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CATCHING AND MISSING FINITE SETS

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ABSTRACT. If T is a 1–1 bimeasurable measure-preserving aperiodic transformation on a probability space X which is a Lebesgue space, then $\{A : A \subset X$ and for almost every pair of finite sets F and G there is an $n \in N$ satisfying $F \subset T^nA$ and $G \cap T^nA = \phi\}$ is dense in the σ-algebra of measurable sets.

Let T be a 1–1 bimeasurable measure-preserving aperiodic transformation on a measure space X whose total measure is finite (by aperiodic we mean $\{x : \exists n \in \mathbb{N}^+$ with $T^n(x) = x\}$ has measure zero). Assume the σ-algebra on X is countably generated mod sets of measure zero and separates points, and that the measure is atomless. For simplicity assume X has measure one. In [3], Steele shows that for any such transformation and $\varepsilon > 0$ there is a set A of measure less than ε such that for every finite $F \subset X$ there is an $n \in \mathbb{N}$ with $F \subset T^nA$. That is, there is a set A as small as we please which can "catch" every finite set, whence the collection of sets which can catch every finite set is dense. The purpose of this paper is to strengthen Steele's result to assert that for any such transformation the collection of sets which can "simultaneously catch and miss" almost any pair of finite sets is dense. The proof which will be given was motivated by Steele's work.

The only fact about aperiodic transformations which will be used is the following version of Rohlin's Theorem, a proof of which appears in [1, p. 71].

LEMMA. *Let T be a 1–1 aperiodic bimeasurable measure-preserving transformation on a probability space. For every $\varepsilon > 0$ and any $n \in \mathbb{N}^+$ there is a measurable set E such that $E, TE, \ldots, T^{n-1}E$ are pairwise disjoint and the measure of $\bigcup_{i=0}^{n-1} T^iE$ exceeds $1 - \varepsilon$.*

Throughout this paper T will denote a 1–1 bimeasurable measure-preserving aperiodic transformation on a probability space X, and μ will denote the measure.

THEOREM 1. *For every $\varepsilon > 0$ there is an $A \subset X$ with $\mu(A) < \varepsilon$ such that for almost every pair of finite sets of points F and G there is an $n \in \mathbb{N}$ satisfying $F \subset T^nA$ and $G \cap T^nA = \emptyset$. ("Almost every" means that for all $p, q \in \mathbb{N}$ with*

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$p+q\geq 1$ the set $\{(x_1,\ldots,x_p,x_{p+1},\ldots,x_{p+q}):\exists n\in\mathbb{N}$ satisfying $\{x_1,\ldots,x_p\}\subset T^nA$ and $\{x_{p+1},\ldots,x_{p+q}\}\cap T^nA=\phi\}$ has measure one in the product measure on X^{p+q}).

Proof. Assume $1>\varepsilon>0$. For each $n\in\mathbb{N}$, let $\varepsilon_n=2^{-n-1}\varepsilon$. Choose increasing sequences of natural numbers $\{v_n\}_0^\infty$ and $\{h_n\}_0^\infty$ satisfying the following conditions. Let $v_0=h_0=1$. Suppose $v_0,\ldots,v_n,h_0,\ldots,h_n$ have been chosen. Choose

$$v_{n+1}>h_n\varepsilon_{n+1}^{-1}. \tag{1}$$

Let $r_{n+1,0}=h_n$, and for $0\leq k\leq n+1$ let

$$s_{n+1,k+1}=2^{k+1}v_{n+1}r_{n+1,k}$$

and

$$r_{n+1,k+1}=s_{n+1,k+1}+r_{n+1,k}.$$

Let $h_{n+1}=r_{n+1,n+2}\prod_{i=1}^{n+2}s_{n+1,i}$. Note that h_n is a factor of h_{n+1} and

$$h_{n+1}>v_{n+1}r_{n+1,n+2}. \tag{2}$$

The construction of A will now commence.

For each $n\in\mathbb{N}^+$ Rohlin's Theorem and (1) and (2) allow us to choose a Rohlin tower for T of height h_{n+1} with base set B_{n+1} (hence the levels of the tower are T^iB_{n+1}, $0\leq i<h_{n+1}$) satisfying

$$\mu\left(\bigcup_{i=0}^{h_{n+1}-1-r_{n+1,n+2}} T^iB_{n+1}\right)>1-\varepsilon_{n+1}h_n^{-1}. \tag{3}$$

Let

$$Z_{n+1}=X-\bigcup_{i=0}^{h_{n+1}-1-r_{n+1,n+2}} T^iB_{n+1}.$$

For each $n\in\mathbb{N}^+$, let

$$C_{n+1}=Z_{n+1}\cup\bigcup_{k=0}^{n}\ \bigcup_{i=0}^{(h_{n+1}/s_{n,k+1})-1}\ \bigcup_{j=0}^{-1+r_{n,k}} T^{j+is_{n,k+1}}B_{n+1}.$$

Then (1) and (3) imply that for all $n\in\mathbb{N}^+$

$$\begin{aligned}\mu(C_{n+1})&<\varepsilon_{n+1}h_n^{-1}+\sum_{k=0}^{n} r_{n,k}s_{n,k+1}^{-1}\\&=\varepsilon_{n+1}h_n^{-1}+\sum_{k=0}^{n}(2^{k+1}v_n)^{-1}\\&<\varepsilon_{n+1}h_n^{-1}+\varepsilon_nh_{n-1}^{-1}<\varepsilon_{n-1}h_{n-1}^{-1}.\end{aligned} \tag{4}$$

Let $D_0=\phi$, and for $n\in\mathbb{N}$ let $D_{n+1}=(D_n\cup C_{2n+2})-C_{2n+3}$. Finally, let

$$A=\bigcup_{k=0}^{\infty}\bigcap_{n=k}^{\infty}D_n.$$

(Equivalently,

$$A=\bigcup_{n=1}^{\infty}\left(C_{2n}-\bigcup_{j=0}^{\infty}C_{2n+1+2j}\right),$$

or, up to a set of measure zero, $A=\lim_{n\to\infty}D_n$). The construction of the C_n implies the following holds:

(*) For all $n\in\mathbb{N}^+$, if $m\le n+1$ then for every set of points $\{x_1,\ldots,x_m\}$ and natural number z, $z\le r_{n+1,n+2}-h_n$, there is a natural number w, $w\le r_{n,n+1}-h_{n-1}$, such that

$$\{x_1,\ldots,x_m\}\subset\bigcap_{i=0}^{h_{n-1}-1}T^{z+w+i}C_{n+1}.$$

To prove (*) fix $n\in\mathbb{N}^+$, $z\le r_{n+1,n+2}-h_n$ and $\{x_1,\ldots,x_m\}$ with $m\le n+1$; for $1\le j\le m$ let t_j be the smallest natural number satisfying

$$x_j\in\bigcap_{i=0}^{-1+r_{n,n+1-j}}T^{z+t_1+\cdots+t_j+i}C_{n+1}.$$

The construction of C_{n+1} implies that each t_j is well-defined and that $t_j\le s_{n,n+2-j}$. Let

$$w=\sum_{j=1}^{m}t_j.$$

Then

$$w\le\sum_{j=1}^{m}s_{n,n+2-j}=r_{n,n+1}-r_{n,n+1-m}$$

and

$$\{x_1,\ldots,x_m\}\subset\bigcap_{i=0}^{-1+r_{n,n+1-m}}T^{z+w+t}C_{n+1}.$$

Since $r_{n,n+1-m}\ge r_{n,0}=h_{n-1}$, (*) is proven.

For $n\in\mathbb{N}^+$ let $Y_{n+1}=\bigcup_{i=r_{n+1,n+2}}^{h_{n+1}-1-r_{n+1,n+2}}T^iB_{n+1}$. Note that Y_{n+1} can be thought of as the set of points contained in the Rohlin tower consisting of $r_{n,n+1}$ levels with base level L_{n+1}, where

$$L_{n+1}=\bigcup_{i=r_{n+1,n+2}r_{n,n+1}^{-1}-1}^{-1+(h_{n+1}-r_{n+1,n+2})r_{n,n+1}^{-1}}T^{ir_{n,n+1}}B_{n+1}.$$

For $n\in\mathbb{N}^+$, if F is a finite subset of X call F $(n+1)$-good if $|F|\leq n+1$ and $F\subset Y_{n+1}$. Equation (3) implies

$$\mu(Y_{n+1})>1-2\varepsilon_{n+1}h_n^{-1}$$

hence with probability one F is n-good for all but finitely many n. The construction of the C_n implies that for each $n\in\mathbb{N}^+$, if F and F' are $(n+1)$-good sets which intersect precisely the same L_{n+1}-levels, then for each natural number u less than $r_{n+1,n+2}$, $F\subset T^uC_{n+1}$ if and only if $F'\subset T^uC_{n+1}$. (That is, if F is $(n+1)$-good and $u<r_{n+1,n+2}$, the containment of F in T^uC_{n+1} is completely determined by the L_{n+1}-levels which contain point(s) in F.) This implies the following:

(**) Let $M\subset X$. For each $n\in\mathbb{N}^+$ and natural number z, $z\leq r_{n+1,n+2}-h_n$, if F is a randomly chosen $(n+1)$-good set and w satisfies (*) (for $n+1$, F, z), the probability that $F\cap T^{z+w}M=\phi$ is at least $1-|F|\,\mu(M)\mu(L_{n+1})^{-1}$. To prove (**), let $\bar{M}=Y_{n+1}\cap T^{z+w}M$; thus $\bar{M}$ and F are contained in the Rohlin tower above L_{n+1}. Let $\hat{M}$ and $\hat{F}$ denote the projections of $\bar{M}$ and F (respectively) down to L_{n+1}, the base of the tower. The fact that w satisfies (*) (for $n+1$, F, z) does not affect the random distribution of $\hat{F}$ in L_{n+1}: this fact gives information about the L_{n+1}-levels F can intersect, but no information about where in those levels F lies. Thus, the probability that $\hat{F}\cap\hat{M}=\phi$ equals $(1-\mu(\hat{M})\mu(L_{n+1})^{-1})^{|\hat{F}|}$. Since $\mu(\hat{M})\leq\mu(M)$, $|\hat{F}|\leq|F|$, and $\hat{F}\cap\hat{M}=\phi$ implies $F\cap T^{z+w}M=\phi$, (**) is proven.

It will now be shown that A satisfies the conditions stated in the theorem. Equation (4) implies

$$\mu(A)\leq\sum_{n=1}^{\infty}\mu(C_{2n})<\sum_{n=0}^{\infty}\varepsilon_n=\varepsilon.$$

Let F and G be two finite sets of points, and assume F is $(2n)$-good and G is $(2n+1)$-good (with probability one F and G are n-good for all n sufficiently large). Then (*) implies that natural numbers z and w can be chosen satisfying $z\leq r_{2n,2n+1}-h_{2n-1}$, $w\leq r_{2n-1,2n}-h_{2n-2}$,

$$G\subset\bigcap_{i=0}^{n_{2n-1}}T^{z+i}C_{2n+1}\tag{5}$$

and

$$F\subset T^{z+w}C_{2n}.\tag{6}$$

Let $u=z+w$ (note u depends on n). Equation (5) implies $G\cap T^uA=\phi$ if

$$G\cap T^u\left(\bigcup_{k=n}^{\infty}C_{2k+2}\right)=\phi\tag{7}$$

and by (6), $F\subset T^uA$ if

$$F\cap T^u\Big(\bigcup_{k=n}^{\infty} C_{2k+1}\Big)=\phi. \tag{8}$$

Inequalities (2) and (4) and (**) imply that the probability that (7) holds exceeds

$$\begin{aligned}
&1-|G|\,\mu\Big(\bigcup_{k=n}^{\infty} C_{2k+2}\Big)\mu(L_{2n+1})^{-1} \\
&>1-|G|\Big(\sum_{k=n}^{\infty}\varepsilon_{2k}h_{2k}^{-1}\Big)(\tfrac{2}{3}r_{2n,2n+1}^{-1})^{-1} \\
&>1-|G|\,(\tfrac{4}{3}\varepsilon_{2n}h_{2n}^{-1})(\tfrac{3}{2}r_{2n,2n+1}) \\
&=1-|G|\,\varepsilon_{2n-1}h_{2n}^{-1}r_{2n,2n+1} \\
&>1-|G|\,\varepsilon_{2n-1}.
\end{aligned} \tag{9}$$

Similarly, (2), (4) and (**) imply the probability that (8) holds exceeds

$$1-|F|\,\varepsilon_{2n-2}. \tag{10}$$

Thus, if F is $(2n)$-good and G is $(2n+1)$-good, (9) and (10) imply the probability that $F\subset T^uA$ and $G\cap T^uA=\phi$ exceeds

$$1-|F|\,\varepsilon_{2n-2}-|G|\,\varepsilon_{2n-1}.$$

Since with probability one F and G will be n-good for all sufficiently large n, with probability one the probability that for some $v\in\mathbb{N}$ $F\subset T^vA$ and $G\cap T^vA=\phi$ exceeds $1-|F|\,\varepsilon_{2n-2}-|G|\,\varepsilon_{2n-1}$ for all n, and the theorem follows.

The proof of Theorem 1 yields more than is stated in Theorem 1. A Borel–Cantelli argument will show that the set constructed actually catches and misses almost every pair of finite sets for infinitely many positive integers and infinitely many negative integers. Furthermore, by setting $D_0=B$ (B arbitrary), the set A constructed will catch and miss almost every pair of finite sets infinitely often and will satisfy $\mu(A\Delta B)<\varepsilon$. These observations are incorporated into the following two theorems, which strengthen Theorem 1.

THEOREM 2. *There is a sequence of sets A_n with $A_{n+1}\subset A_n$ and $\mu(A_n)\to 0$ such that each A_n catches and misses almost every pair of finite sets for infinitely many positive and infinitely many negative integers.*

Proof. Let

$$A_n=\bigcup_{i=n}^{\infty}\Big(\hat{C}_{2i+2}-\bigcup_{j=0}^{\infty}\hat{C}_{2i+3+2j}\Big),$$

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where

$$\hat{C}_n = C_n \cup \bigcup_{i=0}^{-1+r_{n,n+1}} T^i B_n,$$

and change Y_n so that it equals $\bigcup_{i=2r_{n,n+1}}^{h_n-1-2r_{n,n+1}} T^i B_n$. The proof of the Theorem 1 will now go through for T and T^{-1}. Q.E.D.

Actually we need not replace C_n with $\hat{C}_n$, but if we do then no extra argument is needed for the negative integers, since $\hat{C}_n$ satisfy (*) for T^{-1} as well as T.

THEOREM 3. *The sets which can catch and miss almost every pair of finite sets for infinitely many positive and infinitely many negative integers are dense* (*in the σ-algebra of measurable sets*).

Proof. Given a measurable set B, let $B_n = (B - \bigcup_{i=2n+2}^{\infty} \hat{C}_i) \cup A_n$, where $\hat{C}_i$ and A_n are as in the proof of Theorem 2. Then B_n has the desired property, and by (3) and (4)

$$\mu(\hat{C}_{n+1}) < \mu(C_{n+1}) + \varepsilon_{n+1} < \varepsilon_{n-1} + \varepsilon_{n+1}$$

whence

$$\mu(B \Delta B_n) \leq \mu\left(\bigcup_{i=2n+2}^{\infty} \hat{C}_i\right) < \sum_{i=2n+2}^{\infty} (\varepsilon_{i-2} + \varepsilon_i) = 5\varepsilon_{2n+1}.$$

Alternatively, in light of the comment following Theorem 2, in the proof of Theorem 1 simply let $D_0 = B$ to get an A with the desired property satisfying $\mu(B \Delta A) < \varepsilon$.

If T is weakly mixing there is no need for the construction in Theorem 1, for the following is true.

THEOREM 4. *If T is weakly mixing then for every set A with $0 < \mu(A) < 1$, for almost every pair of finite sets of points F and G, $\{n : n \in \mathbb{Z}$ and $F \subset T^n A$ and $G \cap T^n A = \phi\}$ has density $\mu(A)^{|F|}(1-\mu(A))^{|G|}$ in the positive* (*or negative*) *integers.*

Proof. Let $|F| = p$, $|G| = q$, $p + q \geq 1$. If T is weakly mixing, then $\hat{T} = T \times \cdots \times T$ ($p + q$ T's) is ergodic (in fact weakly mixing). Given A with $0 < \mu(A) < 1$, let $\hat{A} = A \times \cdots \times A \times A^c \times \cdots \times A^c$ (p A's and q A^c's); by the Individual Ergodic Theorem [1, p. 18] for almost every $(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q})$ the set $\{n : (x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) \in \hat{T}^n \hat{A}\}$ has density $\mu(\hat{A})$ in the positive or negative integers. Note n is in the above set if and only if $\{x_1, \ldots, x_p\} \subset T^n A$ and $\{x_{p+1}, \ldots, x_{p+q}\} \cap T^n A = \phi$. The theorem follows by choosing $(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q})$ so that $F = \{x_1, \ldots, x_p\}$ and $G = \{x_{p+1}, \ldots, x_{p+q}\}$.

The converse of Theorem 4 is also true: if T is not weakly mixing then there is a set A with $0<\mu(A)<1$ which cannot catch and miss almost every pair of finite sets. Such a set can easily be constructed using the fact that such T do not have continuous spectrum [1, p. 39], hence support a non-constant eigenfunction.

Note that Theorem 1 cannot in general be strengthened to assert the existence of a set of small measure which separates *every* pair of disjoint finite sets. Any such set A would separate points, hence the partition $\{A, A^c\}$ would generate the σ-algebra of measurable sets. However, if the entropy of T exceeds $\varepsilon \log \varepsilon^{-1}+(1-\varepsilon)\log(1-\varepsilon)^{-1}$ and $\mu(A)\leq\varepsilon$ (assume $\varepsilon\leq\frac{1}{2}$) then A cannot generate, hence cannot separate points. Thus, if T has positive entropy it cannot have arbitrarily small sets which catch and miss every pair of disjoint singleton sets, and if the entropy of T is greater than $\log 2$ no set may catch and miss every pair of disjoint singleton sets.

The results proved in this paper can however be strengthened in the following direction. Using Rohlin's Theorem for non-singular aperiodic transformations (for example, Theorem 1.11 of [2]) and a construction similar to but more complicated than that in the proof of Theorem 1, the following result can be established.

THEOREM 5. *Let T be a* 1–1 *aperiodic bimeasurable non-singular transformation on a probability space X, and assume X is a Lebesgue space. Then*

$$\{A\colon \textit{for almost every finite } H\subset X, \textit{ for every } k\in\mathbb{N} \textit{ and sequence } H_0,\ldots,H_k \textit{ of subsets of } H \textit{ there is an } n\in\mathbb{N} \textit{ such that } H_i=H\cap T^{n+i}A \textit{ for } 0\leq i\leq k\}$$

is dense (*in the σ-algebra of measurable sets*).

By setting $H=F\cup G$, $k=0$ and $H_0=F$, the result stated in the abstract is attained.

Theorem 4 showed that if T is measure-preserving and weakly mixing then every non-trivial set A can catch and miss almost every finite set. The collection of sets described in Theorem 5 however can never contain every non-trivial set: its easy to show that if A is in this collection then for every $\varepsilon>0$ there are sets B and C with $B\subset A\subset C$, $u(C-B)<\varepsilon$, and for all D, if $D\subseteq B$ or $C\subseteq D$ then D is not in the collection.

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