



# Integrating $L_\infty$ -algebras

André Henriques

## ABSTRACT

Given a Lie  $n$ -algebra, we provide an explicit construction of its integrating Lie  $n$ -group. This extends work done by Getzler in the case of nilpotent  $L_\infty$ -algebras. When applied to an ordinary Lie algebra, our construction yields the simplicial classifying space of the corresponding simply connected Lie group. In the case of the string Lie 2-algebra of Baez and Crans, we obtain the simplicial nerve of their model of the string group.

## 1. Introduction

### 1.1 Homotopy Lie algebras

Stasheff [Sta92] introduced  $L_\infty$ -algebras, or strongly homotopy Lie algebras, as a model for ‘Lie algebras that satisfy Jacobi up to all higher homotopies’ (see also [HS93]).

A (non-negatively graded)  $L_\infty$ -algebra is a graded vector space  $L = L_0 \oplus L_1 \oplus L_2 \oplus \dots$  equipped with brackets

$$[\ ] : L \rightarrow L, \quad [ , ] : \Lambda^2 L \rightarrow L, \quad [ , , ] : \Lambda^3 L \rightarrow L, \dots \quad (1)$$

of degrees  $-1, 0, 1, 2, \dots$ , where the exterior powers are interpreted in the graded sense. Thereafter, all  $L_\infty$ -algebras will be assumed finite-dimensional in each degree. The axioms satisfied by the brackets can be summarized as follows.

Let  $L^\vee$  be the graded vector space with  $L_{n-1}^* = \text{Hom}(L_{n-1}, \mathbb{R})$  in degree  $n$ , and let  $C^*(L) := \text{Sym}(L^\vee)$  be its symmetric algebra, again interpreted in the graded sense

$$\begin{aligned} C^*(L) = & \mathbb{R} \oplus [L_0^*] \oplus [\Lambda^2 L_0^* \oplus L_1^*] \oplus [\Lambda^3 L_0^* \oplus (L_0^* \otimes L_1^*) \oplus L_2^*] \\ & \oplus [\Lambda^4 L_0^* \oplus (\Lambda^2 L_0^* \otimes L_1^*) \oplus \text{Sym}^2 L_1^* \oplus \dots] \oplus \dots \end{aligned} \quad (2)$$

The transpose of the brackets (1) can be assembled into a degree one map  $L^\vee \rightarrow C^*(L)$ , which extends uniquely to a derivation  $\delta : C^*(L) \rightarrow C^*(L)$ . The equation  $\delta^2 = 0$  then contains all of the axioms that the brackets satisfy.

If the  $k$ -ary brackets are zero for all  $k > 2$ , we recover the usual notion of differential graded Lie algebra. If  $L$  is concentrated in degrees less than  $n$ , we obtain the notion of Lie  $n$ -algebra, also called  $n$ -term  $L_\infty$ -algebra. The  $k$ -ary brackets are then zero for all  $k > n + 1$ .

The case of Lie 2-algebras has been studied by Baez and Crans [BC04]. A Lie 2-algebra consists of two vector spaces  $L_0$  and  $L_1$ , and three brackets  $[ , ]$ ,  $[ , , ]$ ,  $[ , , , ]$  acting on  $L = L_0 \oplus L_1$ , of degrees  $-1, 0, 1$ , respectively. A complete list of the axioms is given in [BC04, Lemma 4.3.3].

Our main example, introduced by Baez and Crans [BC04], is the string Lie 2-algebra  $\mathfrak{str}(\mathfrak{g}) := \mathfrak{g} \oplus \mathbb{R}$  associated to a simple Lie algebra  $\mathfrak{g}$  of compact type. Its only non-zero brackets are  $[X, Y] \in \mathfrak{g}$  for  $X, Y \in \mathfrak{g}$  and  $[X, Y, Z] := \langle [X, Y], Z \rangle \in \mathbb{R}$  for  $X, Y, Z \in \mathfrak{g}$ .

---

Received 26 March 2006, accepted in final form 24 November 2007.

2000 Mathematics Subject Classification 53C05; 55U10; 58B10.

Keywords:  $L_\infty$ -algebra, simplicial manifold, String group, Lie 2-group, Lie 2-algebra, Kan condition.

This journal is © Foundation Compositio Mathematica 2008.

### 1.2 The string group and its Lie algebra

The string group  $String(n)$  is classically defined as the 3-connected cover of  $Spin(n)$ . Extending the above definition, we call  $String = String_G$  the 3-connected cover of any compact simple simply connected Lie group  $G$ .

Stolz and Teichner [Sto96], [ST04] present a few different models for  $String$ , one of which, inspired by work of Anthony Wassermann [Was98], is an extension of  $G$  by the group of projective unitary operators in a type III von Neumann algebra. Another model, due to Brylinski and McLaughlin [BM94, BM96], consists of a  $U(1)$ -gerbe with connection over  $G$ . More recently, Baez *et al.* [BCSS07] constructed a strict Lie 2-group modeling  $String$ , based on the universal central extension of the loop group of  $G$ .

Strict 2-groups, that is group objects in the category of groupoids, are equivalent to crossed modules [BL04, For02]. A crossed module consists of two groups  $H_0, H_1$ , a homomorphism  $\partial : H_1 \rightarrow H_0$ , and an action  $H_0 \curvearrowright H_1$  satisfying

$$\partial(a \cdot b) = a\partial(b)a^{-1} \quad \text{and} \quad \partial(b) \cdot b' = bb'b^{-1}, \quad a \in H_0, b, b' \in H_1.$$

In the language of crossed modules, the Lie 2-group constructed in [BCSS07] is given by

$$String := (\widetilde{\Omega G} \xrightarrow{\partial} Path_*(G)), \tag{3}$$

where  $Path_*(G)$  denotes the based path space of  $G$ , and  $\widetilde{\Omega G}$  the universal central extension of  $\Omega G$ . The homomorphism  $\partial$  is obtained by composing the projection  $\widetilde{\Omega G} \rightarrow \Omega G$  with the inclusion  $\Omega G \hookrightarrow Path_*(G)$ , while the action  $Path_*(G) \curvearrowright \widetilde{\Omega G}$  is given by lifting the pointwise conjugation action  $Path_*(G) \curvearrowright \Omega G$ .

The authors of [BCSS07] did not provide a recipe for integrating Lie 2-algebras: they simply constructed  $String$ , and observed that the corresponding (infinite-dimensional) strict Lie 2-algebra is equivalent to  $\mathfrak{stt}$ . The present paper fills this gap by providing the desired integrating procedure.

### 1.3 Homotopy Lie groups

Let  $\Delta$  be the category whose objects are the sets  $[m] := \{0, \dots, m\}$ ,  $m \geq 0$ , and whose morphisms are the non-decreasing maps. It is generated by  $d^i : [m-1] \rightarrow [m]$  and  $s^i : [m+1] \rightarrow [m]$  given by

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \quad s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

DEFINITION 1.1. A simplicial manifold  $X$  is a contravariant functor from  $\Delta$  to the category of smooth manifolds. We write  $X_m$  for  $X([m])$ , and call  $d_i := X(d^i) : X_m \rightarrow X_{m-1}$  and  $s_i := X(s^i) : X_m \rightarrow X_{m+1}$  the face and degeneracy maps, respectively. We represent a simplicial manifold  $X$  by a diagram

$$X_0 \leftarrow X_1 \rightrightarrows X_2 \rightleftarrows X_3 \cdots$$

in which the arrows stand for the face maps. A simplicial manifold is called reduced if  $X_0 = pt$ .

Kan simplicial sets [May92, § 1.3], an important special class of simplicial sets, are the simplicial sets  $X$  such that any map  $\Lambda[m, j] \rightarrow X$  extends to a map  $\Delta[m] \rightarrow X$ . Here  $\Delta[m]$  denotes the simplicial  $m$ -simplex, and its  $j$ th horn  $\Lambda[m, j] \subset \Delta[m]$  is the union of all of its facets containing the  $j$ th vertex. Similarly to the case of simplicial sets, we restrict our class of simplicial manifolds by imposing a version of the Kan condition.

DEFINITION 1.2. A simplicial manifold  $X$  satisfies the Kan condition if the restriction maps

$$X_m = \text{Hom}(\Delta[m], X) \rightarrow \text{Hom}(\Lambda[m, j], X) \tag{4}$$

are surjective submersions for all  $m, j$ .

*Remark 1.3.* The above generalization of the Kan condition is different from that of Seymour [Sey80], in that he requires the map (4) to have a *global* section, whereas the condition we impose on it only implies the existence of local sections.

Note the similarity between this condition and Verdier’s definition of a hypercover [AGV72b, Exposé V, §7.3] in the smooth topology. The difference is that here we have the horn  $\Lambda[m, j]$ , whereas in the definition of a hypercover, one has the boundary  $\partial\Delta[m]$  of the  $m$ -simplex.

It is well known that the category of groups can be embedded in the category of pointed spaces via the functor  $G \mapsto K(G, 1)$ . If  $G$  is a Lie group, then  $K(G, 1)$  can be interpreted as a simplicial manifold

$$K(G, 1) := (* \leftarrow G \rightrightarrows G^2 \rightleftarrows G^3 \dots), \tag{5}$$

where the faces and degeneracies are given by the well-known formulas

$$\begin{aligned} d_0(g_1, \dots, g_m) &= (g_2, \dots, g_m), & d_i(g_1, \dots, g_m) &= (g_1, \dots, g_i g_{i+1}, \dots, g_m), \\ d_m(g_1, \dots, g_m) &= (g_1, \dots, g_{m-1}), & s_i(g_1, \dots, g_m) &= (g_1, \dots, g_i, e, g_{i+1}, \dots, g_m). \end{aligned}$$

One can then characterize the reduced Kan simplicial manifolds (5) coming from Lie groups as those such that the maps (4) are diffeomorphisms for all  $m > 1$ . This justifies the following definition, which is a manifold analog of Duskin’s notion of  $n$ -group(oid) [Dus79] (see also Glenn [Gle82]).

**DEFINITION 1.4.** A reduced simplicial manifold  $X$  is called a Lie  $n$ -group if in addition to the Kan condition, the map (4) is a diffeomorphism for all  $m > n$ .

Note that our Lie 2-groups are different from those of Baez and Lauda [BL04]. There is a functor from their category to ours, which is fully faithful, but not essentially surjective. To compare our model of the string group with that of [BCSS07], consider the simplicial nerve of *String*:

$$NString = [* \leftarrow Path_*(G) \rightrightarrows Map_*(\widetilde{\partial\Delta^2}, G) \rightleftarrows Map_*(\widetilde{sk_1\Delta^3}, G) \dots].$$

That simplicial manifold agrees with our Lie 2-group (57) integrating  $\mathfrak{str}$ . (Here, the tilde over  $Map_*(sk_1\Delta^m, G)$  indicates the total space of an  $(S^1)^{\binom{m}{2}}$ -principal bundle.)

### 1.4 Integrating $L_\infty$ -algebras

The goal of this paper is to give a procedure that takes an  $L_\infty$ -algebra (a homotopy Lie algebra) and produces a reduced Kan simplicial manifold (a homotopy Lie group). It assigns to  $L$  the simplicial manifold  $\int L$  given by

$$\left(\int L\right)_m := \text{Hom}_{DGA}(C^*(L), \Omega^*(\Delta^m)), \tag{6}$$

where  $C^*(L)$  denotes the generalized Chevalley–Eilenberg cochain complex (2), and  $\Omega^*(\Delta^m)$  the differential graded algebra of de Rham differential forms on the  $m$ -simplex. This is essentially the spatial realization construction of rational homotopy theory (see [BG76, Sul77]) applied to  $C^*(L)$ , used by Hinich in the case of differential graded Lie algebras [Hin97], and by Getzler in the case of  $L_\infty$ -algebras [Get03]. If the  $L_\infty$ -algebra is nilpotent, Getzler then uses a gauge condition to cut down (6) to an equivalent sub-simplicial space, which is finite dimensional in each degree.

In the absence of the nilpotence condition, we prefer not to modify (6) and instead address the analysis inherent to the infinite dimensionality of these spaces. More precisely, we concentrate on their manifold structure. Provided we interpret  $\Omega^*(\Delta^m)$  to mean

$$\Omega^*(\Delta^m) := \{\alpha \mid \text{both } \alpha \text{ and } d\alpha \text{ are of class } C^r\},$$

we prove that the spaces (6) are Banach manifolds, and that  $\int L$  satisfies the Kan condition (Theorem 5.10). We then set up a long exact sequence for computing the simplicial homotopy groups

$\pi_n^{\text{spl}}(\int L)$  (Theorem 6.4), and show that the Lie algebra of  $\pi_n^{\text{spl}}(\int L)$  is canonically isomorphic to the homology group  $H_{n-1}(L; \partial)$  of  $L$ , with respect to the 1-ary bracket  $\partial = []$ .

If  $L$  is a Lie  $n$ -algebra, we also provide a modification  $\tau_{\leq n} \int L$  of  $\int L$  which satisfies the condition in Definition 1.4. It is not always a simplicial manifold, but we give a simple criterion which is equivalent to the spaces  $(\tau_{\leq n} \int L)_m$  being manifolds (Theorem 7.5).

## 2. Kan simplicial objects

In this section, we fix a category  $\mathcal{C}$  equipped with a *Grothendieck pretopology* and consider simplicial objects in that category. We introduce the concepts of Kan simplicial objects and of Kan fibrations for maps between simplicial objects. Finally, we introduce the simplicial homotopy groups of a Kan simplicial object, and develop the theory of Postnikov towers. Our formalism is closely related to Jardine’s work on simplicial sheaves [Jar86].

We assume that  $\mathcal{C}$  has all coproducts, which simplifies the axioms for a Grothendieck pretopology.

DEFINITION 2.1. A Grothendieck pretopology on  $\mathcal{C}$  is a collection of morphisms, called covers, subject the following three axioms.

- (i) Isomorphisms are covers.
- (ii) The composition of two covers is a cover.
- (iii) If  $U \rightarrow X$  is a cover and  $Y \rightarrow X$  is a morphism, then the pullback  $Y \times_X U$  exists, and the natural morphism  $Y \times_X U \rightarrow Y$  is a cover.

According to Grothendieck’s original definition [AGV72a, Exposé II], a cover should rather be a collection of morphisms  $\{U_i \rightarrow X\}$ . Given a pretopology in our sense, one recovers a pretopology in Grothendieck’s sense by declaring  $\{U_i \rightarrow X\}$  to be a cover if  $\coprod_i U_i \rightarrow X$  is a cover. See [Joh77, §0.3], [MM94, ch. III] for other discussions of Grothendieck pretopologies.

We make the following assumptions on our Grothendieck pretopology.

ASSUMPTIONS 2.2. *The category  $\mathcal{C}$  has a terminal object  $*$ , and for any object  $X \in \mathcal{C}$ , the map  $X \rightarrow *$  is a cover.*

*The category  $\mathcal{C}$  is Cauchy complete, which means that for any idempotent morphism  $e : X \rightarrow X$ ,  $e^2 = e$ , there exists a subobject  $i : A \rightarrow X$  and a retraction  $r : X \rightarrow A$ ,  $ri = 1_A$ , such that  $ir = e$ .*

*The retract of a cover is a cover. Namely, if we have a commutative diagram*

$$\begin{array}{ccc}
 A & \overset{\curvearrowright}{\longrightarrow} & X \\
 \downarrow f & & \downarrow g \\
 B & \overset{\curvearrowright}{\longrightarrow} & Y
 \end{array} \tag{7}$$

*where the composites  $A \rightarrow X \rightarrow A$  and  $B \rightarrow Y \rightarrow B$  are identities, then  $g$  being a cover implies that  $f$  is also a cover.*

*Our pretopology is subcanonical, which means that all the representable functors  $T \mapsto \text{hom}(T, X)$  are sheaves.*

We sometimes want to talk about the limit of a diagram in  $\mathcal{C}$ , before knowing that it actually exists. For this purpose, we use the Yoneda functor

$$\begin{aligned}
 \mathbf{y} : \mathcal{C} &\rightarrow \{\text{Sheaves on } \mathcal{C}\} \\
 X &\mapsto (T \mapsto \text{hom}(T, X))
 \end{aligned}$$

to embed  $\mathcal{C}$  in the category of sheaves on  $\mathcal{C}$ . Using  $\mathbf{y}$ , a limit of objects of  $\mathcal{C}$  can always be interpreted as the limit of the corresponding representable sheaves. The limit sheaf is then itself representable if and only if the original diagram had a limit in  $\mathcal{C}$ .

Given a set  $S$ , and an object  $X \in \mathcal{C}$ , we write  $\text{Hom}(S, X)$  for the product  $\prod_S X$ , that is the object  $X \times \cdots \times X$ , where the copies of  $X$  are indexed by  $S$ . If  $S$  is now a simplicial set, and  $X$  a simplicial object, we denote by  $\text{Hom}(S, X)$  the obvious equalizer

$$\text{Hom}(S, X) \rightarrow \prod_{m \geq 0} \text{Hom}(S_m, X_m) \rightrightarrows \prod_{[m] \rightarrow [n]} \text{Hom}(S_n, X_n), \tag{8}$$

which, as explained above, is *a priori* only a sheaf. The sheaf  $\text{Hom}(S, X)$  can also be described by giving its values on the various objects of  $\mathcal{C}$ . Namely, for  $T \in \mathcal{C}$  we have

$$(\text{Hom}(S, X))(T) := \text{hom}(T \times S, X), \tag{9}$$

where  $T \times S$  denotes the simplicial object with  $\coprod_{S_m} T$  in degree  $m$ .

Recall the Kan condition from Definition 1.2. The following definition generalizes it to other categories of simplicial objects.

DEFINITION 2.3. Let  $\mathcal{C}$  be a category equipped with a Grothendieck pretopology. A map  $f : X \rightarrow Y$  between reduced simplicial objects of  $\mathcal{C}$  is a Kan fibration if the natural map from  $X_m = \text{Hom}(\Delta[m], X)$  to the object  $\text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \rightarrow Y)$  of commutative squares

$$\begin{array}{ccc} \Lambda[m, j] & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta[m] & \longrightarrow & Y \end{array} \tag{10}$$

is a cover. A simplicial object  $X$  is called Kan if the map  $X \rightarrow *$  is a Kan fibration.

The object of squares (10) is a certain equalizer, similar to (8). It is best described in the language of sheaves by saying that its value at  $T$  is the set of commutative squares

$$\begin{array}{ccc} T \times \Lambda[m, j] & \longrightarrow & X \\ \downarrow & & \downarrow \\ T \times \Delta[m] & \longrightarrow & Y \end{array} \tag{11}$$

with given vertical arrows. The fact that the above sheaf is representable is not obvious from the definition. It relies on the following lemma.

Recall that a simplicial set  $S$  is *collapsible* if it admits a filtration

$$* = S_0 \subset S_1 \subset \cdots \subset S_k = S \tag{12}$$

such that each  $S_i$  is obtained from the previous one by filling a horn, namely such that  $S_i$  can be written as  $S_i = S_{i-1} \times_{\Lambda[n_i, j_i]} \Delta[n_i]$  for some map  $\Lambda[n_i, j_i] \rightarrow S_{i-1}$ .

LEMMA 2.4. Let  $S \subset \Delta[n]$  be a collapsible simplicial set. Let  $f : X \rightarrow Y$  be a map between reduced simplicial objects that satisfies the conditions of Definition 2.3 for all  $m < n$  (this implicitly means that  $\text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \rightarrow Y)$  is representable). Then the object of commutative squares  $\text{Hom}(S \rightarrow \Delta[n], X \rightarrow Y)$  exists in  $\mathcal{C}$ .

*Proof.* Filter  $S$  as in (12). We show by induction on  $i$  that  $\text{Hom}(S_i \rightarrow \Delta[n], X \rightarrow Y)$  exists in  $\mathcal{C}$ . We begin the induction by noting that  $\text{Hom}(* \rightarrow \Delta[n], X \rightarrow Y) = Y_n$  is an object of  $\mathcal{C}$ .

We now assume that  $\text{Hom}(S_{i-1} \rightarrow \Delta[n], X \rightarrow Y)$  exists in  $\mathcal{C}$ . The bottom row of the pullback diagram

$$\begin{array}{ccc} \text{Hom}(S_i \rightarrow \Delta[n], X \rightarrow Y) & \longrightarrow & \text{Hom}(S_{i-1} \rightarrow \Delta[n], X \rightarrow Y) \\ \downarrow & \lrcorner & \downarrow \\ \text{Hom}(\Delta[n_i], X) & \longrightarrow & \text{Hom}(\Lambda[n_i, j_i] \rightarrow \Delta[n_i], X \rightarrow Y) \end{array}$$

is a cover by hypothesis. It follows that  $\text{Hom}(S_i \rightarrow \Delta[n], X \rightarrow Y)$  exists in  $\mathcal{C}$ . □

Since the horns  $\Lambda[n, j]$  are collapsible, we have the following result.

**COROLLARY 2.5.** *Let  $f : X \rightarrow Y$  be a map satisfying the conditions of Definition 2.3 for all  $m < n$ . Then the object  $\text{Hom}(\Lambda[n, j] \rightarrow \Delta[n], X \rightarrow Y)$  of commutative squares (10) exists in  $\mathcal{C}$ .*

*Remark 2.6.* If we worked with simplicial objects that are not reduced, then Lemma 2.4 would require the additional assumption that  $X_0 \rightarrow Y_0$  be a cover.

We now collect a few easy lemmas to be used in the future.

**LEMMA 2.7.** *The category  $\mathcal{C}$  has all finite products; the product of two covers is a cover.*

*Proof.* Since  $X \rightarrow *$  is a cover, the pullback  $X \times_* Y$  exists. The product of two covers  $X \rightarrow X', Y \rightarrow Y'$  can be factorized as  $X \times Y \rightarrow X \times Y' \rightarrow X' \times Y'$ . The first morphism is a cover because it is the pullback of  $Y \rightarrow Y'$  along the projection  $X \times Y' \rightarrow Y'$ , and similarly for the second morphism. The result follows since covers are closed under composition. □

**LEMMA 2.8.** *The product of two Kan fibrations of simplicial objects is again a Kan fibration.*

*Proof.* This follows from the equality  $\text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \times X' \rightarrow Y \times Y') = \text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \rightarrow Y) \times \text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X' \rightarrow Y')$ , and the fact that the product of two covers is a cover. □

**LEMMA 2.9.** *The composite  $X \rightarrow Y \rightarrow Z$  of two Kan fibrations of simplicial objects is again a Kan fibration.*

*Proof.* In the diagram below, the top left and the bottom horizontal arrows are covers. The square being a pullback, it follows that the top composite is also a cover.

$$\begin{array}{ccc} X_m \longrightarrow \text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \rightarrow Y) & \longrightarrow & \text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], X \rightarrow Z) \\ \downarrow & \lrcorner & \downarrow \\ Y_m \longrightarrow \text{Hom}(\Lambda[m, j] \rightarrow \Delta[m], Y \rightarrow Z) & & \end{array} \quad \square$$

**LEMMA 2.10.** *Let  $A \rightarrow B$  be an acyclic cofibration between finite simplicial sets, and let  $X \rightarrow Y$  be a Kan fibration of simplicial objects. Assume that the object of commutative squares  $\text{Hom}(A \rightarrow B, X \rightarrow Y)$  exists in  $\mathcal{C}$ . Then  $\text{Hom}(B, X)$  exists in  $\mathcal{C}$ , and the map*

$$\text{Hom}(B, X) \rightarrow \text{Hom}(A \rightarrow B, X \rightarrow Y) \tag{13}$$

is a cover.

*Proof.* By [GZ67, § IV.2] (see also [Hov99, Corollary 2.1.15]), there exists a simplicial set  $S$  containing  $B$  such that  $B$  is a retract of  $S$  via maps  $j : B \rightarrow S$  and  $r : S \rightarrow B$  fixing  $A$ , and such that  $S$  admits a filtration  $A = S_0 \subset S_1 \subset \dots \subset S_k = S$  with  $S_i = S_{i-1} \times_{\Lambda[n_i, j_i]} \Delta[n_i]$ . As in Lemma 2.4, we

have pullback diagrams

$$\begin{array}{ccc} \text{Hom}(S_i \rightarrow B, X \rightarrow Y) & \longrightarrow & \text{Hom}(S_{i-1} \rightarrow B, X \rightarrow Y) \\ \downarrow & \lrcorner & \downarrow \\ \text{Hom}(\Delta[n_i], X) & \longrightarrow & \text{Hom}(\Lambda[n_i, j_i] \rightarrow \Delta[n_i], X \rightarrow Y) \end{array}$$

where the maps  $S_i \rightarrow B$  are given by  $r|_{S_i}$ . The bottom arrow is a cover, and therefore so is the top arrow. Composing all of the above maps, we conclude that  $\text{Hom}(S \rightarrow B, X \rightarrow Y)$  exists in  $\mathcal{C}$  and that

$$\text{Hom}(S \rightarrow B, X \rightarrow Y) \rightarrow \text{Hom}(A \rightarrow B, X \rightarrow Y) \tag{14}$$

is a cover. The sheaf  $\text{Hom}(B, X)$  is a retract of  $\text{Hom}(S \rightarrow B, X \rightarrow Y)$  via maps

$$(B \xrightarrow{f} X) \mapsto \left( \begin{array}{ccc} S & \xrightarrow{f \circ r} & X \\ r \downarrow & & \pi \downarrow \\ B & \xrightarrow{\pi \circ f} & Y \end{array} \right) \quad \text{and} \quad \left( \begin{array}{ccc} S & \xrightarrow{\phi} & X_i \\ r \downarrow & & p \downarrow \\ B & \xrightarrow{\psi} & Y \end{array} \right) \mapsto (B \xrightarrow{\phi \circ j} X),$$

where for better readability, we have omitted all of the ‘ $T \times \dots$ ’ that should be there as in (9) and (11). It follows that  $\text{Hom}(B, X)$  is representable in  $\mathcal{C}$ . It is not hard to check that the above maps are compatible with the projections to  $\text{Hom}(A \rightarrow B, X \rightarrow Y)$ . The morphism (13) is therefore a retract of (14), and thus also a cover.  $\square$

**COROLLARY 2.11.** *Let  $A \rightarrow B$  be an acyclic cofibration between finite simplicial sets, and let  $X$  be a Kan simplicial object. Assume that  $\text{Hom}(A, X)$  exists in  $\mathcal{C}$ . Then  $\text{Hom}(B, X)$  exists in  $\mathcal{C}$ , and the map*

$$\text{Hom}(B, X) \rightarrow \text{Hom}(A, X)$$

*is a cover. In particular, if  $B$  be a finite contractible simplicial set and  $X$  a Kan simplicial object, then  $\text{Hom}(B, X)$  exists in  $\mathcal{C}$ .*

Without the assumption that the sheaf  $\text{Hom}(A, X)$  is representable in  $\mathcal{C}$ , we still have the following analog of Corollary 2.11.

**LEMMA 2.12.** *Let  $A \rightarrow B$  be an acyclic cofibration between finite simplicial sets, and let  $X$  be a Kan simplicial object. Then, the map*

$$\text{Hom}(B, X) \rightarrow \text{Hom}(A, X)$$

*is a surjective map of sheaves.*

*Proof.* In the statement of Corollary 2.11, replace  $\mathcal{C}$  by the category of sheaves on  $\mathcal{C}$ , and the notion of cover by the notion of surjective map of sheaves.  $\square$

### 3. Simplicial homotopy groups

Recall [May92, §I.4] that given a reduced Kan simplicial set  $X$ , its homotopy groups are given by

$$\pi_n(X) := \{x \in X_n \mid d_i(x) = * \text{ for all } i\} / \sim, \tag{15}$$

where  $x \sim x'$  if there exists an element  $y \in X_{n+1}$  such that  $d_0(y) = x$ ,  $d_1(y) = x'$ , and  $d_i(y) = *$  for all  $i > 1$ .

More generally, let  $S$  be any simplicial set whose geometric realization is homotopy equivalent to the  $n$ -sphere, and let  $SI$  be any simplicial set equipped with an injective map  $S \vee S \rightarrow SI$  with



the property that the two inclusions  $S \rightarrow SI$  are homotopy equivalences, and are homotopic to each other. Then  $\pi_n(X)$  is isomorphic to the coequalizer

$$\operatorname{coeq}(\operatorname{Hom}(SI, X) \rightrightarrows \operatorname{Hom}(S, X)). \tag{16}$$

Definition (15) is the special case of (16) in which

$$S = \Delta[n]/\partial\Delta[n] \quad \text{and} \quad SI = \Delta[n+1] / \left( \bigcup_{i=2}^n F^i \cup (F^0 \cap F^1) \right),$$

where  $F^i$  denotes the  $i$ th facet of  $\Delta[n+1]$ .

Homotopy groups in the category of simplicial sheaves were introduced in [Jar86]. The following is essentially Jardine’s definition.

DEFINITION 3.1. Let  $X$  be a reduced Kan simplicial object and let  $S, SI$  be as above. The simplicial homotopy groups  $\pi_i^{\operatorname{spl}}(X)$  are the sheaves

$$\pi_n^{\operatorname{spl}}(X) := \operatorname{coeq}(\operatorname{Hom}(SI, X) \rightrightarrows \operatorname{Hom}(S, X)). \tag{17}$$

The fact that this is independent of the choice of  $S$  and  $SI$  is the content of Lemma 3.3.

To make the above definition a more concrete, we write down what it means for the simplest case

$$S = \Delta[n]/\partial\Delta[n], \quad SI = \Delta[n+1] / \left( \bigcup_{i=2}^n F^i \cup (F^0 \cap F^1) \right).$$

In that case, we obtain

$$\pi_n^{\operatorname{spl}}(X)(T) = \left\{ \begin{array}{l} \text{- a cover } U \rightarrow T \\ \text{- a morphism } \\ x : U \rightarrow X_n \end{array} \left| \begin{array}{l} \text{- } d_i \circ x = * \text{ for all } i \\ \text{- there exist } y : U \times_T U \rightarrow X_{n+1} \\ \text{such that } d_0 \circ y = x \circ \operatorname{pr}_1, d_1 \circ y = \\ x \circ \operatorname{pr}_2, \text{ and } d_i \circ y = * \text{ for all } i > 1. \end{array} \right. \right\} / \sim,$$

where  $(U, x) \sim (U', x')$  if there exist a covering  $V$  of  $U \times_X U'$  and a map  $y' : V \rightarrow X_{n+1}$  such that  $d_0 \circ y' = x \circ \operatorname{pr}_1, d_1 \circ y' = x' \circ \operatorname{pr}_2,$  and  $d_i \circ y' = *$  for all  $i > 1$ .

To show that  $\pi_n^{\operatorname{spl}}(X)$  are groups in the category of sheaves, we let  $(U, x), (U', x')$  be two representatives of elements of  $\pi_n^{\operatorname{spl}}(X)(T)$ . By Lemma 2.12, the map

$$\operatorname{Hom}(\Delta[n+1], X) \rightarrow \operatorname{Hom}(\Lambda[n+1, 1], X)$$

is surjective. So we may pick a common refinement  $U \leftarrow V \rightarrow U'$ , and a map  $y : V \rightarrow X_{n+1}$  such that  $d_0 \circ y = x, d_2 \circ y = x'$  and  $d_i \circ y = *$  for  $i > 2$ . The product is then represented by  $(V, d_1 \circ y)$ .

Example 3.2. Let  $G$  be a group object of  $\mathcal{C}$ , and let  $K(G, 1)$  be as in (5). Then  $\pi_1^{\operatorname{spl}}(K(G, 1))$  is the sheaf represented by  $G$ , and  $\pi_n^{\operatorname{spl}}(K(G, 1)) = 0$  for  $n \geq 2$ .

LEMMA 3.3. Let  $S, S'$  be simplicial sets weakly equivalent to the  $n$ -sphere, and let  $S \vee S \rightarrow SI, S' \vee S' \rightarrow SI'$  be monomorphisms with the property that the two inclusions  $S \rightarrow SI$  are homotopic homotopy equivalences, and similarly for the two inclusions  $S' \rightarrow SI'$ . Then the sheaves  $\pi_n^{\operatorname{spl}}(X) := \operatorname{coeq}(\operatorname{Hom}(SI, X) \rightrightarrows \operatorname{Hom}(S, X))$  and  $\pi_n^{\operatorname{spl}}(X)' := \operatorname{coeq}(\operatorname{Hom}(SI', X) \rightrightarrows \operatorname{Hom}(S', X))$  are canonically isomorphic.

Proof. Since  $S$  and  $S'$  are weakly equivalent, we may find a third simplicial set  $S''$ , and acyclic cofibrations  $S \hookrightarrow S'' \hookrightarrow S'$ . Similarly, we may find acyclic cofibrations  $SI \hookrightarrow SI'' \hookrightarrow SI'$  making the



following diagram commute.

$$\begin{array}{ccccc}
 S \vee S & \longrightarrow & S'' \vee S'' & \longleftarrow & S' \vee S' \\
 \downarrow & & \downarrow & & \downarrow \\
 SI & \longrightarrow & SI'' & \longleftarrow & SI'
 \end{array}$$

Letting  $\pi_n^{\text{spl}}(X)'' := \text{coeq}(\text{Hom}(SI'', X) \rightrightarrows \text{Hom}(S'', X))$ , the above diagram induces maps  $\pi_n^{\text{spl}}(X) \rightarrow \pi_n^{\text{spl}}(X)'' \leftarrow \pi_n^{\text{spl}}(X)'$ . We just show that the first map is an isomorphism, the argument being the same for the second map.

The left and middle vertical maps in

$$\begin{array}{ccccc}
 \text{Hom}(SI'', X) & \rightrightarrows & \text{Hom}(S'', X) & \longrightarrow & \pi_n^{\text{spl}}(X)'' \\
 \downarrow (a) & & \downarrow (b) & & \downarrow (c) \\
 \text{Hom}(SI, X) & \rightrightarrows & \text{Hom}(S, X) & \longrightarrow & \pi_n^{\text{spl}}(X)
 \end{array}$$

are surjective by Lemma 2.12. The surjectivity of (b) implies the surjectivity of (c), and the surjectivity (a) implies the injectivity of (c). □

All basic properties of homotopy groups of Kan simplicial sets go through to Kan simplicial objects. For example, we have the following result.

**PROPOSITION 3.4.** *Let  $X \rightarrow Y$  be a Kan fibration of Kan simplicial objects, and let  $F$  be its fiber. Then there is an associated long exact sequence*

$$\dots \rightarrow \pi_{n+1}^{\text{spl}}(Y) \xrightarrow{\partial} \pi_n^{\text{spl}}(F) \rightarrow \pi_n^{\text{spl}}(X) \rightarrow \pi_n^{\text{spl}}(Y) \rightarrow \dots \tag{18}$$

*Proof.* At the level of points, the proof is standard and can be found in [May92, Theorem 7.6]. At the level of sheaves, we just need to keep track of covers and refine them appropriately each time we use the Kan condition.

A more complete argument, including a precise construction of the connecting homomorphism  $\partial$ , can be found in [Van77, § II]. □

We now introduce truncation functors that kill the simplicial homotopy groups above a given range. Recall that for Kan simplicial sets, one has two truncation functors  $\tau_{\leq n}$  and  $\tau_{< n}$  given by

$$(\tau_{\leq n} X)_m = X_m / \sim, \quad (\tau_{< n} X)_m = X_m / \approx$$

where  $x \sim x'$  if the corresponding maps  $\Delta[m] \rightarrow X$  are simplicially homotopic relatively to the  $(n - 1)$ -skeleton of  $\Delta[m]$ , and  $x \approx x'$  if they are equal when restricted to the  $(n - 1)$ -skeleton of  $\Delta[m]$ . The  $i$ th simplicial homotopy group of  $\tau_{< n} X$  (of  $\tau_{\leq n} X$ ) is equal to  $\pi_i^{\text{spl}}(X)$  for  $i < n$  (respectively,  $i \leq n$ ) and zero otherwise. In particular,  $\tau_{< n+1} X \rightarrow \tau_{\leq n} X$  is a weak equivalence.

The functor  $\tau_{< n}$  is due to Moore [Moo58], and the functor  $\tau_{\leq n}$  is apparently due to Duskin. Moore’s functor is explained in detail in [May92] and [GJ99], and Duskin’s in [Gle82, § 3.1], [Bek04, Proposition 1.5] and [Get03, § 2]. Observe that under the Dold–Kan correspondence, these functors applied to simplicial abelian groups become the usual truncation functors on complexes of abelian groups.

If one is willing to leave the category of simplicial objects of  $\mathcal{C}$  and allow simplicial sheaves, then one has analogs of the above truncations functors.

DEFINITION 3.5. Let  $X$  be a reduced Kan simplicial object. Then the truncations  $\tau_{\leq n}X$  and  $\tau_{< n}X$  are the simplicial sheaves given by

$$\begin{aligned} (\tau_{\leq n}X)_m &= \text{coeq}(\text{Hom}(P, X) \rightrightarrows \text{Hom}(\Delta[m], X)), \\ (\tau_{< n}X)_m &= \text{im}(\text{Hom}(\Delta[m], X) \rightarrow \text{Hom}(\text{sk}_{n-1}\Delta[m], X)), \end{aligned}$$

where  $P$  is the following pushout simplicial set.

$$\begin{array}{ccc} \Delta[1] \times \text{sk}_{n-1}\Delta[m] & \longrightarrow & \text{sk}_{n-1}\Delta[m] \\ \downarrow & \lrcorner & \downarrow \\ \Delta[1] \times \Delta[m] & \longrightarrow & P \end{array}$$

We then have a tower

$$\cdots \rightarrow \tau_{< n+1}X \rightarrow \tau_{\leq n}X \rightarrow \tau_{< n}X \rightarrow \cdots \rightarrow \tau_{\leq 1}X \rightarrow * \tag{19}$$

whose inverse limit is  $X$ . As before, the  $i$ th simplicial homotopy group of  $\tau_{< n}X$  (of  $\tau_{\leq n}X$ ) is equal to  $\pi_i^{\text{spl}}(X)$  for  $i < n$  (respectively,  $i \leq n$ ), and zero otherwise.

The truncation functor  $\tau_{\leq n}$  will later be used to construct Lie  $n$ -groups integrating Lie  $n$ -algebras. We can already check that  $\tau_{\leq n}X$  satisfies one of the conditions in Definition 1.4.

LEMMA 3.6. Let  $X$  be a Kan simplicial object, and let  $\tau_{\leq n}X$  be as above. Then

$$(\tau_{\leq n}X)_m \rightarrow \text{Hom}(\Lambda[m, j], \tau_{\leq n}X)$$

is an isomorphism for all  $m > n$ .

*Proof.* We only prove the lemma in the case when the topos of sheaves on  $\mathcal{C}$  has enough points [MM94, § IX.11]. Since stalks of Kan simplicial objects are Kan simplicial sets, and since  $\tau_{\leq n}$  commutes with taking stalks, it is enough to prove the lemma for the case of simplicial sets.

So let  $X$  be a Kan simplicial set. Let  $S$  be the union of all  $n$ -faces of  $\Delta[m]$  containing its  $j$ th vertex. Since  $S$  is weakly homotopy equivalent to  $\Delta[m]$  as simplicial set under  $\text{sk}_{n-1}\Delta[m]$ , and since  $X$  is fibrant, the restriction map

$$\begin{aligned} (\tau_{\leq n}X)_m &= \{\text{maps } \Delta[m] \rightarrow X \text{ modulo homotopy fixing } \text{sk}_{n-1}\Delta[m]\} \\ &\rightarrow \{\text{maps } S \rightarrow X \text{ modulo homotopy fixing } \text{sk}_{n-1}\Delta[m]\} \end{aligned}$$

is a bijection. An element of  $(\tau_{\leq n}X)_m$  can thus be described as a map  $f : \text{sk}_{n-1}\Delta[m] \rightarrow X$ , along with a collection of homotopy classes of nullhomotopies, one for each restriction  $f|_{\partial F}$  to an  $n$ -face  $F$  of  $S$ .

A map  $\Lambda[m, j] \rightarrow \tau_{\leq n}X$  is therefore the same thing as a map  $\text{sk}_{n-1}\Lambda[m, j] \rightarrow X$ , along with a collection of homotopy classes of nullhomotopies, one for each restriction  $f|_{\partial F}$  to an  $n$ -face  $F$  of  $\Lambda[m, j]$  containing the  $j$ th vertex. The result follows since  $\text{sk}_{n-1}\Lambda[m, j] = \text{sk}_{n-1}\Delta[m]$  and since every  $n$ -face  $F$  of  $\Delta[m]$  containing its  $j$ th vertex is also contained in  $\Lambda[m, j]$ .  $\square$

*Remark 3.7.* For the standard argument that shows how to reduce the proof of Lemma 3.6 to the case of simplicial sets, even if the topos does not have enough points, see [Bek04, Section ‘For logical reasons’].

Since  $(\tau_{\leq n}X)_m$  and  $(\tau_{< n}X)_m$  might fail to be representable sheaves, we cannot make sense of our Kan condition, and thus cannot say much about them at this level of generality. However, in our situation of interest  $\mathcal{C} = \{\text{Banach manifolds}\}$ , we will nevertheless be able to analyze the structure of  $\tau_{\leq n}X$  and  $\tau_{< n}X$ .

4. Banach manifolds

From now on, we work with the category of manifolds, by which we mean Banach manifolds, and infinitely differentiable maps in the sense of Fréchet [Lan72, §I.3]. Given two manifolds  $M, N$ , a map  $f : M \rightarrow N$  is called a *submersion* if the induced maps on tangent spaces is split surjective. This condition implies that  $f$  is locally diffeomorphic to a projection  $V \oplus W \rightarrow V$  (see [Lan72, Corollary I.5.2]). Our working definition is the following.

DEFINITION 4.1. A map  $f : M \rightarrow N$  is a submersion if for every point  $x \in M$ , there exists a neighborhood  $U$  of  $f(x)$  and a section  $U \rightarrow M$  sending  $f(x)$  to  $x$ .

In this section, we prove that the category of Banach manifold, equipped with the Grothendieck pretopology in which the covers are the surjective submersions, satisfies Assumptions 2.2.

Recall some basic facts about transversality in Banach manifolds (see [Lan72, §I.5], [AMR88, §3.5] for these and other related results).

LEMMA 4.2. Let  $f : M \rightarrow N$  be a submersion and let  $y \in N$  be a point. Then  $f^{-1}(y)$  is a manifold.

*Proof.* Let  $x \in f^{-1}(y)$  be a point, and let us identify  $M$  locally with its tangent space  $V := T_x M$ . Let  $s : N \rightarrow V$  be a local section of  $f$ . We then have a direct sum decomposition  $V = \ker(T_x f) \oplus \text{im}(T_y s)$ . Let  $p : V \rightarrow \ker(T_x f)$  denote the projection.

The derivative of  $\phi := (s \circ f) + p$  is the identity of  $V$ , so by the inverse function theorem [Lan72, Theorem I.5.1],  $\phi$  is a local diffeomorphism. The result follows since  $f^{-1}(y) = \phi^{-1}(\ker(T_x f))$ .  $\square$

LEMMA 4.3. Let  $f : M \rightarrow N$  be a (surjective) submersion and let  $g : Q \rightarrow N$  be an arbitrary map. Then the pullback  $P := M \times_N Q$  is a manifold and its map to  $Q$  is a (surjective) submersion.

*Proof.* The question being local, we may assume that  $N$  is a vector space. The space  $P$  is then the preimage of the origin under the submersion  $(f, -g) : M \times Q \rightarrow N$  and is a manifold by Lemma 4.2. The fact that  $P \rightarrow Q$  is a (surjective) submersion is easy and left to the reader.  $\square$

COROLLARY 4.4. Surjective submersions form a Grothendieck pretopology on the category of Banach manifolds.

We now show that the category of Banach manifolds satisfies Assumptions 2.2. The first of those assumptions is trivial. The fact that the category of Banach manifolds is Cauchy complete is the content of the following lemma.

LEMMA 4.5. Let  $M$  be a manifold and  $p : M \rightarrow M$  be an idempotent map, then  $p(M)$  is a manifold.

*Proof.* Let  $x \in p(M)$  be a point, and let us identify  $M$  locally with the tangent space  $V := T_x M$ . The linear map  $T_x p : V \rightarrow V$  being idempotent, the vector space  $V$  splits as

$$V = \text{im}(T_x p) \oplus \ker(T_x p).$$

Let  $q$  denote the linear projection  $q : V \rightarrow \ker(T_x p)$ , with kernel  $\text{im}(T_x p)$ . The map  $\phi := p + q$  is smooth and satisfies  $T_x \phi = 1_V$ . So by the inverse function theorem, we conclude that  $\phi$  is a diffeomorphism in a neighborhood of  $x$ . The claim follows since locally  $p(M) = \phi(\text{im}(T_x p))$ , and  $\text{im}(T_x p)$  is a vector space.  $\square$

LEMMA 4.6. The retract of a surjective submersion is a surjective submersion.

*Proof.* The retract of a surjective map is clearly surjective. So we consider a commutative diagram like (7)

$$\begin{array}{ccc}
 A & \xrightarrow{i} & X \\
 \downarrow f & \nearrow r & \downarrow g \\
 B & \xrightarrow{i} & Y
 \end{array}$$

and assume that  $g$  is a submersion. Let  $a \in A$  be a point. Since  $g$  is a submersion, we can find a neighborhood  $U \subset Y$  of  $y := gi(a)$ , and a local section  $s : U \rightarrow X$  of  $g$  sending  $y$  to  $i(a)$ . The preimage  $V := r^{-1}(U)$  is then a neighborhood of  $b := f(a)$  and  $r \circ s \circ i : V \rightarrow A$  is a local section of  $f$  sending  $b$  to  $a$ . This shows that  $f$  is a submersion.  $\square$

Finally, the last of Assumptions 2.2 follows from the fact that surjective submersions can be refined by usual open covers, and that the ‘open covers’ Grothendieck pretopology is subcanonical.

*Remark 4.7.* As an open cover is a special case of a surjective submersion, it follows that each of the ‘surjective submersions’ and the ‘open covers’ Grothendieck pretopologies refines the other. A functor  $\{\text{Banach manifolds}\}^{\text{op}} \rightarrow \{\text{Sets}\}$  is therefore a sheaf for one of the two pretopologies if and only if it is a sheaf for the other.

We finish this section with an easy lemma that is needed in § 7.

LEMMA 4.8. *Let  $f : M \rightarrow N, g : N \rightarrow Q$  be maps between Banach manifolds, with the properties that  $f$  is surjective and  $g \circ f$  is a surjective submersion. Then  $g$  is a surjective submersion.*

*Proof.* The surjectivity of  $g$  is clear. To show that  $g$  is a submersion, we pick a point  $n \in N$  and construct a local section  $Q \supset U \rightarrow N$ , sending  $g(n)$  to  $n$ . Letting  $m \in M$  be any preimage of  $n$ , we may pick a section  $s : U \rightarrow M$  of  $g \circ f$  sending  $gf(m) = g(n)$  to  $m$ . The map  $f \circ s$  is the desired section of  $g$ .  $\square$

### 5. Integrating $L_\infty$ -algebras

In the following sections, all  $L_\infty$ -algebras are assumed to be non-negatively graded, and finite dimensional in each degree.

Given a differential graded algebra  $A$  over  $\mathbb{Q}$ , Sullivan [Sul77] introduced its spatial realization  $\langle A \rangle$  (see also Bousfield–Gugenheim [BG76], in which  $\langle A \rangle$  is called  $FA$ ). The  $n$ -simplices of  $\langle A \rangle$  are given by the set of differential graded algebra homomorphisms from  $A$  to the algebra  $\Omega_{\text{pol}}^*(\Delta^n; \mathbb{Q})$  of polynomial de Rham forms on the  $n$ -simplex with rational coefficients. If  $A$  is a nilpotent differential graded algebra (see [Sul77, § 2]), then  $\langle A \rangle$  is a simplicial set with uniquely divisible homotopy groups, and nilpotent  $\pi_1$ . The idea of applying the above construction to the Chevalley–Eilenberg complex of an  $L_\infty$ -algebra is due to Getzler [Get03], who was inspired by Hinich [Hin97].

We shall be working with algebras over  $\mathbb{R}$ , and without any nilpotence assumption. As suggested in [Sul77], we use an algebra which is bigger than  $\Omega_{\text{pol}}^*$ , and more suitable for this analytical context. One of our results (Theorem 5.10) states the existence of natural manifold structures on the above sets of algebra homomorphisms.

The failure of the implicit function theorem for Fréchet manifolds forces us to work with Banach manifolds instead. This leads us to the following slightly inelegant convention. Fix an integer  $r \geq 1$ . Given a compact finite-dimensional manifold  $M$ , possibly with boundary and corners, let  $\Omega_{C^r}^*(M)$  be the differential graded algebra given by

$$\Omega_{C^r}^n(M) := \{ \alpha \in \Gamma_{C^r}(M; \Lambda^n T^* M) \mid d\alpha \in \Gamma_{C^r}(M; \Lambda^{n+1} T^* M) \},$$

where  $\Gamma_{C^r}(M, \xi)$  denotes the space of  $r$  times continuously differentiable sections of a smooth vector bundle  $\xi$ . The choice of  $r$  is immaterial, and will often be suppressed from the notation.

CONVENTION 5.1. Hereafter,  $\Omega^n$  always means  $\Omega_{C^r}^n$ .

The only properties of  $\Omega_{C^r}^*(M)$  that we use are that it is a differential graded Banach algebra, and that it behaves well under restriction. Any other completion of  $\Omega_{C^\infty}^*(M)$  with the above properties could be used instead.

DEFINITION 5.2. Let  $L$  be an  $L_\infty$ -algebra, and let  $C^*(L)$  be its Chevalley–Eilenberg complex (2). The integrating simplicial manifold  $\int L$  is then given by

$$\left(\int L\right)_m := \text{Hom}_{DGA}(C^*(L), \Omega^*(\Delta^m)), \tag{20}$$

where  $\Omega^*(\Delta^m)$  denotes the differential graded algebra of de Rham forms on the  $m$ -simplex, following Convention 5.1.

Given a morphism  $f : [m] \rightarrow [n]$  in  $\Delta$ , the corresponding map  $(\int L)_n \rightarrow (\int L)_m$  is induced by the pullback  $\Omega^*(\Delta^n) \rightarrow \Omega^*(\Delta^m)$  along  $f_* : \Delta^m \rightarrow \Delta^n$ .

Example 5.3. Let  $L = L_{n-1}$  be an  $L_\infty$ -algebra concentrated in degree  $n - 1$ , and with all brackets vanishing. Then  $(\int L)_m = \Omega_{\text{closed}}^n(\Delta^m; L_{n-1})$ .

Example 5.4. Let  $L = L_n \oplus L_{n-1}$  be a contractible  $L_\infty$ -algebra concentrated in degrees  $n$  and  $n - 1$ . All of its brackets are zero except  $[\ ] : L_n \rightarrow L_{n-1}$ , which is an isomorphism. Then  $(\int L)_m = \Omega^n(\Delta^m; L_n)$ .

Example 5.5 (Sullivan [Sul77, ‘Theorem’(8.1)']). Let  $\mathfrak{g}$  be a Lie algebra, viewed as  $L_\infty$ -algebra concentrated in degree 0, and let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . A homomorphism of graded algebras  $C^*(\mathfrak{g}) \rightarrow \Omega^*(\Delta^m)$  is uniquely determined by a linear map  $\mathfrak{g}^* \rightarrow \Omega^1(\Delta^m)$ . This is the same as a  $\mathfrak{g}$ -valued 1-form  $\alpha \in \Omega^1(\Delta^m; \mathfrak{g})$ , which can then be interpreted as a connection on the trivial  $G$ -bundle  $G \times \Delta^m \rightarrow \Delta^m$ . This graded algebra homomorphism respects the differentials if and only if  $\alpha$  satisfies the Maurer–Cartan equation

$$d\alpha = \frac{1}{2}[\alpha \wedge \alpha], \tag{21}$$

where  $[\alpha \wedge \alpha]$  denotes the image under the bracket map of the  $(\mathfrak{g} \otimes \mathfrak{g})$ -valued 2-form  $\alpha \wedge \alpha$ . Such  $\alpha$  correspond to flat connections on  $G \times \Delta^m \rightarrow \Delta^m$ . Assigning to such a connection the set of all of its horizontal sections establishes an isomorphism between the space of flat connections, and the space of maps  $\Delta^m \rightarrow G$ , modulo translation. Note that  $\alpha$  being in  $\Omega_{C^r}^1(\Delta^m; \mathfrak{g})$  is equivalent to that map being of class  $C^{r+1}$ . So we obtain that

$$\left(\int \mathfrak{g}\right)_m = \text{Map}(\Delta^m, G)/G. \tag{22}$$

where  $\text{Map}$  denotes  $C^{r+1}$  maps.

The goal of this section is to prove that  $\int L$  is a Kan simplicial manifold. For this purpose, we introduce a filtration (23) whose associated graded pieces are the  $L_\infty$ -algebras of Examples 5.3, 5.4, and 5.5.

Given an  $L_\infty$ -algebra  $L$ , let  $\partial_n : L_n \rightarrow L_{n-1}$  denote the components of the 1-ary bracket  $\partial = [\ ]$ . The homology groups  $H_n(L) = \ker(\partial_n)/\text{im}(\partial_{n+1})$  behave very much like homotopy groups, with  $H_{n-1}$  playing the role of  $\pi_n$ .

DEFINITION 5.6. The truncations  $\tau_{\leq n}L$  and  $\tau_{< n}L$  of an  $L_\infty$ -algebra  $L$  are given by

$$(\tau_{\leq n}L)_i = \begin{cases} L_i & \text{if } i < n, \\ \text{coker}(\partial_{n+1}) & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases} \quad (\tau_{< n}L)_i = \begin{cases} L_i & \text{if } i < n, \\ \text{im}(\partial_n) & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases}$$

with brackets inherited from  $L$ .

We then have a tower of projections

$$\cdots \rightarrow \tau_{< n+1}L \rightarrow \tau_{\leq n}L \rightarrow \tau_{< n}L \rightarrow \cdots \rightarrow \tau_{\leq 1}L \rightarrow 0 \tag{23}$$

whose inverse limit is  $L$ . The  $i$ th homology group of  $\tau_{< n}L$  ( $\tau_{\leq n}L$ ) is that of  $L$  for  $i < n$  ( $i \leq n$ ) and zero otherwise, and the maps  $\tau_{< n}L \rightarrow \tau_{\leq n-1}L$  are isomorphisms on homology.

Denote by  $\Lambda^{m,j} \subset \Delta^m$  the geometric realization of  $\Lambda[m, j] \subset \Delta[m]$ .

LEMMA 5.7. *Let  $M$  be a smooth finite-dimensional manifold and let  $X$  be the simplicial manifold given by  $X_m = \text{Map}(\Delta^m, M)$ , where  $\text{Map}$  denotes  $C^{r+1}$  maps. Then  $X$  satisfies the Kan condition.*

*Proof.* A map  $\Lambda[m, j] \rightarrow X$  is the same thing as a map  $\Lambda^{m,j} \rightarrow M$  which is  $C^{r+1}$  when restricted to each face of  $\Lambda^{m,j}$ .

To show that (4) is surjective, we must show that every such map  $f : \Lambda^{m,j} \rightarrow M$  extends to  $\Delta^m$ . Pick a closed embedding  $\iota : M \hookrightarrow \mathbb{R}^N$ , let  $\nu(M) \subset \mathbb{R}^N$  be a tubular neighborhood of  $M$ , and let  $\pi : \nu(M) \rightarrow M$  be the normal projection. As before, let  $[m] = \{0, 1, \dots, m\}$ . For  $\{j\} \subsetneq I \subset [m]$ , let  $p_I : \Delta^m \rightarrow \Lambda^{m,j}$  denote the affine projection given on vertices by

$$p_I(e_i) := \begin{cases} e_j & \text{if } i \in I, \\ e_i & \text{if } i \notin I. \end{cases} \tag{24}$$

For  $f : \Lambda^{m,j} \rightarrow M$ , let  $\tilde{f} : \Delta \rightarrow \mathbb{R}^N$  be the sum

$$\tilde{f} := \sum_{\{j\} \subsetneq I \subset [m]} (-1)^{|I|} \iota \circ f \circ p_I. \tag{25}$$

One easily checks that  $\tilde{f}|_{\Lambda^{m,j}} = \iota \circ f$ . Given  $\epsilon > 0$ , let  $\Lambda_\epsilon^{m,j} \subset \Delta^m$  be the  $\epsilon$ -neighborhood of  $\Lambda^{m,j}$  and let  $\epsilon(f) := \inf\{\epsilon : \tilde{f}|_{\Lambda_\epsilon^{m,j}} \subset \nu(M)\}$ . Pick smooth maps  $r_\epsilon : \Delta^m \rightarrow \Lambda_\epsilon^{m,j}$  fixing  $\Lambda^{m,j}$  and depending smoothly on  $\epsilon$ . An extension  $\bar{f} : \Delta^m \rightarrow M$  is then given by the formula

$$\bar{f} := \pi \circ \tilde{f} \circ r_{\epsilon(f)}.$$

This provides a global lift of (4).

To show that (4) is submersive, we modify the above construction so that the image  $\bar{f}_0$  of a chosen map  $f_0$  is specified. Instead of (25), we set

$$\tilde{f}' := \sum (-1)^{|I|} \iota \circ f \circ p_I + \left( \iota \circ \bar{f}_0 - \sum (-1)^{|I|} \iota \circ f_0 \circ p_I \right). \tag{26}$$

Let us now further assume that  $r_\epsilon$  has been chosen so that  $r_\epsilon = 1$  if  $\Lambda_\epsilon^{m,j} = \Delta^m$ . Letting  $\epsilon'(f) := \inf\{\epsilon : \tilde{f}'|_{\Lambda_\epsilon^{m,j}} \subset \nu(M)\}$ , we now have a new extension

$$\bar{f}' := \pi \circ \tilde{f}' \circ r_{\epsilon'(f)},$$

which is compatible with our initial choice  $\bar{f}_0$ . □

LEMMA 5.8. *The quotient of a Kan simplicial manifold  $X$  by the proper free action of a Lie group  $G$  is again a Kan simplicial manifold.*

*Proof.* Since  $\Lambda[m, j]$  is simply connected, we have

$$\text{Hom}(\Lambda[m, j], X/G) = \text{Hom}(\Lambda[m, j], X)/G.$$

The result then follows since the quotient of a surjective submersion by a proper free action is again a surjective submersion.  $\square$

Combining the above result with Lemma 5.7, we see that the simplicial manifolds constructed in Example 5.5 are Kan.

LEMMA 5.9. *The simplicial manifold  $X$  given by  $X_m = \Omega^n(\Delta^m)$  is Kan, and so is that given by  $X_m = \Omega_{\text{closed}}^n(\Delta^m)$ .*

Note that these simplicial manifolds are integrals of explicit abelian  $L_\infty$ -algebras, namely  $\mathbb{R}[n] \xrightarrow{\sim} \mathbb{R}[n - 1]$  and  $\mathbb{R}[n - 1]$ , as explained in Examples 5.4 and 5.3.

*Proof.* Given a (closed) form  $\alpha$  on  $\Lambda^{m,j}$ , we can extend it to  $\Delta^m$  by the formula

$$\bar{\alpha} := \sum_{\{j\} \subsetneq I \subset [m]} (-1)^{|I|} p_I^* \alpha. \tag{27}$$

where  $p_I$  are the projections introduced in the proof of Lemma 5.7. This shows that (4) is surjective. If we have a chosen extension  $\bar{\alpha}_0 \in \Omega^n(\Delta^m)$  of a form  $\alpha_0 \in \Omega^n(\Lambda^{m,j})$ , we can modify (27) in a way similar to (26)

$$\bar{\alpha}' := \sum (-1)^{|I|} p_I^* \alpha + \left( \bar{\alpha}_0 - \sum (-1)^{|I|} p_I^* \alpha_0 \right)$$

so that the extension of  $\alpha_0$  is exactly  $\bar{\alpha}_0$ . This shows that (4) is a submersion.  $\square$

We are now in position to prove our main theorem.

THEOREM 5.10. *Let  $L$  be an  $L_\infty$ -algebra. Then each  $(\int L)_m$  is a smooth manifold, and  $\int L$  satisfies the Kan condition (4). Moreover, all the maps*

$$\cdots \rightarrow \int \tau_{\leq 1} L \rightarrow \int \tau_{< 1} L \rightarrow \int \tau_{\leq 0} L \rightarrow * \tag{28}$$

are Kan fibrations.

*Proof.* By Lemma 2.9, that the maps (28) are Kan fibrations implies that each  $\int \tau_{< n} L$  is Kan. Since  $(\int L)_m = (\int \tau_{< n} L)_m$  for  $n \geq m$ , this also implies the Kan condition for  $\int L$ .

We first note that  $\int \tau_{\leq 0} L$  is Kan. Indeed,  $\tau_{\leq 0} L$  is a Lie algebra. Its integrating simplicial manifold has been computed in Example 5.5 and it is Kan by Lemmas 5.7 and 5.8.

To see that  $\int \tau_{< n} L \rightarrow \int \tau_{\leq n-1} L$  is a Kan fibration, note that  $C^*(\tau_{< n} L)$  is freely generated as a differential graded algebra over  $C^*(\tau_{\leq n-1} L)$  by the vector space  $\text{im}(\partial_n)^*$ , put in degree  $n$ . It follows that  $\int \tau_{< n} L$  is the product of  $\int \tau_{\leq n-1} L$  with the simplicial manifold of Example 5.4. Assuming by induction that the sheaves  $(\int \tau_{< n} L)_m$  are manifolds, it follows that the sheaves  $(\int \tau_{\leq n-1} L)_m$  are also manifolds. By Lemmas 2.8 and 5.9, we also deduce that  $\int \tau_{< n} L \rightarrow \int \tau_{\leq n-1} L$  is a Kan fibration.

We now show that

$$\int \tau_{\leq n-1} L \longrightarrow \int \tau_{< n-1} L \tag{29}$$

is a Kan fibration. Let us use the shorthand notation  $H := H_{n-1}(L)$  and  $\mathfrak{g} := H_0(L)$ . Pick a splitting of the inclusion  $H \rightarrow (\tau_{\leq n-1} L)_{n-1}$ . The graded algebra  $C^*(\tau_{\leq n-1} L)$  is freely generated over  $C^*(\tau_{< n-1} L)$  by the vector space  $H^*$ . A graded algebra homomorphism  $\varphi : C^*(\tau_{\leq n-1} L) \rightarrow \Omega^*(\Delta^k)$  is therefore the same thing as a homomorphism  $\psi : C^*(\tau_{< n-1} L) \rightarrow \Omega^*(\Delta^k)$  along with an  $H$ -valued  $n$ -form  $\beta$ . We now characterize the pairs  $(\psi, \beta)$  such that  $\varphi = \varphi(\psi, \beta)$  respects the differential.



Clearly,  $\psi$  needs to respect the differential, so we assume that this is the case and investigate the condition on  $\beta$ . By the Leibnitz rule, it is enough to verify on  $H^*$  whether  $d \circ \varphi = \varphi \circ \delta$ .

Let us write  $\delta|_{H^*}$  as  $\delta_0 + \delta_1$ , where  $\delta_0$  lands in  $L_0^* \otimes H^*$  and  $\delta_1$  lands in  $C^*(\tau_{<n-1}L)$ . Since  $\partial = [ ]$  is a derivation of  $[ , ]$  the component of  $[ , ]$  that maps  $\text{im}(\partial_1) \otimes H$  to  $H$  vanishes. It follows that  $\delta_0$  only lands in  $\mathfrak{g}^* \otimes H^*$ . Let  $\alpha \in \Omega^1(\Delta^k; \mathfrak{g})$  be the restriction of  $\varphi$  to  $\mathfrak{g}^*$ , and  $\gamma \in \Omega^{n+1}(\Delta^k; H)$  be the composite  $\varphi \circ \delta_1$ . We then have  $d \circ \varphi|_{H^*} = d\beta$  and  $\varphi \circ \delta|_{H^*} = \varphi \circ \delta_0 + \varphi \circ \delta_1 = [\alpha \wedge \beta] + \gamma$ . It follows that  $\varphi(\psi, \beta)$  respects the differentials if and only if  $\beta$  satisfies

$$d\beta = [\alpha \wedge \beta] + \gamma. \tag{30}$$

Let  $G$  be the simply connected Lie group integrating  $\mathfrak{g}$ . Since  $H$  is a  $\mathfrak{g}$ -module, we can integrate it to a  $G$ -module. The 1-form  $\alpha$  satisfies the Maurer–Cartan equation  $d\alpha = \frac{1}{2}[\alpha, \alpha]$  so we can integrate it to a map  $f : \Delta^k \rightarrow G$  satisfying  $-\alpha = f^{-1}df$ . Fix the image of some vertex, so as to make  $f$  uniquely defined. Let now

$$\beta' := f\beta, \quad \gamma' := f\gamma. \tag{31}$$

Clearly, once  $\psi$  is fixed, the data of  $\beta$  and  $\beta'$  are equivalent to each other. However, in terms of  $\beta'$ , (30) becomes much simpler:

$$d\beta' = \gamma'. \tag{32}$$

We want to solve (32) for  $\beta'$ , so we first check that  $\gamma'$  is closed. Indeed, we have

$$d\gamma' = [df f^{-1} \wedge f\gamma] + fd\gamma = -[f\alpha \wedge f\gamma] + fd\gamma = f(d\gamma - [\alpha \wedge \gamma]) = 0$$

since

$$\begin{aligned} d\gamma &= d(d\beta - [\alpha \wedge \beta]) = [\alpha \wedge d\beta] - [d\alpha \wedge \beta] \\ &= [\alpha \wedge \gamma] + [\alpha \wedge [\alpha \wedge \beta]] - \frac{1}{2}[[\alpha \wedge \alpha] \wedge \beta] = [\alpha \wedge \gamma], \end{aligned}$$

where the last equality holds because  $[ , ] : \mathfrak{g} \otimes H \rightarrow H$  is a Lie algebra action.

Returning to our problem of showing that (29) is a Kan fibration, we must show that

$$\left( \int \tau_{\leq n-1}L \right)_m \longrightarrow \text{Hom} \left( \Lambda[m, j] \rightarrow \Delta[m], \int \tau_{\leq n-1}L \rightarrow \int \tau_{<n-1}L \right) \tag{33}$$

is a surjective submersion. An element in the right-hand side of (33) consists of two compatible homomorphisms  $\psi : C^*(\tau_{<n-1}L) \rightarrow \Omega^*(\Delta^m)$  and  $\varphi : C^*(\tau_{\leq n-1}L) \rightarrow \Omega^*(\Lambda^{m,j})$ . Let  $P$  be the pullback of

$$\Omega^n(\Lambda^{m,j}) \xrightarrow{d} \Omega_{\text{closed}}^{n+1}(\Lambda^{m,j}) \longleftarrow \Omega_{\text{closed}}^{n+1}(\Delta^m),$$

and let  $\Omega^n(\Delta^m) \rightarrow P$  be the obvious map. We then have the following commutative square.

$$\begin{array}{ccc} \left( \int \tau_{\leq n-1}L \right)_m & \xrightarrow{\hspace{10em}} & \Omega^n(\Delta^m) \\ \downarrow & & \downarrow \\ \text{Hom} \left( \Lambda[m, j] \rightarrow \Delta[m], \int \tau_{\leq n-1}L \rightarrow \int \tau_{<n-1}L \right) & \xrightarrow{\hspace{10em}} & P \end{array} \tag{34}$$

The bottom map assigns to a pair  $(\varphi, \psi)$  the forms  $\beta' \in \Omega^n(\Lambda^{m,j})$ ,  $\gamma' \in \Omega_{\text{closed}}^{n+1}(\Delta^m)$  defined in (31). Then the top map assigns to  $\bar{\varphi} : C^*(\tau_{\leq n-1}L) \rightarrow \Omega^*(\Delta^m)$  the corresponding form  $\bar{\beta}' \in \Omega^n(\Delta^m)$ . By the previous discussion, an extension  $\bar{\varphi}$  of  $\varphi$  compatible with a given  $\psi$ , is equivalent to an extension  $\bar{\beta}'$  of  $\beta'$  compatible with a given  $\gamma'$ . In other words, (34) is a pullback square. The right vertical map is a surjective submersion by Lemma 5.11, therefore so is the left vertical map.

Lastly, we show by induction on  $m$  that  $(\int \tau_{\leq n-1} L)_m$  is a manifold. The space on the lower left corner of (34) is a manifold by Corollary 2.5. The result then follows from Lemma 5.11.  $\square$

LEMMA 5.11. *Let  $P$  be the pullback of*

$$\Omega^n(\Lambda^{m,j}) \xrightarrow{d} \Omega_{\text{closed}}^{n+1}(\Lambda^{m,j}) \longleftarrow \Omega_{\text{closed}}^{n+1}(\Delta^m).$$

*Then the natural map  $\Omega^n(\Delta^m) \rightarrow P$  is a surjective submersion.*

*Proof.* We first show surjectivity. Let  $\alpha \in \Omega^n(\Lambda^{m,j})$ ,  $\beta \in \Omega_{\text{closed}}^{n+1}(\Delta^m)$  be forms satisfying  $d\alpha = \beta|_{\Lambda^{m,j}}$ . We must find an extension  $\bar{\alpha} \in \Omega^n(\Delta^m)$  of  $\alpha$  satisfying  $d\bar{\alpha} = \beta$ .

Let  $p_I : \Delta^m \rightarrow \Lambda^{m,j}$  be as in (24). For  $s \in [0, 1]$ , let  $p_I^s := s \cdot Id + (1 - s)p_I$ , and let  $v_I$  be the vector field given by  $(v_I)_x = x - p_I(x)$ . The form

$$\bar{\alpha}_I := p_I^* \alpha + \int_{s=0}^1 (p_I^s)^* \iota_{v_I} \beta \, ds \tag{35}$$

then satisfies  $\bar{\alpha}_I|_{p_I(\Delta^m)} = \alpha$ , and

$$\bar{\alpha}_I|_{p_I(\Delta^m)} = \bar{\alpha}_{I'}|_{p_I(\Delta^m)}$$

when  $p_I(\Delta^m) \cap p_{I'}(\Delta^m) = p_I(\Delta^m) \cap p_{I'}(\Delta^m)$ . Its differential is given by

$$d\bar{\alpha}_I = p_I^* d\alpha + \int_{s=0}^1 (p_I^s)^* \mathcal{L}_{v_I} \beta \, ds \tag{36}$$

where  $\mathcal{L}$  denotes the Lie derivative. Since  $\beta$  is closed and  $p_I^s$  is the flow generated by  $v_I$ , this becomes

$$d\bar{\alpha}_I = p_I^* d\alpha + (p_I^1)^* \beta - (p_I^0)^* \beta = \beta. \tag{37}$$

Now let  $\bar{\alpha}$  be given by

$$\bar{\alpha} = \sum_{\{j\} \subsetneq I \subset [m]} (-1)^{|I|} \bar{\alpha}_I. \tag{38}$$

By inclusion exclusion, we have  $\bar{\alpha}|_{\Lambda^{m,j}} = \alpha$ , and by (37) it satisfies  $d\bar{\alpha} = \beta$ .

We now show that  $\Omega^n(\Delta^m) \rightarrow P$  is a submersion. Let  $(\alpha^0, \beta^0) \in P$ , and let  $\bar{\alpha}^0 \in \Omega^n(\Delta^m)$  be a preimage. We modify (38) to produce a section  $P \rightarrow \Omega^n(\Delta^m)$  sending  $(\alpha^0, \beta^0)$  to  $\bar{\alpha}^0$ . This can again be done by using an explicit formula:

$$(\alpha, \beta) \mapsto \bar{\alpha}^0 + \sum (-1)^{|I|} (\bar{\alpha}_I - \bar{\alpha}_I^0). \tag{39}$$

### 6. The homotopy groups of $\int L$

From a purely homotopy theoretic point of view, the simplicial manifold  $\int L$  is not very interesting. Each manifold  $(\int L)_m$  is contractible, and so is the geometric realization of  $\int L$ . Indeed, a nullhomotopy  $h_t : \Delta^m \rightarrow \Delta^m$ ,  $t \in [0, 1]$ , induces nullhomotopies  $h_t^* : \Omega^*(\Delta^m) \rightarrow \Omega^*(\Delta^m)$ , and

$$h_t^* \circ - : \text{Hom}_{DGA}(C^*(L), \Omega^*(\Delta^m)) \rightarrow \text{Hom}_{DGA}(C^*(L), \Omega^*(\Delta^m)),$$

proving that each  $(\int L)_m$  is contractible.

The simplicial homotopy groups of  $\int L$  carry much more information. In general, these simplicial homotopy groups can fail to be representable sheaves (see Example 7.10). However, the worse that can happen is that  $\pi_n^{\text{sp1}}(\int L)$  is the quotient of a Lie group by a non-discrete finitely generated subgroup (Theorem 6.4). In that sense, it is never very far from being a Lie group.

If  $X$  is a simplicial manifold, then the space

$$\{x \in X_n \mid d_i(x) = * \forall i\} / \sim \quad (\sim \text{ as in (15)}) \tag{39}$$

is the best possible approximation of  $\pi_n^{\text{spl}}(X)$  by a topological space. If we assume that  $\{x \in X_n \mid d_i(x) = *\forall i\}$  is a submanifold of  $X_n$ , that  $\sim$  is a Hausdorff equivalence relation given by a foliation, and that the space

$$\{x_1, x_2 \in X_n \mid d_i(x_1) = d_i(x_2) = *, x_1 \sim x_2\}$$

admits local lifts into  $X_{n+1}$ , then (39) is actually a manifold, and that manifold represents  $\pi_n^{\text{spl}}(X)$ .

Below are some examples of simplicial manifolds whose simplicial homotopy groups are representable. We just check that the spaces (39) are Lie groups.

*Example 6.1.* Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Then  $\pi_1^{\text{spl}}(\int \mathfrak{g})$  is the universal cover of  $G$ , while  $\pi_n^{\text{spl}}(\int \mathfrak{g}) = \pi_n(G)$  for  $n \geq 2$ .

Indeed, an element of  $\pi_n^{\text{spl}}(\int \mathfrak{g})$  is represented by a map  $f : \Delta^n \rightarrow G$ , which is constant on each facet of  $\Delta^n$ . Two maps are identified if one is a translate of the other, or if they are homotopic relative to  $\partial\Delta^n$ . For  $n \geq 2$ , the above condition means that  $f|_{\partial\Delta^n}$  is constant. There is a unique translate of  $f$  sending  $\partial\Delta^n$  to  $e \in G$ , and we recover the usual definition of  $\pi_n(G)$ . For  $n = 1$ , there is no condition on  $f$ . Again, there is a unique translate sending a given vertex to  $e$ . The group  $\pi_1^{\text{spl}}(\int \mathfrak{g})$  is then the based paths space of  $G$  modulo homotopy fixing the endpoints, that is, the universal cover of  $G$ .

*Example 6.2.* Let  $L$  be as in Example 5.3. Then  $\pi_n^{\text{spl}}(\int L) = L_{n-1}$ , and all other homotopy groups vanish.

An element of  $\pi_m^{\text{spl}}(\int L)$  is represented by a closed  $L_{n-1}$ -valued  $n$ -form  $\omega$  on  $\Delta^m$  that vanishes when restricted to  $\partial\Delta^m$ . Two forms  $\omega, \omega'$  are then equivalent if  $\omega \sqcup \omega'$ , defined on the  $m$ -sphere  $S^m = \Delta^m \sqcup_{\partial\Delta^m} \Delta^m$ , extends to a closed form  $\bar{\omega}$  on the  $(m + 1)$ -disk.

If  $m \neq n$ , then any two forms are equivalent. Indeed, since  $\omega \sqcup \omega'$  is zero in  $H^n(S^m; L_{n-1})$ , we may pick a form  $\alpha$  such that  $d\alpha = \omega$ . Using formulas such as (27) and partitions of unity, we can extend  $\alpha$  to a form  $\bar{\alpha}$  on the  $(m + 1)$ -disk. The desired extension is then given by  $\bar{\omega} = d\bar{\alpha}$ . This shows that  $\pi_m(\int L) = 0$  for  $m \neq n$ .

If  $m = n$ , then by Stokes' theorem any extension  $\bar{\omega}$  must satisfy

$$\int_{D^{n+1}} d\bar{\omega} = \int_{\Delta^n} \omega - \int_{\Delta^n} \omega'. \tag{40}$$

A necessary condition for the existence of a closed extension  $\bar{\omega}$  is for the right-hand side of (40) to vanish. If it does vanish, then the same argument as above shows that  $\omega$  and  $\omega'$  are equivalent. We conclude that  $\omega \sim \omega'$  if and only if their integrals agree, and hence that  $\pi_n(\int L) = L_{n-1}$ .

*Example 6.3.* Let  $L$  be as in Example 5.4. Then all of the simplicial homotopy groups of  $\int L$  vanish.

As above, an element of  $\pi_m^{\text{spl}}(\int L)$  is given by a form  $\omega$  on  $\Delta^m$ , and two forms  $\omega, \omega'$  are equivalent if  $\omega \sqcup \omega'$  extends to the  $(m + 1)$ -disk. There are no obstructions to finding such an extension, and thus any two forms are equivalent.

Even though  $\pi_n^{\text{spl}}(\int L)$  might fail to be a Lie group, it still belongs to a class of group sheaves for which it makes sense to take the Lie algebra. These are quotient sheaves of the form  $G/A$ , where  $G$  is a finite-dimensional Lie group, and  $A$  is a finitely generated subgroup of  $G$ , possibly non-discrete. We call them *finite-dimensional diffeological groups*. The universal cover of a finite-dimensional diffeological group being a Lie group [Sou92], we can define its Lie algebra to be that of its universal cover.

**THEOREM 6.4.** *Let  $L$  be an  $L_\infty$ -algebra, and let  $G$  be the simply connected Lie group integrating  $H_0(L)$ . Then the simplicial homotopy groups  $\pi_n^{\text{spl}}(\int L)$  are finite-dimensional diffeological groups.*

We have  $\pi_1^{\text{spl}}(\int L) = G$ , and a long exact sequence

$$\cdots \rightarrow \pi_{n+1}^{\text{spl}}\left(\int L\right) \rightarrow \pi_{n+1}(G) \rightarrow H_{n-1}(L) \rightarrow \pi_n^{\text{spl}}\left(\int L\right) \rightarrow \pi_n(G) \rightarrow \cdots \tag{41}$$

for  $n \geq 2$ .

*Proof.* We work by induction on the tower (28). The initial case  $L = \tau_{\leq 0}L$  is the computation done in Example 6.1.

Assuming that the theorem holds for  $\tau_{\leq n-1}L$ , we show that it also holds for  $\tau_{< n}L$ . Indeed, the fiber of  $\int \tau_{< n}L \rightarrow \int \tau_{\leq n-1}L$  is the simplicial manifold of Example 5.4. Its homotopy groups being zero by Example 6.3, we conclude by Proposition 3.4 that  $\pi_m^{\text{spl}}(\int \tau_{< n}L) = \pi_m^{\text{spl}}(\int \tau_{\leq n-1}L)$ . Similarly, we have  $\pi_m(\tau_{< n}L) = \pi_m(\tau_{\leq n-1}L)$ . The sequence (41) is therefore the same as that for  $\tau_{\leq n-1}L$ .

Now, we assume that the result holds for  $\tau_{< n-1}L$  and show that it also holds for  $\tau_{\leq n-1}L$ . The fiber of  $\int \tau_{\leq n-1}L \rightarrow \int \tau_{< n-1}L$  is as in Example 5.3, but with  $H_{n-1}(L)$  instead of  $L_{n-1}$ . By Example 6.2, its unique homotopy group is then  $H_{n-1}(L)$  in dimension  $n$ . It follows from Proposition 3.4 that  $\pi_m^{\text{spl}}(\int \tau_{\leq n-1}L) = \pi_m^{\text{spl}}(\int \tau_{< n-1}L)$  for  $m \neq n, n + 1$ , and that we have an exact sequence

$$\begin{aligned} 0 \rightarrow \pi_{n+1}^{\text{spl}}\left(\int \tau_{\leq n-1}L\right) &\rightarrow \pi_{n+1}^{\text{spl}}\left(\int \tau_{< n-1}L\right) \\ &\rightarrow H_{n-1}(L) \rightarrow \pi_n^{\text{spl}}\left(\int \tau_{\leq n-1}L\right) \rightarrow \pi_n^{\text{spl}}\left(\int \tau_{< n-1}L\right) \rightarrow 0. \end{aligned}$$

By assumption, we also have  $\pi_{n+1}^{\text{spl}}(\int \tau_{< n-1}L) = \pi_{n+1}(G)$ . A diagram chase shows the exactness of (41).

To see that  $\pi_n^{\text{spl}}(\int L)$  is a finite-dimensional diffeological group, consider the exact sequence

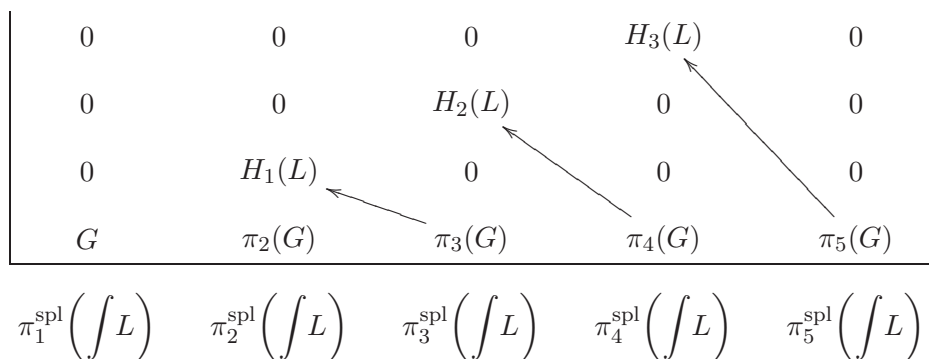
$$0 \rightarrow C \rightarrow H_{n-1}(L) \rightarrow \pi_n^{\text{spl}}\left(\int L\right) \rightarrow K \rightarrow 0,$$

where  $C$  and  $K$  are the finitely generated groups  $\text{coker}(\pi_{n+1}^{\text{spl}}(\int L) \rightarrow \pi_{n+1}(G))$  and  $\text{ker}(\pi_n(G) \rightarrow H_{n-2}(L))$ , respectively. The sheaf  $\pi_n^{\text{spl}}(\int L)$  is an extension of the discrete group  $K$  by the finite-dimensional diffeological group  $H_{n-1}(L)/C$ , and is thus itself a finite-dimensional diffeological group.  $\square$

An alternative, more conceptual proof of Theorem 6.4 can be given using the language of spectral sequences. The Postnikov tower of  $L$  induces a filtration of  $\int L$ . The associated spectral sequence is then

$$E_{m,n}^1 = \pi_m^{\text{spl}}\left(\int \tau_{=n-1}L\right) \Rightarrow \pi_{m+n}^{\text{spl}}\left(\int L\right),$$

where  $\tau_{=n-1}L$  denotes the  $L_\infty$ -algebra with only  $H_{n-1}(L)$  in degree  $n - 1$ . By the computations performed in Examples 6.1 and 6.2, the spectral sequence looks as follows.



As it is sparse, it may be reinterpreted as a long exact sequence.

COROLLARY 6.5. *The Lie algebra of  $\pi_n^{\text{spl}}(\int L)$  is canonically isomorphic to  $H_{n-1}(L)$ .*

*Proof.* Apply the Lie algebra functor to the sequence (41). □

### 7. Integrating Lie $n$ -algebras

A Lie  $n$ -algebra is an  $L_\infty$ -algebra concentrated in degrees  $\{0, \dots, n - 1\}$ . As explained before, the geometric realization of  $\int L$  is contractible. However, if  $L$  is a Lie  $n$ -algebra, there is a modification of  $\int L$  which has a more interesting homotopy type. It is obtained by applying the truncation functor  $\tau_{\leq n}$  of Definition 3.5 to  $\int L$ .

Following our previous convention, our Lie  $n$ -algebras will all be finite dimensional.

DEFINITION 7.1. Let  $L$  be a Lie  $n$ -algebra. Its integrating Lie  $n$ -group is then given by  $\tau_{\leq n} \int L$ .

It can happen that  $(\tau_{\leq n} \int L)_m$  fails to be a manifold (Example 7.10), in which case calling  $\tau_{\leq n} \int L$  a Lie  $n$ -group is an abuse of terminology. However, we know from Lemma 3.6 that the condition in Definition 1.4 is at least satisfied.

Example 7.2. Let  $\mathfrak{g}$  be a Lie algebra, viewed as a Lie 1-algebra, and let  $G$  be the simply connected Lie group with Lie algebra  $\mathfrak{g}$ . Then its integrating Lie 1-group  $\tau_{\leq 1} \int \mathfrak{g}$  is the simplicial manifold  $K(G, 1)$  described in (5).

We just check it at the level of points; it is not much harder to check it at the level of sheaves. We know from (22) that  $(\int \mathfrak{g})_m = \text{Map}(\Delta^m, G)/G$ . Two points of  $(\int \mathfrak{g})_m$  are identified in  $(\tau_{\leq 1} \int \mathfrak{g})_m$  if they are simplicially homotopic relatively to  $\text{sk}_0 \Delta^m$ . As the group  $G$  is simply connected, this happens if and only if their restrictions to  $\text{sk}_0 \Delta^m$  agree. We thus obtain

$$\left( \tau_{\leq 1} \int \mathfrak{g} \right)_m = \text{Map}(\text{sk}_0 \Delta^m, G)/G,$$

which is just another notation for the  $m$ th manifold in (5).

Remark 7.3. For  $n = 1$ , we may generalize the above definition to Lie algebroids. The Lie groupoid corresponding to  $\tau_{\leq 1} \int L$  is then exactly the same as that constructed by Crainic and Fernandez in [CF03]. In that same context, a variant of  $\tau_{\leq 2} \int L$  has been studied by Tseng and Zhu [TZ06] using the formalism of (weak) groupoid objects in the category of smooth étale stacks.

To work better with simplicial sheaves, we extend the Grothendieck pretopology of § 4 to the category of sheaves on manifolds.

DEFINITION 7.4. Let  $X, Y$  be sheaves on the category of Banach manifolds. A map  $X \rightarrow Y$  is called a submersion if for every Banach manifold  $U$ , point  $u \in U$ , element  $y \in Y(U)$ , and lift  $q \in X(\{u\})$  of  $y|_{\{u\}}$ , there exists a neighborhood  $U' \subset U$  of  $u$ , and a lift  $x \in X(U')$  of  $y|_{U'}$  such that  $x|_{\{u\}} = q$ .

$$\begin{array}{ccc} \{u\} & \xrightarrow{q} & X \\ \downarrow & \nearrow x & \downarrow \\ U' \subset U & \xrightarrow{y} & Y \end{array}$$

A map is called a surjective submersion if it is surjective and it is a submersion.

Our next goal is the following result.

**THEOREM 7.5.** *If  $L$  is a Lie  $n$ -algebra, then the sheaf  $(\tau_{\leq n} \int L)_m$  is a manifold for every  $m < n$ .*

Let  $\partial_n : \pi_{n+1}(G) \rightarrow H_{n-1}(L)$  denote the boundary map appearing in the long exact sequence (41). Then  $(\tau_{\leq n} \int L)_n$  is a manifold if and only if  $\partial_n$  has discrete image.

Moreover, if  $\partial_n$  has discrete image, then  $(\tau_{\leq n} \int L)_m$  is a manifold for every  $m$ , and the simplicial manifold  $\tau_{\leq n} \int L$  satisfies the Kan condition.

Given a simplicial manifold  $X$  let us denote by  $Z_n$  the sheaf  $\text{Hom}(\partial\Delta[n+1], X)$ , and by  $B_n$  the image sheaf  $B_n := \text{im}(X_{n+1} \rightarrow Z_n)$ . Letting  $S = \partial\Delta[n+1]$  in Definition 3.1, we obtain an obvious surjection  $p : Z_n \rightarrow \pi_n^{\text{spl}}(X)$ .

**LEMMA 7.6.** *Let  $X$  be a reduced Kan simplicial manifold whose simplicial homotopy groups  $\pi_m^{\text{spl}}(X)$  are finite-dimensional diffeological groups for all  $m \leq n$ . Then  $Z_n$  and  $B_n$  are manifolds, and the map  $p : Z_n \rightarrow \pi_n^{\text{spl}}(X)$  is a surjective submersion.*

*Proof.* We prove the lemma by induction on  $n$ . For  $n = 0$ , the result is trivial. So let  $n \geq 1$ , and let us assume that  $B_{n-1}$  is known to be a manifold. Let  $D[n]$  be the disk

$$D[n] := \Delta[n] \cup (\partial\Delta[n] \times \Delta[1]), \tag{42}$$

and note that  $\partial D[n] = \partial\Delta[n]$ . By Lemma 7.7, the bottom arrow in the pullback

$$\begin{array}{ccc} \text{Hom}(D[n]/\partial D[n], X) & \longrightarrow & \{*\} \\ \downarrow & \lrcorner & \downarrow \\ \text{Hom}(D[n], X) & \longrightarrow & B_{n-1} \end{array}$$

is a submersion. As the sheaf  $\text{Hom}(D[n], X)$  is a manifold by Corollary 2.11, it follows that  $\text{Hom}(D[n]/\partial D[n], X)$  is a manifold.

Let  $f : D[n]/\partial D[n] \rightarrow \partial\Delta[n+1]$  be the obvious weak equivalence, and let  $C_f$  be its mapping cylinder. Applying Corollary 2.11 to the inclusion  $D[n]/\partial D[n] \hookrightarrow C_f$ , we deduce that  $\text{Hom}(C_f, X)$  is a manifold. Since  $\partial\Delta[n+1]$  is a retract of  $C_f$ ,  $\text{Hom}(\partial\Delta[n+1], X)$  is a retract of  $\text{Hom}(C_f, X)$ , and so by Lemma 4.5, it follows that  $Z_n = \text{Hom}(\partial\Delta[n+1], X)$  is a manifold.

Let us use  $\pi_n$  as shorthand notation for  $\pi_n^{\text{spl}}(X)$ . We now show that  $p : Z_n \rightarrow \pi_n$  satisfies the condition in Definition 7.4. Let  $U$  be a manifold,  $u \in U$  be a point, and let us consider  $\gamma \in \pi_n(U)$ ,  $z \in Z_n$  satisfying  $p(z) = \gamma|_{\{u\}}$ . Since  $p$  is surjective,  $\gamma$  lifts to an element  $\tilde{\gamma} \in Z_n(U')$  for some neighborhood  $U' \subset U$  of  $u$ . The two maps

$$z, \tilde{\gamma}|_{\{u\}} : \partial\Delta[n+1] \rightarrow X$$

represent the same element in  $\pi_n(\{u\})$ , so there exists a simplicial homotopy  $h : \partial\Delta[n+1] \times \Delta[1] \rightarrow X$  between them. Let  $H$  be the manifold  $\text{Hom}(\partial\Delta[n+1] \times \Delta[1], X)$  and let  $r_0, r_1 : H \rightarrow Z_n$  be the two restrictions, which are submersions by Corollary 2.11. The element  $h \in H$  maps to  $\tilde{\gamma}|_{\{u\}}$  under  $r_0$  and thus extends to a map  $\theta : U'' \rightarrow H$  satisfying  $\theta(u) = h$ ,  $r_0 \circ \theta = \tilde{\gamma}|_{U''}$ , for some neighborhood  $U'' \subset U'$  of  $u$ . The element

$$\delta := r_1 \circ \theta \in \text{Map}(U'', Z_n) = Z_n(U'')$$

then satisfies  $\delta|_{\{u\}} = z$  and  $p(\delta) = \gamma|_{U''}$ .

We now show that  $B_n$  is a manifold. By assumption,  $\pi_n^{\text{spl}}(X)$  is of the form  $\widetilde{\pi_n^{\text{spl}}(X)}/A$ , with





where  $S \subset \Delta[m]$  is the union of all of the  $n$ -faces of  $\Delta[m]$  incident to a given vertex; it is then a manifold by Lemma 7.8.

The Kan condition in dimensions greater than  $n$  is trivial since the maps (4) are then always isomorphisms. □

We can now finish the proof of our main theorem.

*Proof of Theorem 7.5.* For  $m < n$ , the sheaf  $(\tau_{\leq n} \int L)_m$  is equal to  $(\int L)_m$ , and is thus a manifold by Theorem 5.10.

Note that  $(\tau_{\leq n} \int L)_n$  is a  $\pi_n^{\text{spl}}(\int L)$ -principal bundle over  $(\tau_{< n} \int L)_n$ . We know from Proposition 7.9 that  $(\tau_{< n} \int L)_n$  is a manifold. Hence,  $(\tau_{\leq n} \int L)_n$  is a manifold if and only if  $\pi_n^{\text{spl}}(\int L)$  is a Lie group. This, in turn, is the case if and only if  $\partial_n : \pi_{n+1}(G) \rightarrow H_{n-1}(L)$  has discrete image. Similarly, for  $m > n$ , the sheaf  $(\tau_{\leq n} \int L)_m$  is a  $\pi_n^{\text{spl}}(\int L)^{(m)}$ -principal bundle over  $(\tau_{< n} \int L)_m$  and is thus a manifold under the same assumption.

We now show that  $\tau_{\leq n} \int L$  satisfies the Kan condition. For  $m < n$ , the map

$$\left(\tau_{\leq n} \int L\right)_m \rightarrow \text{Hom}\left(\Lambda[m, j], \tau_{\leq n} \int L\right) \tag{44}$$

is a surjective submersion by Theorem 5.10 since  $\tau_{\leq n} \int L = \int L$  in that range of dimensions. For  $m > n$ , (44) is an isomorphism by Lemma 3.6. Finally, for  $m = n$ , the result follows by Theorem 5.10 and Lemma 4.8 since the composite

$$\left(\int L\right)_n \twoheadrightarrow \left(\tau_{\leq n} \int L\right)_n \rightarrow \text{Hom}\left(\Lambda[n, j], \tau_{\leq n} \int L\right) = \text{Hom}\left(\Lambda[n, j], \int L\right)$$

is then a surjective submersion. □

To complete the picture, we present an example where the hypotheses of Theorem 7.5 are not satisfied, and thus  $\tau_{\leq n} \int L$  is not composed of manifolds.

*Example 7.10.* Let  $\mathfrak{str}$  be as in Definition 8.1, let  $p, q$  be  $\mathbb{Q}$ -linearly independent real numbers, and let  $L$  be the quotient of  $\mathfrak{str} \oplus \mathfrak{str}$  given by  $L_0 = \mathfrak{g} \oplus \mathfrak{g}$  and  $L_1 = \mathbb{R}^2 / (p, q)\mathbb{R}$ . By Lemma 8.2, the boundary homomorphism  $\partial : \pi_3(G \times G) \rightarrow H_1(L)$  is the composite of the standard inclusion  $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$  with the projection  $\mathbb{R}^2 \rightarrow \mathbb{R}^2 / (p, q)\mathbb{R}$ . It does not have a discrete image and so by Theorem 7.5 the sheaf  $(\tau_{\leq 2} \int L)_2$  is not a manifold.

Indeed, anticipating (52), we can write  $(\tau_{\leq 2} \int (\mathfrak{str} \oplus \mathfrak{str}))_2$  as the total space of an  $(S^1 \times S^1)$ -principal bundle. The space  $(\tau_{\leq 2} \int L)_2$  is then the quotient of the above principal bundle by the non-closed subgroup  $(p, q)\mathbb{R} \subset S^1 \times S^1$ . Note also that  $\pi_2^{\text{spl}}(\int L) = \text{coker}(\partial : \pi_3(G \times G) \rightarrow H_1(L))$  is not a Lie group.

### 8. The string Lie 2-algebra

This is our motivating example, which was introduced in [BC04], and studied in [BCSS07]. Let  $\mathfrak{g}$  be a simple real Lie algebra of compact type, and let  $\langle, \rangle$  be the inner product on  $\mathfrak{g}$  such that the norm of its short coroots is 1.

DEFINITION 8.1 (Baez *et al.* [BCSS07]). Let  $\mathfrak{g}$  be a simple Lie algebra of compact type. Its string Lie 2-algebra is the Lie 2-algebra  $\mathfrak{str} = \mathfrak{str}(\mathfrak{g})$  given by

$$\mathfrak{str}_0 = \mathfrak{g}, \quad \mathfrak{str}_1 = \mathbb{R}$$

and brackets

$$\begin{aligned} [(X, c)] &= 0, & [(X_1, c_1), (X_2, c_2)] &= ([X_1, X_2], 0), \\ [(X_1, c_1), (X_2, c_2), (X_3, c_3)] &= (0, \langle [X_1, X_2], X_3 \rangle). \end{aligned}$$

The string Lie 2-algebra should be thought as a central extension of the Lie algebra  $\mathfrak{g}$ , but which is controlled by  $H^3(\mathfrak{g}, \mathbb{R})$  as opposed to  $H^2(\mathfrak{g}, \mathbb{R})$ . The Chevalley–Eilenberg complex of  $\mathfrak{str}$  is given by

$$C^*(\mathfrak{str}) = \mathbb{R} \oplus [\mathfrak{g}^*] \oplus [\Lambda^2 \mathfrak{g}^* \oplus \mathbb{R}] \oplus [\Lambda^3 \mathfrak{g}^* \oplus \mathfrak{g}^*] \oplus [\Lambda^4 \mathfrak{g}^* \oplus \Lambda^2 \mathfrak{g}^* \oplus \mathbb{R}] \oplus \dots$$

Following (20), we study

$$\begin{aligned} \text{Hom}_{DGA}(C^*(\mathfrak{str}), \Omega^*(\Delta^m)) \\ = \{ \alpha \in \Omega^1(\Delta^m; \mathfrak{g}), \beta \in \Omega^2(\Delta^m; \mathbb{R}) \mid d\alpha = \frac{1}{2}[\alpha, \alpha], d\beta = \frac{1}{6}[\alpha, \alpha, \alpha] \}. \end{aligned} \tag{45}$$

The 1-form  $\alpha$  satisfies the Maurer–Cartan equation, so we can integrate it to a map  $f : \Delta^m \rightarrow G$ , defined up to translation, and satisfying  $f^*(\theta_L) = -\alpha$ , where  $\theta_L \in \Omega^1(G; \mathfrak{g})$  is the left invariant Maurer–Cartan form on  $G$ . The 3-form  $\frac{1}{6}[\alpha, \alpha, \alpha]$  is then the pullback of minus the Cartan 3-form

$$\eta = -\frac{1}{6}\langle [\theta_L, \theta_L], \theta_L \rangle \in \Omega^3(G; \mathbb{R}),$$

which represents a generator of  $H^3(G, \mathbb{Z})$ . So we can rewrite (45) as

$$\left( \int \mathfrak{str} \right)_m = \{ f \in \text{Map}(\Delta^m, G)/G, \beta \in \Omega^2(\Delta^m) \mid d\beta = f^*(\eta) \}. \tag{46}$$

In order to understand the simplicial homotopy groups of  $\int \mathfrak{str}$ , we use the long exact sequence constructed in Theorem 6.4. First recall the well-known facts  $\pi_2(G) = 0$ ,  $\pi_3(G) = \mathbb{Z}$ , which hold for any simple compact Lie group. Let us also note that  $H_0(\mathfrak{str}) = \mathfrak{g}$ ,  $H_1(\mathfrak{str}) = \mathbb{R}$  and  $H_n(\mathfrak{str}) = 0$  for  $n \geq 2$ . It follows that  $\pi_n(\int \mathfrak{str}) = \pi_n(G)$  for  $n \geq 4$  and that we have an exact sequence

$$0 \rightarrow \pi_3^{\text{spl}}\left(\int \mathfrak{str}\right) \rightarrow \pi_3(G) \rightarrow H_1(\mathfrak{str}) \rightarrow \pi_2^{\text{spl}}\left(\int \mathfrak{str}\right) \rightarrow 0. \tag{47}$$

LEMMA 8.2. *The boundary homomorphism  $\partial : \pi_3(G) \rightarrow H_1(\mathfrak{str})$  is the inclusion  $\mathbb{Z} \hookrightarrow \mathbb{R}$ .*

*Proof.* Given an element  $a \in \pi_3(G) = \mathbb{Z}$  represented by a map  $f : \Delta^3 \rightarrow G$  sending  $\partial\Delta^3$  to the identity of  $G$ , we lift it to an element  $(f, \beta) \in (\int \mathfrak{str})_3$ . When restricted to  $\partial\Delta^3$ , the map  $f$  becomes constant and we are left with the 2-form  $\beta|_{\partial\Delta^3}$ . This form then represents the element  $\partial(a) = \int_{\partial\Delta^3} \beta \in \mathbb{R} = H_1(\mathfrak{str})$ . From Stokes’ theorem, we then have

$$\partial(a) = \int_{\partial\Delta^3} \beta = \int_{\Delta^3} f^*(\eta) = a$$

as desired. □

Thus we have proven the following.

THEOREM 8.3. *The simplicial homotopy groups of  $\int \mathfrak{str}$  are given by*

$$\begin{aligned} \pi_1^{\text{spl}}\left(\int \mathfrak{str}\right) &= G, & \pi_2^{\text{spl}}\left(\int \mathfrak{str}\right) &= S^1, & \pi_3^{\text{spl}}\left(\int \mathfrak{str}\right) &= 0, \\ \pi_n^{\text{spl}}\left(\int \mathfrak{str}\right) &= \pi_n(G) & \text{for } n &\geq 4. \end{aligned} \tag{48}$$

Moreover, it can be shown [BM94, BM96] that the first  $k$ -invariant of  $\int \mathfrak{str}$  is a generator of  $H^3(K(G, 1), S^1) = H^4(BG, \mathbb{Z}) = \mathbb{Z}$ . Here,  $H^3(K(G, 1), S^1)$  denotes simplicial manifold cohomology<sup>1</sup>

---

<sup>1</sup>Not to be confused with the continuous cohomology of  $G$ !

with coefficients in the sheaf of smooth  $S^1$ -valued functions (see [Fri82, chs 2 and 3] for definitions), while  $H^4(BG, \mathbb{Z})$  denotes the usual topological cohomology of  $BG = |K(G, 1)|$ .

We now study the Lie 2-group  $\tau_{\leq 2} \int \mathfrak{str}$  integrating  $\mathfrak{str}$ , and show that its geometric realization has the homotopy type of  $BString$ . From (48), we have

$$\pi_1^{\text{spl}}\left(\tau_{\leq 2} \int \mathfrak{str}\right) = G, \quad \pi_2^{\text{spl}}\left(\tau_{\leq 2} \int \mathfrak{str}\right) = S^1,$$

and all other simplicial homotopy groups vanish. The Postnikov tower (19) thus induces a fiber sequence

$$K(S^1, 2) \rightarrow \tau_{\leq 2} \int \mathfrak{str} \rightarrow \tau_{< 2} \int \mathfrak{g}. \tag{49}$$

The base  $\tau_{< 2} \int \mathfrak{str} = \tau_{< 2} \int \mathfrak{g}$  is given by  $(\tau_{< 2} \int \mathfrak{g})_m = \text{Map}(\text{sk}_1 \Delta^m, G)/G$ , and the fiber is given by  $(K(S^1, 2))_m = Z^2(\Delta[m], S^1)$ , where  $Z^2(\Delta[m], S^1)$  denotes the space of simplicial 2-cocycles on  $\Delta[m]$  with values in  $S^1$ :

$$K(S^1, 2) = (* \leftarrow * \leftarrow S^1 \leftarrow (S^1)^3 \leftarrow (S^1)^6 \dots).$$

Note that the fiber of the map  $\tau_{< 2} \int \mathfrak{g} \rightarrow \tau_{\leq 1} \int \mathfrak{g} = K(G, 1)$  is simplicially contractible, and that therefore  $|\tau_{< 2} \int \mathfrak{g}|$  is a model for  $BG$ . Also, by applying twice the simplicial path-loop fibration twice, one easily checks that the geometric realization of  $K(S^1, 2)$  is a model for  $K(\mathbb{Z}, 3)$ . Upon geometric realization and up to homotopy, the fiber sequence (49) thus becomes

$$K(\mathbb{Z}, 3) \rightarrow \left| \tau_{\leq 2} \int \mathfrak{str} \right| \rightarrow BG. \tag{50}$$

A version of the following theorem has already appeared in [BCSS07]. We include it here for the sake of completeness.

**THEOREM 8.4.** *The space  $|\tau_{\leq 2} \int \mathfrak{str}|$  has the homotopy type of  $BString$ , where  $String = String_G$  denotes the 3-connected cover of  $G$ .*

*Proof.* Let  $\tau_{> 2} \int \mathfrak{str}$  denote the fiber of the projection  $\int \mathfrak{str} \rightarrow \tau_{\leq 2} \int \mathfrak{str}$ . Its simplicial homotopy groups are then given by

$$\pi_n^{\text{spl}}\left(\tau_{> 2} \int \mathfrak{str}\right) = \begin{cases} \pi_n^{\text{spl}}\left(\int \mathfrak{str}\right) & \text{for } n > 2, \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

Since  $\pi_n^{\text{spl}}(\tau_{> 2} \int \mathfrak{str})$  are discrete groups, the simplicial Postnikov tower of  $\tau_{> 2} \int \mathfrak{str}$  induces the usual Postnikov tower of  $|\tau_{> 2} \int \mathfrak{str}|$  upon geometric realization. It then follows from (48) and (51) that  $\pi_n |\tau_{> 2} \int \mathfrak{str}| = 0$  for  $n \leq 3$ . The total space in the fiber sequence

$$\left| \tau_{> 2} \int \mathfrak{str} \right| \rightarrow \left| \int \mathfrak{str} \right| \rightarrow \left| \tau_{\leq 2} \int \mathfrak{str} \right|$$

is contractible as observed in §6. It follows that  $\pi_n |\tau_{\leq 2} \int \mathfrak{str}| = \pi_{n-1} |\tau_{> 2} \int \mathfrak{str}|$ , and, in particular, that  $\pi_n |\tau_{\leq 2} \int \mathfrak{str}| = 0$  for  $n \leq 4$ .

Since  $\pi_4 |\tau_{\leq 2} \int \mathfrak{str}| = 0$ , the fiber sequence (50) identifies  $|\tau_{\leq 2} \int \mathfrak{str}|$  with the 4-connected cover of  $BG$ . □

Our next task is to understand in explicit terms the manifolds composing  $\tau_{\leq 2} \int \mathfrak{str}$ . Note that for  $m \leq 1$ , the manifold  $(\tau_{\leq 2} \int \mathfrak{str})_m = (\int \mathfrak{str})_m$  has been computed in (46). So we concentrate on  $(\tau_{\leq 2} \int \mathfrak{str})_2$ . Two 2-simplices  $(f_0, \beta_0), (f_1, \beta_1) \in (\int \mathfrak{str})_2$  are identified in  $(\tau_{\leq 2} \int \mathfrak{str})_2$  if they are simplicially homotopic relatively to  $\partial \Delta[2]$ . This happens if there is a homotopy  $F : \Delta^2 \times [0, 1] \rightarrow G$

from  $f_0$  to  $f_1$ , fixing  $\partial\Delta^2$ , and a 2-form  $B$  on  $\Delta^2 \times [0, 1]$  satisfying  $B|_{\Delta^2 \times \partial[0,1]} = \beta_0 \sqcup \beta_1$  and  $dB = F^*\eta$ . By Stokes theorem, such a 2-form  $B$  exists if and only if

$$\int_{\Delta^2 \times [0,1]} F^*\eta = \int_{\Delta^2} \beta_1 - \int_{\Delta^2} \beta_0.$$

So we can forget about the actual 2-form  $\beta$ , and only keep its integral  $b := \int_{\Delta^2} \beta$ . We then obtain

$$\left(\tau_{\leq 2} \int \mathbf{str}\right)_2 = (Map(\Delta^2, G)/G) \times \mathbb{R}/\sim,$$

where  $(f_0, b_0)$  and  $(f_1, b_1)$  are identified if there exists a homotopy  $F$  between  $f_0$  and  $f_1$ , fixing  $\partial\Delta^2$ , and such that  $\int F^*\eta = b_1 - b_0$ . Note that since  $F$  can wrap around  $\pi_3(G) = \mathbb{Z}$ , we have  $(f, b) \sim (f, b + 1)$ , so  $(\tau_{\leq 2} \int \mathbf{str})_2$  is the total space of an  $S^1$ -principal bundle

$$S^1 \rightarrow \left(\tau_{\leq 2} \int \mathbf{str}\right)_2 \rightarrow Map(\partial\Delta^2, G)/G. \tag{52}$$

The fiber of the map  $(Map(\Delta^2, G)/G) \times \mathbb{R} \rightarrow (\tau_{\leq 2} \int \mathbf{str})_2$  is the universal cover of  $\Omega^2 G = Map(\Delta^2/\partial\Delta^2, G)/G$ . Since  $(Map(\Delta^2, G)/G) \times \mathbb{R}$  is contractible, it follows that  $(\tau_{\leq 2} \int \mathbf{str})_2$  is homotopy equivalent to the 2-connected cover of  $\Omega G$ , and that the Chern class of (52) is a generator of  $H^2(Map(\partial\Delta^2, G)/G, \mathbb{Z}) = H^2(\Omega G, \mathbb{Z}) = \mathbb{Z}$ .

We now describe  $(\tau_{\leq 2} \int \mathbf{str})_m$  for arbitrary  $m$ . Given a 2-face  $\sigma : \Delta^2 \rightarrow \Delta^m$ , let  $\mathcal{L}_\sigma$  denote the pullback of (52) along the map  $\sigma^* : Map(\text{sk}_1\Delta^m, G)/G \rightarrow Map(\text{sk}_1\Delta^2, G)/G$ . Given a 3-face  $\tau : \Delta^3 \rightarrow \Delta^m$ , there is a preferred section  $\phi_\tau$  of the tensor product

$$\mathcal{L}_{d_0\tau} \otimes \mathcal{L}_{d_1\tau}^* \otimes \mathcal{L}_{d_2\tau} \otimes \mathcal{L}_{d_3\tau}^*, \tag{53}$$

which we now describe. Let us identify the dual of an  $S^1$ -bundle with the bundle itself equipped with the opposite  $S^1$ -action, and let us call  $a^* \in \mathcal{L}^*$  the element corresponding to  $a \in \mathcal{L}$ . The section of (53) is given by assigning to  $f : \text{sk}_1\Delta^m \rightarrow G$  the element

$$\phi_\tau(f) := (d_0(\tilde{f} \circ \tau), b_0) \otimes (d_1(\tilde{f} \circ \tau), b_1)^* \otimes (d_2(\tilde{f} \circ \tau), b_2) \otimes (d_3(\tilde{f} \circ \tau), b_3)^*$$

defined as follows. Pick an extension  $\tilde{f} : \Delta^m \rightarrow G$  of  $f$  and let  $d_i(\tilde{f} \circ \tau) : \Delta^2 \rightarrow G$  be the  $i$ th face of  $\tilde{f} \circ \tau$ . Then let  $b_i \in \mathbb{R}$  be any numbers satisfying  $b_0 - b_1 + b_2 - b_3 = \int_{\Delta^3} (\tilde{f} \circ \tau)^*\eta$ . The sections  $\phi_\tau$  are well defined and satisfy the cocycle condition saying that, for every 4-face  $\nu : \Delta^4 \rightarrow \Delta^m$  the section  $\phi_{d_0\nu} \otimes \phi_{d_1\nu}^* \otimes \phi_{d_2\nu} \otimes \phi_{d_3\nu}^* \otimes \phi_{d_4\nu}$  is equal to the trivial section in the appropriate trivialized  $S^1$ -bundle.

Given the above structure on the  $S^1$ -bundles  $\mathcal{L}_\sigma$ , we now have the following description of  $(\tau_{\leq 2} \int \mathbf{str})_m$ .

**PROPOSITION 8.5.** *Let  $\mathcal{L}_\sigma$  be the above  $S^1$ -bundle over  $Map(\text{sk}_1(\Delta^m, G)/G)$ , and let  $\mathcal{L}_{\sigma,f}$  denote its fiber over a given point  $f$ . We then have*

$$\left(\tau_{\leq 2} \int \mathbf{str}\right)_m = \{f \in Map(\text{sk}_1\Delta^m, G)/G, (c_\sigma \in \mathcal{L}_{\sigma,f}) \mid \delta c = \phi\}, \tag{54}$$

where  $\delta c = \phi$  means that for every 3-face  $\tau : \Delta^3 \rightarrow \Delta^m$  we have

$$c_{d_0\tau} \otimes c_{d_1\tau}^* \otimes c_{d_2\tau} \otimes c_{d_3\tau}^* = \phi_\tau. \tag{55}$$

*Proof.* Recall that  $(\tau_{\leq 2} \int \mathbf{str})_m$  is a quotient of  $(\int \mathbf{str})_m$  where two elements are identified if they are simplicially homotopic relatively to  $\text{sk}_1\Delta[m]$ . As explained in (46), an element of  $(\int \mathbf{str})_m$  is represented by a pair  $(f, \beta)$  where  $f$  is a map  $\Delta^m \rightarrow G$  and  $\beta$  is a 2-form on  $\Delta^m$  satisfying  $d\beta = f^*\eta$ . Integrating  $\beta$  over the various 2-faces  $\sigma$  of  $\Delta^m$  produces real numbers  $b_\sigma \in \mathbb{R}$ , and the

equivalence class of  $(f, \beta)$  only depends on  $(f, (b_\sigma))$ . By Stokes' theorem, we then have, for every 3-face  $\tau$ , the relation

$$b_{d_0\tau} - b_{d_1\tau} + b_{d_2\tau} - b_{d_3\tau} = \int_{\Delta^3} \tau^* f^* \eta. \tag{56}$$

Given  $(f, (b_\sigma))$  representing an element in the left-hand side of (54), we can assign to it the pair  $(f|_{\text{sk}_1\Delta^m}, (c_\sigma))$ , where  $c_\sigma \in \mathcal{L}_{\sigma,f}$  is the element represented by  $(d_i(f \circ \tau), b_i)$ . Equation (56) then implies (55) by the definition of  $\phi_\tau$ .

Conversely, if we are given  $(f, (c_\sigma))$  in the right-hand side of (54), we first pick an extension  $\tilde{f} : \Delta^m \rightarrow G$ . Let  $S \subset \Delta^m$  be the union of all 2-faces incident to a given vertex. It is a maximal contractible subcomplex of  $\text{sk}_2\Delta^m$ . Since  $S$  is contractible, there are no obstructions to picking a 2-form on  $\Delta^m$  with prescribed integrals on the various 2-faces of  $S$ . So we may pick  $\beta \in \Omega^2(\Delta^m)$  such that for every face  $\sigma : \Delta^2 \rightarrow S$ , the pair  $(\tilde{f} \circ \sigma, \int_{\Delta^2} \sigma^* \beta)$  is a representative of  $c_\sigma$ . From (55) and Stokes' theorem, we conclude that  $(\tilde{f} \circ \sigma, \int_{\Delta^2} \sigma^* \beta)$  is also a representative of  $c_\sigma$  for the other 2-faces  $\sigma$  of  $\Delta^m$ . It follows that the assignment  $(f, (c_\sigma)) \mapsto (\tilde{f}, \beta)$  is an inverse to the previous construction.  $\square$

**COROLLARY 8.6.** *The manifold  $(\tau_{\leq 2} \int \mathfrak{str})_m$  is the total space of an  $(S^1)^{\binom{m}{2}}$ -principal bundle over  $\text{Map}(\text{sk}_1\Delta^m, G)/G$ .*

Note that the abelian group  $(S^1)^{\binom{m}{2}}$  can be naturally identified with  $S^1 \otimes H^1(\text{sk}_1\Delta^m, \mathbb{Z})$ . So we can write our computation of  $\tau_{\leq 2} \int \mathfrak{str}$  schematically as

$$\tau_{\leq 2} \int \mathfrak{str} = [* \Leftarrow \text{Path}(G)/G \Leftarrow \widetilde{\text{Map}(\partial\Delta^2, G)}/G \Leftarrow \widetilde{\text{Map}(\text{sk}_1\Delta^3, G)}/G \cdots] \tag{57}$$

where the tilde indicates that  $\text{Map}(\text{sk}_1\Delta^m, G)$  has been replaced by the total space of an  $(S^1 \otimes H^1(\text{sk}_1\Delta^m, \mathbb{Z}))$ -principal bundle.

If we apply the simplicial loop functor to (57), we obtain another simplicial manifold whose homotopy type is now that of  $\text{String}_G$ :

$$\Omega^{\text{spl}}(\tau_{\leq 2} \int \mathfrak{str}) = [\text{Path}(G)/G \Leftarrow \widetilde{\text{Loop}(G)}/G \Leftarrow \widetilde{\text{Map}(\text{sk}_1(\Sigma\Delta^2), G)}/G \Leftarrow \cdots],$$

or, more pictorially,

$$\text{Path}_*(G) \Leftarrow \widetilde{\text{Loop}_*(G)} \Leftarrow \widetilde{\text{Map}_*(\ominus, G)} \Leftarrow \widetilde{\text{Map}_*(\ominus, G)} \cdots, \tag{58}$$

where the stars denote pointed mapping spaces.

*Remark 8.7.* It was a pleasant surprise to see that the manifolds in (58) come with natural group structures, and that these group structures are preserved by the face and degeneracy maps. Therefore (58) is a simplicial Banach Lie group, and its geometrical realization is a topological group. We do not know whether this is a general phenomenon whether or it is a special feature of the Lie 2-algebra  $\mathfrak{str}$ .

The manifolds in (57) can also be endowed with group structures, but not all face maps are group homomorphisms.

ACKNOWLEDGEMENTS

I would like to thank Allen Knutson for first telling me about the *String* group, Jacob Lurie for explaining to me what  $n$ -groupoids are, Michael Hopkins for showing me simplicial sheaves, Ezra Getzler for a pleasant stay at Northwestern University and for a careful reading of this paper,

Dmitry Roytenberg for conversations at an early stage of this work, and Walter Paravicini for his help with Banach manifolds.

## REFERENCES

- AMR88 R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second edition, Applied Mathematical Sciences, vol. 75 (Springer, New York, 1988).
- AGV72a M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 1, Théorie des topos*. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, vol. 269 (Springer, Berlin, 1972).
- AGV72b M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 2, Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat*, Lecture Notes in Mathematics, vol. 270 (Springer, Berlin, 1972).
- BC04 J. Baez and A. Crans, *Higher-dimensional algebra VI. Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–538 (electronic).
- BCSS07 J. Baez, A. Crans, U. Schreiber and D. Stevenson, *From loop groups to 2-groups*, Homology, Homotopy, and Applications **9** (2007), 101–135.
- BL04 J. Baez and A. D. Lauda, *Higher-dimensional algebra V. 2-groups*, Theory Appl. Categ. **12** (2004), 423–491. (electronic).
- Bek04 T. Beke, *Higher Čech theory*, K-Theory **32**(4) (2004), 293–322.
- BG76 A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179.
- BM94 J.-L. Brylinski and D. A. McLaughlin, *The geometry of degree-four characteristic classes and of line bundles on loop spaces. I*, Duke Math. J. **75**(3) (1994), 603–638.
- BM96 J.-L. Brylinski and D. A. McLaughlin, *The geometry of degree characteristic classes and of line bundles on loop spaces. II*, Duke Math. J. **83**(1) (1996), 105–139.
- CF03 M. Crainic and R. L. Fernandes, *Integrability of lie brackets*, Ann. of Math. **157**(2) (2003), 575–620.
- Dus79 J. Duskin, *Higher-dimensional torsors and the cohomology of topoi: the abelian theory*, Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977), Lecture Notes in Mathematics, vol. 753 (Springer, Berlin, 1979), 255–279.
- For02 M. Forrester-Barker, *Group objects and internal categories*, Preprint (2002), math.CT/0212065.
- Fri82 E. Friedlander, *Étale homotopy of simplicial schemes*, in Annals of Mathematics Studies, vol. 104 (Princeton University Press, Princeton, NJ, University of Tokyo Press, Tokyo, 1982).
- GZ67 P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (Springer, New York, 1967).
- Get03 E. Getzler, *Lie theory for nilpotent  $L_\infty$ -algebras*, Ann. of Math. (2), to appear.
- Gle82 P. G. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure Appl. Algebra **25**(1) (1982), 33–105.
- GJ99 P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174 (Birkhäuser, Basel, 1999).
- Hin97 V. Hinich, *Descent of Deligne groupoids*, Internat. Math. Res. Notices **5** (1997), 223–239.
- HS93 V. Hinich and V. Schechtman, *Homotopy Lie algebras*, in *I. M. Gel'fand Seminar*, Advances in Soviet Mathematics, vol. 16, part 2 (American Mathematical Society, Providence, RI, 1993).
- Hov99 M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63 (American Mathematical Society, Providence, RI, 1999).
- Jar86 J. Jardine, *Simplicial objects in a Grothendieck topos*, in *Applications of algebraic K-theory to algebraic geometry and number theory*, Contemporary Mathematics, vol. 55, Part I (1986), 193–239.



- Joh77 P. T. Johnstone, *Topos theory*, London Mathematical Society Monographs, vol. 10 (Academic Press, New York, 1977).
- Lan72 S. Lang, *Differential manifolds*, (Addison-Wesley, Reading, MA, 1972).
- MM94 S. M. Lane and I. Moerdijk, *Sheaves in geometry and logic. A first introduction to topos theory*, Universitext (Springer, New York, 1994), corrected reprint of the 1992 edition.
- May92 J. P. May, *Simplicial objects in algebraic topology*, Chicago Lectures in Mathematics, (University of Chicago Press, Chicago, IL, 1992), Reprint of the 1967 original.
- Moo58 J. C. Moore, *Semi-simplicial complexes and postnikov systems*, in *Symposium internacional de topología algebraica (International symposium on algebraic topology)*, Universidad Nacional Autónoma de México and UNESCO, Mexico City (1958), 232–247.
- Sey80 R. M. Seymour, *Kan fibrations in the category of simplicial spaces*, *Fund. Math.* **106**(2) (1980), 141–152.
- Sou92 J.-M. Souriau, *Groupes différentiels*, Lecture Notes in Mathematics, vol. 336 (Springer, Berlin, 1992).
- Sta92 J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, in *Quantum groups*, Lecture Notes in Mathematics, vol. 1510 (Springer, Berlin, 1992).
- Sto96 S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, *Math. Ann.* **304**(4) (1996), 785–800.
- ST04 S. Stolz and P. Teichner, *What is an elliptic object?* in *Topology, geometry and quantum field theory*, Mathematical Society Lecture Note Series, vol. 308 (Cambridge University Press, Cambridge, London 2004), 247–343.
- Sul77 D. Sullivan, *Infinitesimal computations in topology*, *Publ. Math. Inst. Hautes Études Sci.* **47** (1977), 269–331.
- TZ06 H.-H. Tseng and C. Zhu, *Integrating lie algebroids via stacks*, *Compositio Math.* **142**(1) (2006), 251–270.
- Van77 D. H. Van Osdol, *Simplicial homotopy in an exact category*, *Amer. J. Math.* **6**(99) (1977), 1193–1204.
- Was98 A. Wassermann, *Operator algebras and conformal field theory. III. Fusion of positive energy representations of  $LSU(N)$  using bounded operators*, *Invent. Math.* **133** (1998), 467–538.

André Henriques A.G.Henriques@uu.nl

Mathematisch Instituut, Universiteit Utrecht, PO Box 80.010 NL-3508, TA Utrecht,  
The Netherlands