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FINITE SUBGROUPS IN INTEGRAL GROUP RINGS

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ABSTRACT. A p-subgroup version of the conjecture of Zassenhaus is proved for some finite solvable groups including solvable groups in which any Sylow p-subgroup is either abelian or generalized quaternion, solvable Frobenius groups, nilpotent-by-nilpotent groups and solvable groups whose orders are not divisible by the fourth power of any prime.

1. Introduction Let $\mathcal{U}_1\mathbb{Z}G$ denote the group of units of augmentation one of the integral group ring of a finite group G. The Zassenhaus conjecture (ZC3) says that any finite subgroup of $\mathcal{U}_1\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to a subgroup of G (see [19, Chapter 5]). Its particular case (ZC1) states that the same is true for torsion units of $\mathcal{U}_1\mathbb{Z}G$. We know that (ZC3) holds for nilpotent groups [22] and for split metacyclic groups ([15], [21]). K. W. Roggenkamp and L. Scott have shown that the Zassenhaus conjecture is false and a counterexample is a finite metabelian group [11]. However, somewhat weaker statements hold for large families of finite and infinite groups (see [19, Chapters 5 and 6] and [1], [3], [4], [7], [8], [9], [10], [12]). In the present paper we consider the following p-subgroup version of (ZC3).

$(p - ZC3)$ If H is a p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ then there exists a unit $\alpha \in \mathbb{Q}G$ such that $\alpha^{-1}H\alpha \subset G$.

In particular, if (p-ZC3) is true for a group G then any Sylow p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ is rationally conjugate to a p-subgroup of G. Conjugation of those Sylow subgroups of $\mathcal{U}_1\mathbb{Z}G$ which can be embedded into a group basis is investigated in [9], [10].

In this paper all groups G are assumed to be finite. In Section 2 we establish a reduction modulo a normal subgroup. We apply it to generalize a result of [16] and to prove (p-ZC3) for nilpotent-by-nilpotent groups. In particular, this conjecture is true for both metabelian and supersolvable groups. We also give a partial solution of Problem 32 of [19] and point out that (p-ZC3) implies a positive solution of that problem. In Sections 3 and 4 we establish (ZC3) for S_4 and a covering group of it, the Binary Octahedral Group. We apply these results in Section 5 to prove (p-ZC3) for solvable groups in which any Sylow subgroup is either abelian or generalized quaternion. As a consequence we deduce (p-ZC3) for solvable Frobenius groups. We also prove (p-ZC3) for a family of groups including those solvable groups whose orders are not divisible by the fourth power of any prime.

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2. **A reduction step and some applications.** For an element $\alpha = \sum_{g\in G}\alpha(g)g$ of $\mathbb{Z}G$ we put $\tilde{\alpha}(g) = \sum_{h\in C_g}\alpha(h)$ where C_g is the conjugacy class of $g \in G$. We use symbol $\sim$ to denote conjugation in a group.

Let N be a normal subgroup of G, $\bar{G} = G/N$, $\Psi\colon \mathbb{Z}G \longrightarrow \mathbb{Z}(G/N)$ the natural map, $\bar{g} = \Psi(g)$ for $g \in G$. This notation will be used in all what follows.

LEMMA 2.1. *Let $\alpha \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit, $\beta = \Psi(\alpha)$ and $\big(o(\alpha), |N|\big) = 1$. If the order of $g \in G$ is relatively prime to $|N|$ then $\tilde{\alpha}(g) = \tilde{\beta}(\bar{g})$.*

PROOF. Set $S_g = \{h \in G : \bar{h} \sim \bar{g} \text{ in } \bar{G}\}$ and $S'_g = \{h \in S_g : o(h) = o(g)\}$. We see that $\tilde{\beta}(\bar{g}) = \sum_{h\in S_g}\alpha(h)$. Note that if h is not in S'_g then $\big(o(h), |N|\big) \neq 1$ and consequently there is a prime p such that $p \mid o(h)$ but p does not divide $o(\alpha)$. By [19, Lemma 38.11], $\tilde{\alpha}(h) = 0$. Since the complement of S'_g in S_g is a normal subset of G, we have that $\tilde{\beta}(\bar{g}) = \sum_{h\in S'_g}\alpha(h)$. It suffices to show that the elements of S'_g are conjugate to g. Indeed, if $h \in S'_g$ then $t^{-1}ht = g\theta$ for some $t \in G$, $\theta \in N$ and the equality $o(h) = o(g)$ implies that the cyclic subgroups $\langle g\rangle$ and $\langle g\theta\rangle$ are complements for N in $N \rtimes \langle g\rangle$. Since $\big(o(g), |N|\big) = 1$, we get, by Schur-Zassenhaus Theorem, that $g\theta$ is conjugate to g. The result follows. ∎

The next result generalizes [7, Lemma 2.3].

THEOREM 2.2. *Let H be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G$ such that $(|H|, |N|) = 1$ and G_0 be a subgroup of G with $(|G_0|, |N|) = 1$. Then H is rationally conjugate to G_0 iff $\Psi(H)$ is conjugate to $\Psi(G_0)$ in $\mathbb{Q}\bar{G}$.*

PROOF. We only have to prove the converse. Let $\bar{H} = \Psi(H)$ and $\bar{G}_0 = \Psi(G_0)$. Let $\gamma^{-1}\bar{H}\gamma = \overline{G_0}$ for some $\gamma \in \mathbb{Q}\bar{G}$, $\alpha \in H$ and β be as above. We see that $h_\alpha = \gamma^{-1}\beta\gamma$ is, up to conjugacy, the unique element of $\bar{G}$ with $\tilde{\beta}(h_\alpha) \neq 0$. From [19, Lemma 38.11] it follows that $\big(o(h_\alpha), |N|\big) = 1$. From the Schur-Zassenhaus Theorem it follows that we can choose $g_\alpha \in G$ such that $h_\alpha = \Psi(g_\alpha)$ and $\big(o(g_\alpha), |N|\big) = 1$. Then it follows from [19, Lemma 38.11] and Lemma 2.1 that, up to conjugacy, g_α is the unique element of G with $\tilde{\alpha}(g_\alpha) \neq 0$. Since $(|G_0|, |N|) = 1$, the restriction of Ψ to G_0 gives an isomorphism between G_0 and $\bar{G}_0$. Denote by Ψ_1 the inverse of this isomorphism and define a homomorphism $\phi\colon H \longrightarrow G_0$ by setting $\phi(\alpha) = \Psi_1(\gamma^{-1}\beta\gamma)$. Since $\Big(o\big(\phi(\alpha)\big), |N|\Big) = 1$, Lemma 2.1 implies that $\tilde{\alpha}\big(\phi(\alpha)\big) = \tilde{\beta}\big(\Psi\phi(\alpha)\big) = \tilde{\beta}(h_\alpha) \neq 0$ and $\phi(\alpha)$ is conjugate to g_α. It follows by [19, Lemma 41.4] that H is rationally conjugate to G_0. ∎

REMARK. We have proved that if $H < \mathcal{U}_1\mathbb{Z}G$ and $(|H|, |N|) = 1$ then Ψ is injective on H.

As a consequence we have the following:

COROLLARY 2.3. *Suppose that (ZC3) holds for the factor group G/N. Then any finite subgroup $H \subset \mathcal{U}_1\mathbb{Z}G$ whose order is relatively prime to the order of N is rationally conjugate to a subgroup of G.*

We also obtain some consequences for split extensions.

COROLLARY 2.4. *Let G be an extension of a nilpotent group N by a group X which satisfies (p-ZC3). If the orders of N and X are relatively prime then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. If p does not divide the order of N then we use Theorem 2.2 and the assumption on X. If p does divide $|N|$ then G has a normal Sylow p-subgroup and hence, by [19, Theorem 41.12], we obtain that H is rationally conjugate to a subgroup of G. ∎

We give now an improvement of Lemma 37.13 of [19].

LEMMA 2.5. *Let $G = N \rtimes X$, where the orders of N and X are relatively prime, and let $\alpha = vw \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit with $v \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and $w \in \mathcal{U}_1\mathbb{Z}X$. If $\big(o(\alpha), |N|\big) = 1$ then α and w are rationally conjugate.*

PROOF. We observe that $\tilde{\alpha}(g) = \tilde{w}(g)$ for all $g \in G$. Indeed, if $\big(o(g), |N|\big) \neq 1$ then it follows from [19, Lemma 38.11] that $\tilde{\alpha}(g) = \tilde{w}(g) = 0$. If $\big(o(g), |N|\big) = 1$ then, by Schur-Zassenhaus Theorem, we may suppose that $g \in X$ and apply Lemma 2.1.

Now let d be a divisor of $o(\alpha)$. Then $\alpha^d = v_d w^d$ with $v_d \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and we use the same reasoning for the units α^d, w^d. Hence, according to [13, Theorem 2], α and w are conjugate in $\mathbb{Q}G$. ∎

The next result is a modification of Lemma 37.6 of [19].

LEMMA 2.6. *Let H_1 and H_2 be isomorphic finite subgroups of $\mathcal{U}_1\mathbb{Z}G$ with a given isomorphism $\varphi: H_1 \longrightarrow H_2$. Suppose that $\chi(h) = \chi\big(\varphi(h)\big)$ for all $h \in H_1$ and all absolutely irreducible characters χ of G. Then H_1 is conjugate to H_2 in $\mathbb{Q}G$.*

PROOF. We extend the representation $\Gamma: G \longrightarrow \mathrm{Gl}(n,\mathbb{C})$ corresponding to χ linearly to $\Gamma_1: H_1 \longrightarrow \mathrm{Gl}(n,\mathbb{C})$. By assumption the characters of Γ_1 and $\Gamma_1\varphi$ are equal and, consequently, the images of H_1 and H_2 are conjugate in any simple component of $\mathbb{C}G$. Hence H_1 is conjugate to H_2 in $\mathbb{C}G$ and Lemma 37.5 of [19] implies that the conjugation can be taken in $\mathbb{Q}G$. ∎

Now we extend Theorem 37.17 of [19].

THEOREM 2.7. *Let G be as in Lemma 2.5. Then any finite subgroup H of $\mathcal{U}_1\mathbb{Z}G$ with $(|H|, |N|) = 1$ is rationally conjugate to a subgroup of $\mathcal{U}_1\mathbb{Z}X$.*

PROOF. For $\alpha \in H$ we write $\alpha = vw$ with $v \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and $w \in \mathcal{U}_1\mathbb{Z}X$. By Lemma 2.5 the isomorphism $H \ni \alpha = vw \longrightarrow w$ satisfies the hypothesis of Lemma 2.6. Hence H is conjugate to H_0 in $\mathbb{Q}G$, where H_0 is the image of H in $\mathcal{U}_1\mathbb{Z}X$. ∎

COROLLARY 2.8. *Let N be a normal subgroup of G and H be a finite subgroup in $1 + \Delta(G,N)$. If p is a prime which divides $|H|$ then p divides $|N|$. In particular, if N is a Hall subgroup of G then $|H|$ divides $|N|$.*

PROOF. We already know that $|H|$ is a divisor of $|G|$. Suppose that there exists a prime p that divides the order of H and does not divide $|N|$. Let $\alpha \in H$ be a unit of order p. Then $\Psi(\alpha) = 1$ and, by Theorem 2.2, we have that α is rationally conjugate to 1, a contradiction. ∎

THEOREM 2.9. *Let G be a nilpotent-by-nilpotent group. Then (p-ZC3) holds for G.*

PROOF. Let H be a p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ and G_1 be a normal nilpotent subgroup of G so that G/G_1 is nilpotent. If G_1 is not a p-group, then G possesses a normal p'-subgroup N. It follows from Theorem 2.2 and induction on the order of G that H is conjugate in $\mathbb{Q}G$ to a subgroup of G. If G_1 is a p-group, then the Sylow p-subgroup of G is normal and [19, Lemma 41.12] implies that H is rationally conjugate to a subgroup of G. ∎

The proof of the following lemma can be found in [7] (see Lemma 1.5).

LEMMA 2.10. *Let G be a solvable group and P an abelian Sylow p-subgroup of G. If P is not normal in G then* $O_{p'}(G) \neq 1$.

PROPOSITION 2.11. *Let P be an abelian Sylow p-subgroup of a solvable group G. If H is a finite p-subgroup of* $\mathcal{U}_1\mathbb{Z}G$ *then H is rationally conjugate to a subgroup of G.*

PROOF. By [19, Theorem 41.12] we may assume that P is not normal in G. It follows from the preceding lemma that $N = O_{p'}(G) \neq 1$. Since the factor group G/N satisfies our hypothesis we can use Theorem 2.2 and induction to conclude that H is rationally conjugate to a subgroup of G. ∎

S. K. Sehgal has proposed the following question [19, Problem 32]:
Let $u \in \mathcal{U}\big(1 + \triangle(G,N)\big)$ *be a torsion unit with* $N \lhd G$. *Does* $o(u)$ *divide* $|N|$?
Now we point out that (p-ZC3) implies a positive solution of this question.

PROPOSITION 2.12. *Let N be a normal subgroup of a group G which satisfies (p-ZC3). If H is a finite subgroup of* $\mathcal{U}\big(1 + \triangle(G,N)\big)$ *then* $|H|$ *divides* $|N|$.

PROOF. Let H_p be a Sylow p-subgroup of H. By (p-ZC3), H_p is rationally conjugate to a subgroup H_0 of G. Going down modulo N we see that $H_0 \subset N$. Hence $|H_0|$ divides $|N|$, and consequently $|H|$ divides $|N|$. ∎

Note that Corollary 2.8 gives a partial solution of this problem.

3. (ZC3) for S_4.

The Zassenhaus conjecture for cyclic subgroups in $\mathbb{Z}S_4$ was proved in [5]. In this section we prove the following:

THEOREM 3.1. *(ZC3) holds for* S_4.

PROOF. Let $G = S_4$ and let H be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G$. It is known that G has a faithful irreducible complex representation $\Gamma\colon G \longrightarrow \mathrm{Gl}(3,\mathbb{C})$ such that the trace of $\Gamma\big((12)\big)$ is 1. We denote also by Γ the extension of this representation to $\mathbb{Z}G$. Since (ZC1) holds for G it follows that Γ is faithful on H. Therefore

$$|\Gamma(H)| = |H|. \tag{3.2}$$

Denoting by F the Fitting subgroup of G we have that $F = \langle (12)(34), (13)(24) \rangle$ and $G/F \cong S_3$. Since F is abelian, there exists an invertible matrix X such that $X^{-1}\Gamma(F)X$ has a diagonal form. It is easy to see that

$$X^{-1}\Gamma(F)X = \{I, \mathrm{diag}(-1,-1,1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1)\}. \tag{3.3}$$

Denote by Ψ the natural map $\mathbb{Z}G \longrightarrow \mathbb{Z}G/F$, $\bar{H} = \Psi(H)$ and $H_0 = H \bigcap \big(1 + \triangle(G, F)\big)$. In view of (ZC1), going down modulo F, we obtain that

(3.4) $h \in H_0$ if and only if $\gamma^{-1}h\gamma \in F$ for some unit $\gamma \in \mathbb{Q}G$.

We may also assume that H is not cyclic. According to Lemma 2.6 it suffices to find a monomorphism $\varphi: H \longrightarrow G$ such that $h \sim \varphi(h)$ in $\mathbb{Q}G$ for all $h \in H$. We consider several cases.

CASE 1. $H = \langle u, v\rangle$ is isomorphic to the Klein four group. Since $|H|$ divides $|G|$, the order of $\bar{H}$ divides 6 and $[H : H_0] = 1$ or 2.

If the index is 1 then, by (3.4), the map $\varphi: H \longrightarrow F$ defined by $\varphi(u) = (12)(34)$, $\varphi(v) = (13)(24)$ is a group isomorphism such that $h \sim \varphi(h)$ in $\mathbb{Q}G$ for all $h \in H$. Thus, H is rationally conjugate to F.

Suppose now that $[H : H_0] = 2$. Choose generators u, v such that $u \notin H_0$ and $H_0 = \langle v\rangle$. We have that $u \sim (12)$ and $v \sim (12)(34)$ in $\mathbb{Q}G$. Clearly $uv \notin H_0$ and, therefore, $uv \sim (12) \sim (34)$ in $\mathbb{Q}G$. We now define an isomorphism $\varphi: H \longrightarrow \langle (12), (12)(34)\rangle$ by putting $\varphi(u) = (12)$, $\varphi(v) = (12)(34)$. Then h is rationally conjugate to $\varphi(h)$ for all $h \in H$ and consequently H is conjugate in $\mathbb{Q}G$ to a subgroup of G.

CASE 2. The order of H is 8. Note that in this case $[H : H_0] = 2$. We show that H is not abelian. First suppose that H is elementary abelian and let u_1, u_2, u_3 be generators of H such that $H_0 = \langle u_2, u_3\rangle$. There exists a matrix Y such that $Y^{-1}\Gamma(H)Y$ consists of diagonal matrices. For $h \in H$ we put $d(h) = Y^{-1}\Gamma(h)Y$. Note that $Y^{-1}H_0Y$ consists of the diagonal matrices given in (3.3). So there is $u \in H_0$ so that $d(u) = \mathrm{diag}(1, -1, -1)$. Now since u_1 does not belong to H_0 we may suppose that $d(u_1) = \mathrm{diag}(-1, 1, 1)$. Hence $d(uu_1) = \mathrm{diag}(-1, -1, -1)$, a contradiction since uu_1 is rationally conjugate to (12).

Let $H = \langle u, v\rangle$, where $o(u^2) = o(v) = 2$. Note that u does not belong to H_0 and, consequently, H_0 is generated by u^2 and v. Let Y be such that $Y^{-1}\Gamma(H)Y$ is in diagonal form. As above, the diagonal form of H_0 consists of the matrices given in (3.3). Since $u^2 \in H_0$ we may assume that $d(u^2) = \mathrm{diag}(-1, -1, 1)$. Hence, $d(u) = \mathrm{diag}(\pm i, \pm i, \pm 1)$. Choose $w \in H_0$ so that $d(w) = \mathrm{diag}(1, -1, -1)$. The element uw has order 4 so, since (ZC1) holds for G, we see that uw is rationally conjugate to (1234). Hence, $d(u)$ and $d(uw)$ are conjugate. However, it is easy to check that the matrices $\mathrm{diag}(\pm i, \pm i, \pm 1)$ and $\mathrm{diag}(\pm i, \pm i, \pm 1)\,\mathrm{diag}(1, -1, -1)$ are not conjugate in $\mathrm{Gl}(3, \mathbb{C})$, a contradiction.

Thus H is not abelian and since H_0 has exponent 2 we see that H must be isomorphic to the dihedral group of order 8. Let $H = \langle u, v : u^4 = v^2 = 1, v^{-1}uv = u^3\rangle$. Then u is not in H_0 and we may choose v such that H_0 is generated by u^2 and uv. By (3.4), the nontrivial elements of H_0 are conjugate to (12)(34). Since (ZC1) holds for G we have that the other elements of order 2 are rationally conjugate to (13) $\sim$ (24) and those of order 4 are conjugate to (1234). Put $H_1 = \langle (1234), (13)\rangle$ and define an isomorphism of H to H_1 given by $\varphi(u) = (1234)$, $\varphi(v) = (13)$. Then it is clear that h and $\varphi(h)$ are rationally conjugate for all $h \in H$, and hence $H \sim H_1$ in $\mathbb{Q}G$.

CASE 3. The order of H is 6. Since (ZC1) holds for G we must have that H is isomorphic to S_3. Let $H = \langle u, v \rangle$ with $u^3 = v^2 = 1$. Note that H_0 has to be trivial, otherwise H would be cyclic. Hence the elements of order 2 in H are rationally conjugate to (12). Define a monomorphism $\varphi: H \longrightarrow G$ by $\varphi(u) = (123)$ and $\varphi(v) = (12)$. Then it is clear that h and $\varphi(h)$ are rationally conjugate for all $h \in H$ and hence H is conjugate in $\mathbb{Q}G$ to a subgroup of G.

CASE 4. The order of H is 12. Since $\mathcal{U}_1\mathbb{Z}G$ does not have elements of order 6 we have, by [2, pp. 134–135], that H is isomorphic to A_4. Then the elements of order 2 are pairwise conjugate in H and case 1 implies that H_0 is rationally conjugate to F. If $\varphi: H \longrightarrow A_4$ is any isomorphism then, clearly, h is rationally conjugate to $\varphi(h)$ for all $h \in H$. Hence, H is rationally conjugate to A_4.

CASE 5. H is a group basis. We shall show that H is isomorphic to S_4. First note that H is solvable. Put $H_1 = O_{2'}(H)$. Note that H_0 is normal and has order 4 in this case. So if H_1 is not trivial then H would have an element of order 6 which is, obviously, a contradiction. According to case 2, the Sylow 2-subgroups of H are dihedral of order 8. Hence, [6, p. 462] implies that H is isomorphic to S_4. Denote by φ the extension of any isomorphism $G \cong H$ to the integral group rings. It follows from [19, Theorem 43.6] that φ is an inner automorphism induced by a unit of $\mathbb{Q}G$. Consequently, H is rationally conjugate to G. ∎

4. (ZC3) for the binary octahedral group. Let G be the Binary Octahedral group. We know that the center $\mathcal{Z}(G)$ of G is cyclic of order 2, $G/\mathcal{Z}(G) \cong S_4$, the Sylow 2-subgroups of G are generalized quaternion of order 16 and that any group with these properties is isomorphic to G (see, for example, [20, 2.1.14]). Moreover, the Fitting subgroup F of G is isomorphic to the quaternion group of order8 and $G/F \cong S_3$. Let $N = \mathcal{Z}(G) = \langle z \rangle$ and let $\Psi: \mathbb{Z}G \longrightarrow \mathbb{Z}G/N$ be the natural map.

LEMMA 4.1. *We can choose a Sylow 2-subgroup* $P = \langle a, b : a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1} \rangle$ *of G and its generators so that* $\Psi(a) = (1234)$, $F = \langle a^2, ab \rangle$ *and* $a^2 \sim ab$ *in G.*

PROOF. Obviously, we can take a P with $\Psi(P) = \langle (1234), (13) \rangle$. Since F is the inverse image of $\langle (12)(34), (13)(24) \rangle$, we see that $F = \langle a^2, ab \rangle$. Let $x \in G$ be an element of order 3. Then $x^{-1}a^2x \neq a^6$. Going down modulo N to S_4 we see that $x^{-1}a^2x \in \{a^{2k+1}b\}$ and consequently $a^2 \sim ab$ in G. ∎

We also note that a is not conjugate to a^5 in G. For if $x^{-1}ax = a^5$ for some $x \in G$ then $x^{-1}Px = P$ as $\langle a^2, ab \rangle$ is the Fitting subgroup. However, $N_G(P) = P$ and consequently $x \in P$, a contradiction.

In all that follows in this section we choose P and its generators as in the lemma above. If $c \in G$ is an element of order 3, then we obtain, looking at S_4, the following representatives of the conjugacy classes of G:

order of an element	1	2	3	4	6	8
representatives	1	z	c	a^2, b	zc	a, a^5

We note that Ψ maps the two conjugacy classes of elements of order 4 of G to the two classes of elements of order 2 in S_4. We begin by proving that the Zassenhaus conjecture holds for cyclic subgroups in $\mathbb{Z}G$. We say that $\alpha \in \mathbb{Z}G$ satisfies the unique trace property if there exists a $g \in G$, unique up to conjugacy in G, such that $\tilde{\alpha}(g) \neq 0$.

PROPOSITION 4.2. *(ZC1) holds for G.*

PROOF. Let $\alpha \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit, β its image in $\mathbb{Z}S_4$ and $g \in G$. Denote by $\bar{g} = \Psi(g)$. Since (ZC1) holds for S_4 we have that

$$\tilde{\beta}(\bar{g}) \in \{0, 1\}. \tag{4.3}$$

Note first that the unit group $\mathcal{U}_1\mathbb{Z}G$ has a unique element of order 2. So we may suppose that the order of α is not 2. If $o(\alpha) = 3$ then we apply Theorem 2.2. If the order of α is 6 then we may write $\alpha = z\alpha_0$, where the order of α_0 is 3 and so we are done by the previous case. Going down modulo N we see easily that the only possibilities left for the order of α are 4 and 8.

Let α be a 2-element such that $o(\alpha) \geq 4$. We want to show that every element of $\langle\alpha\rangle$ has the unique trace property. Note that z does not belong to the support of α. If g has order 3 or 6 then [19, Lemma 38.11] implies that $\tilde{\alpha}(g) = 0$. So we may suppose that g is of order 4 or 8. Let g and g_0 be elements of G whose orders are 4 and 8 respectively. Going down modulo N it is easy to see that

$$\begin{aligned} \tilde{\beta}(\bar{g}) &= \tilde{\alpha}(g), \\ \tilde{\beta}(\bar{g}_0) &= \tilde{\alpha}(g_0) + \tilde{\alpha}(g_0^5). \end{aligned} \tag{4.4}$$

Since g_0 is not conjugate to g_0^5 in G, there exists an absolutely irreducible character χ of G so that $\chi(g_0) \neq \chi(g_0^5)$. It is easy to see that the degree of χ divides 4 and χ is not zero on an element of order 8. Moreover, χ is faithful as $\Psi(a) = \Psi(a^5)$. Let Γ be the representation associated with χ. Then $\Gamma(z) = -I$ and therefore

$$\chi({g_0}^5) = -\chi(g_0). \tag{4.5}$$

We now treat separately the remaining two cases.

Assume first that α has order 4. It follows from (4.3) and (4.4) that $\tilde{\alpha}(g_0) + \tilde{\alpha}({g_0}^5) = 0$ and that there exists a unique, up to conjugacy, element $g_1 \in G$ of order 4 such that $\tilde{\alpha}(g_1) \neq 0$. Applying χ to α and using (4.3) and (4.5) we obtain that $\chi(\alpha) = \chi(g_1) + 2\tilde{\alpha}(g_0)\chi(g_0)$. It follows from the equalities $g_1^2 = z = \alpha^2$ that the eigenvalues of $\Gamma(\alpha)$ and $\Gamma(g_1)$ are $\pm i$. Note that in G every element is conjugate to its inverse so χ is real-valued. Consequently, $\chi(\alpha) = \chi(g_1) = 0$ and so $\tilde{\alpha}(g_0) = 0$. Thus any element of $\langle\alpha\rangle$ has the unique trace property and in view of [19, Lemma 41.5] $\alpha \sim g_1$ in $\mathbb{Q}G$.

Finally assume that $o(\alpha) = 8$. By the same reasoning we obtain that $\tilde{\alpha}(g) = 0$ if $o(g) \neq 8$ and $\tilde{\alpha}(g_0) + \tilde{\alpha}(g_0^5) = 1$. Hence,

$$\chi(\alpha) = [\tilde{\alpha}(g_0) - \tilde{\alpha}(g_0^5)]\chi(g_0) = [2\tilde{\alpha}(g_0) - 1]\chi(g_0). \tag{4.6}$$

The equalities $\alpha^4 = z = g_0^4$ imply that the eigenvalues of α and g_0 are primitive roots of unity of degree 8. Since χ is real-valued and $\chi(g_0) \neq 0$ we see easily that the only possibilities for $\chi(\alpha)$ and $\chi(g_0)$ are $\pm\sqrt{2}$ and $\pm 2\sqrt{2}$. Using this fact and (4.6) we obtain that $2\tilde{\alpha}(g_0) - 1 = \pm 1$ and so $\tilde{\alpha}(g_0)$ is 0 or 1. It follows from the former case, that every element of $\langle\alpha\rangle$ has the unique trace property and so, by [19, Lemma 41.5], either $\alpha \sim g_0$ or $\alpha \sim g_0^5$ in $\mathbb{Q}G$. ∎

THEOREM 4.7. *G satisfies (ZC3).*

PROOF. As we already mentioned $\mathcal{U}_1\mathbb{Z}G$ has a unique element z of order 2, which is central, and we denoted $N = \langle z\rangle$. So if H is a finite non-cyclic subgroup of $\mathcal{U}_1\mathbb{Z}G$ then the Sylow 2-subgroups of H are either cyclic, or quaternion of order 8 or generalized quaternion of order 16. Moreover, since (ZC3) holds for S_4 and this group does not have subgroups of order 6, $\mathcal{U}_1\mathbb{Z}G$ does not contain subgroups of order 12.

Let $|H| = 8$. Suppose first that $H < \mathcal{U}\big(1 + \triangle(G,F)\big)$. Then, by (ZC1), any $1 \neq h \in H$ is conjugate in $\mathbb{Q}G$ to $a^2 \sim ab$. Therefore, if $\varphi: H \longrightarrow F$ is any isomorphism, h is rationally conjugate to $\varphi(h)$ for all $h \in H$, and Lemma 2.6 implies that H and F are conjugate in $\mathbb{Q}G$.

If H is not contained in $\mathcal{U}\big(1 + \triangle(G,F)\big)$ then it is easily seen that, going modulo N, we may choose generators h_0, h_1 of H such that $h_0 \sim b$ and $h_1 \sim a^2$ in $\mathbb{Q}G$. We now define a homomorphism $\varphi: H \longrightarrow \langle a^2, b\rangle$ by $\varphi(h_0) = b$, $\varphi(h_1) = a^2$. Since $\Psi(h_1h_0) \notin \mathcal{U}\Big(1 + \triangle\big(S_4, \mathrm{Fit}(S_4)\big)\Big)$ it follows that $h_1h_0 \sim a^2b$ in $\mathbb{Q}G$ and $h_1^3h_0 = zh_1h_0 \sim za^2b = a^6b$ in $\mathbb{Q}G$. Hence h and $\varphi(h)$ are rationally conjugate for all $h \in H$ and consequently so are H and $\langle a^2, b\rangle$.

Suppose now that the order of H is 16. We have that $H \cong P$. Choose generators u, v for H so that $\Psi(u) \sim (1234)$ and $\Psi(v) \sim (13)$ in $\mathbb{Q}S_4$. It follows, by proposition 4.2, that $v \sim b$ in $\mathbb{Q}G$ and either $u \sim a$ or $u \sim a^5$ in $\mathbb{Q}G$. In the later case we consider a^5 instead of a, so we may suppose that $u \sim a$. Define an isomorphism $\varphi: H \longrightarrow P$ by $\varphi(u) = a$, $\varphi(v) = b$. Observe that $\Psi(u^kv)$ is rationally conjugate to $(1234)^k(13)$. So if k is even then $\Psi(u^kv) \sim (24) \sim (13)$ and consequently $u^kv \sim b$ in $\mathbb{Q}G$. If k is odd, then $\Psi(u^kv) \sim (14)(23)$ in $\mathbb{Q}S_4$ and, hence $u^kv \sim a^2 \sim ab$ in $\mathbb{Q}G$. So we proved that $h \sim \varphi(h)$ for all $h \in H$ and, therefore, H and P are rationally conjugate.

Let $|H| = 24$. Since S_4 satisfies $(ZC3)$ it follows that $\Psi(H) \sim A_4$ in $\mathbb{Q}S_4$. Since A_4 has a normal Sylow 2-subgroup it follows that H also has a normal Sylow 2-subgroup H_0. Hence $H = H_0 \rtimes \langle v\rangle$ with $v^3 = 1$. Clearly H_0 is the quaternion group of order 8 and as $\Psi(H_0) \sim \Psi(F)$ in $\mathbb{Q}S_4$, going down modulo F, it is easily seen that $H_0 < \mathcal{U}\big(1+\triangle(G,F)\big)$. Consequently, H_0 is rationally conjugate to F. Let $c \in G$ be an element of order 3, $G_1 = F \rtimes \langle c\rangle$ and $\varphi: H \longrightarrow G_1$ any isomorphism. Recall that the conjugacy classes of elements of order 3 and 6 are respectively represented by zc and c. From this it easily follows that $\varphi(h) \sim h$ in $\mathbb{Q}G$ for every $h \in H$ and hence H and G_1 are rationally conjugate.

Finally let $|H| = 48$. It follows from the information above that $H/\mathcal{Z}(H) \cong S_4$ and the Sylow 2-subgroups of H are isomorphic to P. Hence, H must be the Binary Octahedral

Group. Let $\varphi: H \longrightarrow G$ be any isomorphism. Theorem 3.1 and Proposition 4.2 imply that $\varphi(h) \sim h$ in $\mathbb{Q}G$ for every $h \in H$ with $o(h) \neq 8$. Let $o(h) = 8$ and suppose that $\varphi(h)$ is not rationally conjugate to h. We have that $G = \langle P, c \rangle$, $c^3 = 1$ and $G_1 = F \rtimes \langle c \rangle$ has index 2 in G. Define a map θ by $a \longrightarrow a^5$ and $g \longrightarrow g$ for any $g \in G_1$. Since the elements of G_1 are fixed by this map it follows that it is an automorphism of G. It is easy to check now that if we replace φ by $\varphi\theta$, we get $\varphi(h) \sim h$ in $\mathbb{Q}G$ for all $h \in H$ and consequently H and G are rationally conjugate. ∎

5. (p-ZC3) for some solvable groups.

THEOREM 5.1. *Let G be a solvable group such that any Sylow subgroup of G is either abelian or generalized quaternion. Then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. In view of Proposition 2.11 we may assume that $p = 2$ and the Sylow 2-subgroups of G are generalized quaternion. If the Fitting subgroup F of G is not a 2-group, then G contains a non-trivial normal subgroup N of odd order. Since the factor group G/N satisfies the assumption of the theorem we use Theorem 2.2 and induction on $|G|$.

Let F be a 2-group. Since G is solvable, $C_G(F) = \mathcal{Z}(F)$ [18, p. 144] and, consequently, $G/\mathcal{Z}(F) = N_G(F)/C_G(F)$ is a subgroup of Aut(F). According to [17, Proposition 9.10] if F is not isomorphic to Q_8, the quaternion group of order 8, then Aut(F) is a 2-group and the result follows from [22]. Let $F \cong Q_8$. Then Aut(F) $\cong S_4$, $|\mathcal{Z}(F)| = 2$ and, hence, $|G|$ divides 48. By [22] we may suppose that G is not nilpotent. If $|G| = 24$ then G has a normal Sylow 2-subgroup and we can use Theorem 2.9. If $|G| = 48$ then G is the Binary Octahedral group. In this case we apply Theorem 4.7. ∎

COROLLARY 5.2. *A finite solvable Frobenius group satisfies (p-ZC3).*

PROOF. By [18, 10.5.6] $G = N \rtimes X$ where N is nilpotent, $(|N|, |X|) = 1$ and the Sylow p-subgroups of X are either abelian or generalized quaternion. Hence, the result follows from Corollary 2.4 and Theorem 5.1. ∎

THEOREM 5.3. *Let G be a finite solvable group and $L = L(G)$ the last non-trivial term of the lower central series of G. If p^4 does not divide $|G|$ for any prime p dividing $|L|$, then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. If p does not divide $|L|$ then, since G/L is nilpotent, we apply Theorem 2.2 and the theorem of Weiss [22].

Let p divide $|L|$ and let F be the Fitting subgroup of G. If F is not a p-group, then $N = O_{p'}(F)$ is a non-identity normal subgroup of G. It is easy to see that the factor group G/N satisfies the hypothesis of the theorem, so we may use Theorem 2.2 and induction on the order of G.

Let F be a p-group and P a Sylow p-subgroup of G. In view of Proposition 2.11 and [19, Theorem 41.12] we may assume that P is not abelian and not normal in G. In fact $|P| = p^3$ because p^4 does not divide $|G|$. Now the same arguments as in [7, pp. 4908–4909] shows that $p = 2$ and $G \cong S_4$. Thus, the result follows from Theorem 3.1. ∎

REMARK. The proof of the theorem shows that if $H \subset \mathcal{U}_1\mathbb{Z}G$ is a finite subgroup whose order is relatively prime to that of L then H is rationally conjugate to a subgroup of G.

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REFERENCES

1. A. Bovdi, Z. Marciniak and S. K. Sehgal, *Torsion Units in Infinite Group Rings*, J. Number Theory **47**(1994), 284–299.
2. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, Berlin, 1980.
3. M. A. Dokuchaev, *Torsion units in integral group ring of nilpotent metabelian groups*, Comm. Algebra (2) **20**(1992), 423–435.
4. M. A. Dokuchaev and S. K. Sehgal, *Torsion Units in Integral Group Rings of Solvable Groups*, Comm. Algebra (12) **22**(1994), 5005–5020.
5. N. A. Fernandes, *Torsion Units in the Integral Group Ring of* S_4, Bol. Soc. Brasil. Mat. (1) **18**(1987), 1–10.
6. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968.
7. S. O. Juriaans, *Torsion units in integral group rings*, Comm. Algebra (12) **22**(1994), 4905–4913.
8. ———, *Torsion Units in Integral Group Rings II*, Canad. Math. Bull. to appear.
9. W. Kimmerle and K.W. Roggenkamp, *Projective Limits of Group Rings*, J. Pure Appl. Algebra **88**(1993), 119–142.
10. W. Kimmerle, K. W. Roggenkamp and A. Zimmerman, DMV-*Seminar Part 1 Group Rings: Units and the Isomorphism Problem.*
11. L. Klinger, *Construction of a counterexample to a conjecture of Zassenhaus*, Comm. Algebra **19**(1993), 2303–2330.
12. A. I. Lichtman and S. K. Sehgal, *The elements of finite order in the group of units of group rings of free products of groups*, Comm. Algebra **17**(1989), 2223–2253.
13. I. S. Luthar and I. B. S. Passi, *Zassenhaus conjecture for* A_5, Proc. Indian Acad. Sci. (1) **99**(1989), 1–5.
14. Z. Marciniak, J. Ritter, S. K. Sehgal and A. Weiss, *Torsion units in integral group rings of some metabelian groups, II*, J. Number Theory **25**(1987), 340–352.
15. C. Polcino Milies, J. Ritter and S. K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings II*, Proc. Amer. Math. Soc. (2) **97**(1986), 206–210.
16. C. Polcino Milies and S. K. Sehgal, *Torsion Units in integral group rings of metacyclic groups*, J. Number Theory **19**(1984), 103–114.
17. D. S. Passman, *Permutation groups,* W. A. Benjamin, Inc., New York, 1968.
18. D. J. J. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
19. S. K. Sehgal, *Units of Integral Group Rings*, Longman's, Essex, 1993.
20. M. Shirvani and B. A. F. Wehrfritz, *Skew Linear Groups*, Cambridge, Univ. Press, 1986.
21. A. Valenti, *Torsion Units in Integral Group Rings*, Proc. Amer. Math. Soc. (1) **120**(1994), 1–4.
22. A. Weiss, *Torsion units in integral group rings*, J. Reine Angew. Math. **415**(1991), 175–187.

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