0)$ satisfying $(u(0,x), v(0,x)) = (0,0)$ since $a_2(x)/\rho(x) = 1/x - 1$ is holomorphic. Similarly, if we choose $a(x) = 1$ instead of $a(x) = x$ in (3.18) then the new system has no holomorphic solutions $(u(t,x), v(t,x))$ near $(t,x) = (0,0)$ satisfying $(u(0,x), v(0,x)) = (0,0)$ since $a_2(x)/\rho(x) = 1/(x(x-1))$ is not holomorphic.

Consider (3.4) with $a(x) = \rho(x) = f(t,x,u,v,u_x,v_x) = 0$, $q = 2$, $b(x) = \eta(x) = x$ and $g(t,x,u,v,u_x,v_x) = tv$. This gives the system

$$\begin{cases} tu_t = 2u, \\ tv_t = xt + xv + tv. \end{cases} \tag{3.19}$$

With the change of variables

$$u = t\tilde{u}, \quad v = t\Big(\tilde{v} + \frac{x}{1-x}\Big),$$

the system (3.19) becomes

$$\begin{cases} t\tilde{u}_t = \tilde{u}, \\ t\tilde{v}_t = \dfrac{x}{1-x}t + (x-1)\tilde{v} + t\tilde{v}. \end{cases}$$

We conclude using Theorem 3.1(2) that (3.19) has infinitely many holomorphic solutions $(u(t,x), v(t,x))$ near $(t,x) = (0,0)$ satisfying $(u(0,x), v(0,x)) = (0,0)$ since $a_2(x) \equiv 0$. In fact $u(t,x)$ can take the form of $t^2h(x)$ for any holomorphic function $h(x)$. Similarly, if we choose $a(x) \neq 0$ in (3.19), then the new system has no holomorphic solutions $(u(t,x), v(t,x))$ near $(t,x) = (0,0)$ satisfying $(u(0,x), v(0,x)) = (0,0)$ since $a_2(x) \neq 0$.

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