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A. J. G. BARCLAY, Esq., M.A., President, in the Chair.

On the form in Polar co-ordinates of certain expressions occurring in Elastic Solids and in Hydrodynamics.

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In this paper is considered the form taken in Polar co-ordinates by the equivalents of certain well known expressions in Cartesian co-ordinates which occur in Elastic Solids and in Hydrodynamics.

The accompanying figure (fig. 29) will enable us to transform from Cartesians to Polars by simple geometry.

Let the co-ordinates of the point P referred to O, to a fixed plane, and the axis ON be r, ϕ, θ. Let PQSR be an element of surface of a sphere centre O and radius r, while P′Q′S′R′ lies on a sphere of radius $r+\delta r$; the several points being taken so that the space between these elements as shown in the figure represents the ordinary polar element of volume. Thus co-ordinates of Q are r, $\theta+\delta\theta$, ϕ

.... R are r, θ, $\phi+\delta\phi$.

Let PN and RN be drawn perpendicular to the axis; and let PT, RT be the tangents to the arcs PQ, RS respectively. We are to regard as the *fundamental* directions at any point P the radius vector PP′, the tangent to the arc PQ and the perpendicular to these two, *i.e.*, the tangent to arc PR.

We wish to find the relations between the fundamental directions at P and at the adjacent point S′.

From the figure we see at once that the angle PNR $=\delta\phi$

.... POR $=\sin\theta\delta\phi$

.........PTR $=\cos\theta\delta\phi$.

Thus the following scheme shows at once the direction cosines of the one set of fundamental directions referred to the other—

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$$
\begin{array}{ccc|c|c|c|}
 & & & \multicolumn{3}{c}{\overbrace{\hphantom{\sin\theta\delta\phi \quad \cos\theta\delta\phi \quad -\sin\theta\delta\phi}}^{\text{at S}'}} \\
 & & & r & \theta & \phi \\
\hline
 & & r & 1 & -\delta\theta & -\sin\theta\delta\phi \\
\hline
\text{at P} & \left\{\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right. & \theta & \delta\theta & 1 & -\cos\theta\delta\phi \\
\hline
 & & \phi & \sin\theta\delta\phi & \cos\theta\delta\phi & 1 \\
\hline
\end{array}
$$

neglecting the squares and products of small quantities of the order $\delta\theta$ or $\delta\phi$.

Regard now the fundamental directions at P as forming a system of fixed rectangular Cartesian co-ordinates to which is referred a given elastic solid.

Let u, v, w denote the displacements at P along these axes, then the displacements at S′ *along the fundamental directions at* S′ are respectively

$$
\left.\begin{aligned}
&u+\frac{du}{dr}\,\delta r+\frac{du}{d\theta}\,\delta\theta+\frac{du}{d\phi}\,\delta\phi\\
&v+\frac{dv}{dr}\,\delta r+\frac{dv}{d\theta}\,\delta\theta+\frac{dv}{d\phi}\,\delta\phi\\
&w+\frac{dw}{dr}\,\delta r+\frac{dw}{d\theta}\,\delta\theta+\frac{dw}{d\phi}\,\delta\phi
\end{aligned}\right\}\qquad(1)
$$

Now, let U′, V′, W′ denote the displacements at S′ relative to the axes at P, then using the above scheme for the direction cosines and neglecting the squares and products of small quantities we get

$$
\mathrm{U}'=u+\frac{du}{dr}\,\delta r+\frac{du}{d\theta}\delta\theta+\frac{du}{d\phi}\,\delta\phi-v\delta\theta-w\sin\theta\delta\phi
$$

or as it may be written

$$
\mathrm{U}'-u=\frac{du}{dr}\delta r+\left(\frac{1}{r}\frac{du}{d\theta}-\frac{v}{r}\right)r\delta\theta+\left(\frac{du}{r\sin\theta d\phi}-\frac{w}{r}\right)r\sin\theta\delta\phi
$$

Similarly we find

$$
\mathrm{V}'-v=\frac{dv}{dr}\delta r+\left(\frac{1}{r}\,\frac{dv}{d\theta}+\frac{v}{r}\right)r\delta\theta+\left(\frac{1}{r\sin\theta}\,\frac{dv}{d\phi}-\frac{w}{r}\,\cot\theta\right)r\sin\theta\delta\phi
$$

$$
\mathrm{W}'-w=\frac{dw}{dr}\,\delta r+\frac{dw}{rd\theta}r\delta\theta+\left(\frac{dw}{r\sin\theta d\phi}+\frac{u}{r}+\frac{v}{r}\,\cot\theta\right)r\sin\theta\delta\phi
$$

These may be written thus

$$
\begin{aligned}
\mathrm{U}'-u=\frac{du}{dr}\delta r&+\tfrac{1}{2}\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\,\frac{du}{d\theta}\right)r\delta\theta+\tfrac{1}{2}\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\,\frac{du}{d\phi}\right)r\sin\theta\delta\phi\\
&-\tfrac{1}{2}\left(\frac{dv}{dr}+\frac{v}{r}-\frac{1}{r}\,\frac{du}{d\theta}\right)r\delta\theta+\tfrac{1}{2}\left(\frac{1}{r\sin\theta}\,\frac{du}{d\phi}-\frac{dw}{dr}-\frac{w}{r}\right)r\sin\theta\delta\phi\qquad(2)
\end{aligned}
$$

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$$V' - v = \tfrac{1}{2}\left(\frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}\right)\delta r + \left(\frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r}\right)r\delta\theta$$
$$+ \tfrac{1}{2}\left(\frac{1}{r\sin\theta}\frac{dv}{d\phi} - \frac{w}{r}\cot\theta + \frac{1}{r}\frac{dw}{d\theta}\right)r\sin\theta\delta\phi.$$
$$- \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\sin\theta\delta\phi + \tfrac{1}{2}\left(\frac{dv}{dr} + \frac{v}{r} - \frac{1}{r}\frac{du}{d\theta}\right)\delta r \quad (3)$$

$$W' - w = \tfrac{1}{2}\left(\frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right)\delta r + \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\delta\theta$$
$$+ \left(\frac{1}{r\sin\theta}\frac{dw}{d\phi} + \frac{v}{r}\cot\theta + \frac{u}{r}\right)r\sin\theta\delta\phi$$
$$- \tfrac{1}{2}\left(\frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{dw}{dr} - \frac{w}{r}\right)\delta r + \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\delta\theta \quad (4)$$

Let us now adopt, for shortness, the following notation

$$\frac{du}{dr} = e\,; \quad \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r} = f\,; \quad \frac{1}{r\sin\theta}\frac{dw}{d\phi} + \frac{v}{r}\cot\theta + \frac{u}{r} = g \qquad (5)$$

$$\left.\begin{aligned} \frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta} &= 2c \\ \frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi} &= 2b \\ \frac{1}{r\sin\theta}\frac{dv}{d\phi} + \frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta &= 2a \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} \frac{dv}{dr} + \frac{v}{r} - \frac{1}{r}\frac{du}{d\theta} &= 2\zeta \\ \frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{dw}{dr} - \frac{w}{r} &= 2\eta \\ \frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi} &= 2\xi \end{aligned}\right\} \qquad (7)$$

Then the displacements may be written

$$\left.\begin{aligned} U' - u &= e\delta r + cr\delta\theta + br\sin\theta\delta\phi - \zeta r\delta\theta + \eta r\sin\theta\delta\phi \\ V' - v &= c\delta r + fr\delta\theta + ar\sin\theta\delta\phi - \xi r\sin\theta\delta\phi + \zeta\delta r \\ W' - w &= b\delta r + ar\delta\theta + gr\sin\theta\delta\phi - \eta\delta r + \xi r\delta\theta \end{aligned}\right\} \qquad (8)$$

The last two terms in each expression are obviously the displacements due to the rotation of the body as a whole through the elementary angles ξ, η, ζ, about the three fundamental directions through P.

The first three terms in each expression indicate a relative displacement of points surrounding P such that, taking the fundamental directions at P as three Cartesian axes of x, y, z, every point on the

quadric $ex^2+fy^2+gz^2+2ayz+2bzx+2cxy=\text{constant}$.................(9)
is displaced in the direction of the normal to the quadric.

Since the rotatory displacement first considered causes no relative displacement of the particles of the solid, it introduces no elastic forces or stresses. These stresses, depending as they do on relative displacements, are thus functions only of the six coefficients e, f, g, a, b, c, of the above quadric.

This quadric has been called the Elongation Quadric, and its properties are well known.

The stresses must operate in obedience to the principles of Conservation of Energy, and so must be derivable from a potential which is a quadratic function of the relative displacements or strains, the potential representing the work done by the stresses during the corresponding strains. The potential, as representing physical properties, must for an isotropic medium be independent of all systems of axes, and so must consist of such quadratic expressions involving e, f, g, a, b, c, as are independent of the axes chosen.

From the elongation quadric we see that such independent quadratic expressions must be derivable from $e+f+g$ and $a^2+b^2+c^2-(ef+eg+fg)$ the two first invariants of the above quadric. Thus for the potential we have an expression of the form $2W=A(e+f+g)^2+B\{a^2+b^2+c^2-(ef+eg+fg)\}$ where A and B are constants.

This is at once seen to be the same as the form used by * Thomson and Tait, writing $B=4n$, $A=k+\frac{4}{3}n$, and noticing that their a, b, and c are double of mine.

Adopting their usual constants m and n, and writing $\frac{a}{2}$ for a etc. in the above, we get the most convenient form

$$2W=(m+n)(e+f+g)^2+n\{a^2+b^2+c^2-4(ef+eg+fg)\}\text{.........}(10)$$

and so W is completely known from (5) and (6) in terms of Polars, noticing now to drop the 2 in the second side of (6).

From W we can at once deduce the elastic forces at each point by regarding the fundamental directions at that point as forming three rectangular axes. Thus using Thomson and Tait's notation the stresses are P, Q, R, S, T, U,

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* (7) of § 695, Part II.

$$\left.\begin{aligned} \text{P being } &= \frac{dw}{de} \text{ etc.,} \\ \text{S} &= \frac{dw}{da} \text{ etc.,} \end{aligned}\right\} \quad (11)$$

The surface conditions are those given by * Thomson and Tait.

It will be found safer to use λ, μ, ν, however, for the direction cosines of the normal. It should be noticed that here the normal is referred to the fundamental directions at each point of the surface which vary from point to point. If, however, the surface be spherical $\mu = 0 = \nu$ and $\lambda = 1$, and the conditions are much simplified.

The quantity $\Delta \equiv e + f + g$ occurs frequently; its value is from (5)

$$\begin{aligned} \Delta &\equiv \frac{du}{dr} + \frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi} \\ &\equiv \frac{1}{r^2}\left\{\frac{d(ur^2)}{dr} + \frac{1}{\sin\theta}\frac{d(vr\sin\theta)}{d\theta} + \frac{1}{\sin^2\theta}\frac{d(wr\sin\theta)}{d\phi}\right\} \end{aligned} \quad (12)$$

it indicates the expansion of unit volume during the strain.

The values of ξ, η, ζ in (7) can be written more concisely thus

$$\left.\begin{aligned} 2\zeta &= \frac{1}{r}\left\{\frac{d(vr)}{dr} - \frac{du}{d\theta}\right\} \\ 2\eta &= \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{du}{d\phi} - \frac{d(wr)}{dr}\right\} \\ 2\xi &= \frac{1}{r\sin\theta}\left\{\frac{d(w\sin\theta}{d\theta} - \frac{dv}{d\phi}\right\} \end{aligned}\right\} \quad (13)$$

It will be found that the equations given by † Lamé for the equilibrium or motion of an Elastic Solid in Polars become, when the change in notation is allowed for, the following

$$\left.\begin{aligned} (m+n)r^2\sin\theta\frac{d\Delta}{dr} - 2n\frac{d(\zeta r\sin\theta)}{d\theta} + 2n\frac{d(\eta r)}{d\phi} &= \rho r^2\sin\theta\left(\frac{d^2u}{dt^2} - \text{R}\right) \\ (m+n)\sin\theta\frac{d\Delta}{d\theta} - 2n\frac{d\xi}{d\phi} + 2n\frac{d(\zeta r\sin\theta)}{dr} &= \rho\sin\theta\left(\frac{d^2v}{dt^2} - \Theta\right) \\ (m+n)\frac{1}{\sin\theta}\frac{d\Delta}{d\phi} - 2n\frac{d(\eta r)}{dr} + 2n\frac{d\xi}{d\theta} &= \rho\frac{1}{\sin\theta}\left(\frac{d^2w}{dt^2} - \Phi\right) \end{aligned}\right\} \quad (14)$$

where ρ is the density of the solid, and R, Θ, Φ are the components, along the fundamental directions at the point considered, of the external forces. Thus the expressions ξ, η, ζ, are of great importance.

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* § 662, (1); § 670, (10); and § 734 of Part II.

† Leçons sur l'Elasticité.

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In Hydrodynamics we are to regard u, v, w as the velocities at any point in the fluid.

The equations (8) then give us the relative velocities of the fluid at two adjacent points along the fundamental directions at the first of the points. The two last terms in each indicate a motion of rotation in adjacent points in virtue of which the element of fluid moves as a solid body. Unless these terms vanish there is vortex motion, and ξ, η, ζ are the components of the vorticity about the fundamental directions at the point considered.

If ξ, η, ζ all vanish, we get as before, the quadric (9).

Its hydrodynamical property is that at every point on its surface the fluid is moving along the normal.

This corresponds exactly to the quadric given by * Lamb noticing the difference of notation.

Thus the conditions for irrotational motion are

$$\left.\begin{aligned} \frac{d(vr)}{dr} - \frac{du}{d\theta} = 0 \\ \frac{1}{\sin\theta}\,\frac{du}{d\phi} - \frac{d(wr)}{dr} = 0 \\ \frac{d(w\sin\theta)}{d\theta} - \frac{dv}{d\phi} = 0 \end{aligned}\right\} \tag{15}$$

The quantity Δ, see (12), is termed the † *expansion* of the fluid; for an incompressible fluid it is zero.

From (13) it is easily found that

$$\frac{d(\xi r^2)}{dr} + \frac{1}{\sin\theta}\,\frac{d(\eta r\sin\theta)}{d\theta} + \frac{1}{\sin^2\theta}\,\frac{d(\zeta r\sin\theta)}{d\phi} = 0 \tag{16}$$

$\therefore$ ξ, η, ζ might be the components along the fundamental directions of the velocities of an incompressible fluid.

The conditions (15) may be verified by transferring directly to Polars the ordinary expressions for the components of vorticity, and resolving them about the fundamental directions at the point considered. They indicate that $udr + vrd\theta + wr\sin\theta d\phi$ is a complete differential.

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* See Lamb's "Motion of Fluids," Chap. III.

† *Ibid.*, p. 6.