A GENERAL HAMILTON-JACOBI EQUATION AND ASSOCIATED PROBLEM OF LAGRANGE

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It is well known [1] that the variational problem of minimizing

$$\lambda = \int F\{x^i, \dot{x}^i\}\,dt \qquad i = 1, \ldots, n, \tag{1}$$

where F is positive homogeneous of degree one in $\dot{x}^i$ (henceforth abbreviated to "plus-one" in $\dot{x}^i$) leads to a Hamiltonian $H\{x^i, p_i\}$ and corresponding Hamilton-Jacobi equation

$$H\{x^i, p_i\} = 1 \qquad \text{where} \quad p_i = \partial_i \lambda = \frac{\partial \lambda}{\partial x^i} . \tag{2}$$

Here H is also plus-one in p_i. The geodesic equations of (1) are characteristic equations for (2) and the Monge cones associated with (2) are given by the integrand in (1); the cone at (x^i_0, λ_0) being given by $\lambda - \lambda_0 = F\{x^i_0, x^i - x^i_0\}$. The purpose of this note is simply to point out that the more general equation

$$H\{x^i, p_i, \lambda\} = 1 \qquad \text{where} \quad p_i = \partial_i \lambda , \tag{3}$$

subject to the condition that H be plus-one in p_i and that

$$\det\left(\frac{\partial^2 H^2}{\partial p_j \partial p_i}\right) = \det\left(H^2{}_{p_i p_j}\right) \neq 0 , \tag{4}$$

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is always associated with a variational problem which can be put in either of the forms: minimize

$$(5)\quad \begin{cases} \int_{t_0}^{t_1} F\{x^i, \dot{x}^i, \lambda\}\, dt \\ \text{subject to the restraint} \\ \dot{\lambda} - F\{x^i, \dot{x}^i, \lambda\} = 0, \quad \lambda(t_0) = 0 \end{cases}$$

or: minimize

$$(5^*)\qquad \lambda(t_1) = \int_{t_0}^{t_1} F\{x^i, \dot{x}^i, \lambda\}\, dt$$

relative to curves $x^i(t)$ joining two given points x^i_0 and x^i_1. The form (5*) is easily obtained from (5) by integrating the restraining equation. In the form (5) the problem is clearly analogous to a problem of Lagrange.

LEMMA. A necessary condition for an extremal curve for the problem (5) is that the $x^i(t)$ satisfy

$$(6)\qquad \frac{d}{dt} F_{\dot{x}^i} - F_{x^i} = F_{\dot{x}^i} F_\lambda .$$

<u>Proof.</u> Considering $\mu = \mu(t)$ as a Lagrange multiplier, set $G = F + \mu(\dot{\lambda} - F)$. Then the Euler equations

$$\frac{d}{dt} G_{\dot{x}^i} - G_{x^i} = 0, \qquad \frac{d}{dt} G_{\dot{\lambda}} - G_\lambda = 0$$

become

$$\dot{\mu} F_{\dot{x}^i} = (1-\mu)\left(\frac{d}{dt} F_{\dot{x}^i} - F_{x^i}\right), \quad \dot{\mu} = (1-\mu) F_\lambda ,$$

and eliminating $\dot{\mu}$ will yield the lemma.

The Hamiltonian and Hamilton-Jacobi equation were originally derived directly from (5*) by a geometric construction, but the following development is considerably shorter and derives (5*) from the generalized Hamilton-Jacobi equation (3). (The author is indebted to Prof. H. Rund for pointing out this reversed process.)

The partial differential equation (3) has as characteristic equations [2]

(7.1) $$\dot{x}^i = H_{p_i}$$

(7.2) $$\dot{p}_i = -H_{x^i} - p_i H_\lambda$$

(7.3) $$\dot{\lambda} = \sum_{i=1}^{N} p_i H_{p_i} = H \text{ by homogeneity.}$$

The development consists in constructing a Lagrangian F and showing that (7.3) for H implies (5*) for F while equations (7.1) and (7.2) for H imply (6) for F.

To this end consider the equation

(8) $$\dot{x}^i = \frac{1}{2} H^2_{p_i} \quad \left(= \frac{1}{2}\frac{\partial H^2}{\partial p_i}\right)$$

which by (4) may always be solved for p_i, obtaining say

(9) $$p_i = \psi_i (x^j, \dot{x}^j, \lambda) .$$

Define

(10) $$F(x^i, \dot{x}^i, \lambda) = H\{x^i, \psi_i(x^j, \dot{x}^j, \lambda), \lambda\} .$$

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It follows that for a set $\{\dot{x}^i\}$ satisfying $F\{x^i, \dot{x}^i, \lambda\} = 1$ the corresponding p_i given by (9) satisfy $H\{x^i, p_i, \lambda\} = 1$; expanding (8) in the form $\dot{x}^i = H H_{p_i}$, these $\dot{x}^i$ and p_i also satisfy (7.1). Finally, since $H^2_{p_i}$ is plus-one in p_i, equations (8) and (9) imply that ψ_i, and hence also F, are plus-one in $\dot{x}^i$.

THEOREM. Let $H(x^i, p_i, \lambda)$ be plus-one in p_i, assume $\det(H^2_{p_i p_j}) \neq 0$, where the second derivatives are continuous, and consider the partial differential equation (3). With H is associated a Lagrangian F defined by (10) satisfying

$$(11) \qquad F_{x^i} = -H_{x^i} \qquad F_\lambda = -H_\lambda ,$$

and such that the curves satisfying the characteristic equations (7.1) are also extremals of (5*) in that they also satisfy (6).

Proof. By (8), since H^2 has continuous second derivatives, the matrix $(\partial \dot{x}^i / \partial p_j)$ is symmetric, and by (4) has a symmetric inverse matrix clearly given by $(\partial p_i / \partial \dot{x}^j)$. Hence by (8), (10) and the homogeneity of ψ_i, (using the summation convention for $j = 1, \ldots, N$)

$$(12) \qquad \frac{1}{2} F^2_{\dot{x}^i} = \frac{1}{2} H^2_{p_j} \frac{\partial \psi_j}{\partial \dot{x}^i} = \dot{x}^j \frac{\partial \psi_i}{\partial \dot{x}^j} = \psi_i = p_i .$$

It follows from the plus-one homogeneity of F and $F_{\dot{x}^j}$ that

$$H H_{p_j} \frac{\partial \psi_j}{\partial x^i} = \dot{x}^j \frac{\partial}{\partial x^i} (F . F_{\dot{x}^j}) = \dot{x}^j (F_{\dot{x}^j} F_{x^i} + F F_{x^i \dot{x}^j}) = 2 F F_x$$

from which, using a similar argument for λ, one obtains

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$$H_{p_j} \frac{\partial \psi_j}{\partial x^i} = 2 F_{x^i} \qquad H_{p_j} \frac{\partial \psi_j}{\partial \lambda} = 2 F_\lambda .$$

But differentiating (10) yields

$$F_{x^i} = H_{x^i} + H_{p_j} \frac{\partial \psi_j}{\partial x^i} = H_{x^i} + 2 F_{x^i} ,$$

and a similar argument for λ proves (11). Substituting from (11) and (12) into (7.2) yields

$$\frac{d}{dt}(F . F_{\dot{x}^i}) = F_{x^i} + (F F_{\dot{x}^i}) F_\lambda .$$

Using a parameter consistent with (7.1) so that $H = F = 1$, this reduces to (6), proving the theorem.

REFERENCES

1. H. Rund, The Differential Geometry of Finsler Spaces, Springer, 1958.

2. Fritz John, Partial Differential Equations, New York University, 1952-53, p. 36.

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