BULL. AUSTRAL. MATH. SOC. 30D50
VOL. 33 (1986), 253-260.

A NOTE ON THE COEFFICIENTS OF MIXED NORMED SPACES

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For $0 < p,q < \infty$, $\alpha > -1$, $A^{p,q,\alpha}$ denotes the space of all holomorphic functions in the unit disc satisfying

$$||f||_{p,q,\alpha}^{p} = \int_0^1 M_q(r,f)^p (1-r)^\alpha \, dr < \infty ,$$

where

$$M_q(r,f)^q = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta .$$

In this paper, we find a sufficient condition for the multipliers from $A^{p,q,\alpha}$ into ℓ^s, $1 \le s \le \infty$, $1 \le q \le 2$, which interpolates the results of Patrick Ahern and Miroljub Jevtić. As a corollary, we can calculate

$$(A^{p,q,\alpha}, \ \ell^s)$$

for $q' \le s \le \infty$, $1/q + 1/q' = 1$. Also, we can find a sharp coefficient condition for H^p functions.

1. Introduction.

$H(U)$ denotes the space of all holomorphic functions in the unit disc U. For a function $f(z) \in H(U)$ and for $\alpha > -1$, $0 < p,q < \infty$,

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Received 2 June 1985.

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we let

$$M_p(r,f)^p = 1/2\pi \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta ,$$

$$||f||_p = \sup_r M_p(r,f)$$

and

$$||f||^p_{p,q,\alpha} = \int_0^1 M_q(r,f)^p \, (1-r)^\alpha \, dr .$$

The spaces $H^p(U)$ and $A^{p,q,\alpha}(U)$ are defined to be

$$H^p(U) = \{f \in H(U) \; ; \; ||f||_p < \infty \}$$

$$A^{p,q,\alpha} = \{f \in H(U) \; ; \; ||f||_{p,q,\alpha} < \infty \}.$$

These spaces form Banach spaces or Frechet spaces. We refer to [3], [1] for properties of these spaces.

Let A, B be two vector spaces of complex sequences. A sequence $\lambda = \{\lambda_n\}$ is said to be a multiplier from A into B if $\{\lambda_n a_n\} \in B$ for any $\{a_n\} \in A$. The space of all such multipliers is denoted by (A,B). We want to calculate multipliers from H^p or $A^{p,q,\alpha}$ into $\ell(s,t)$, the space defined below.

DEFINITION. For $1 \le s,t \le \infty$, we denote by $\ell(s,t)$ the set of those sequences $\{a_k\}_0^\infty$ for which

$$\{ (\sum_{k \in I_n} |a_k|^s)^{1/s} \}_{n=0}^\infty \in \ell^t \quad (s < \infty)$$

and

$$\{ \sup_{k \in I_n} |a_k| \}_{n=0}^\infty \in \ell^t \quad (s = \infty) ,$$

where $I_n = \{k \; ; \; 2^n < k \le 2^{n+1}\}$ $(n = 1,2,\dots)$ and $I_0 = \{0\}$.

The $\ell(s,t)$ form normed spaces. For dual spaces and multipliers between these spaces we refer to [5]. We follow Anderson and Shields [2] for notation and many results. Let A be a sequence space. A^a is defined to be the space of sequences $\{a_n\}$ for which

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$$\lim_{r \to 1-} \Sigma \; a_n r^n$$

exists and A^k is defined to be (A, ℓ^1) . $s(A)$ is defined to be the largest subspace of A having the property that if $\{a_n\} \varepsilon \; s(A)$ and $|b_n| \leq |a_n|$ then $\{b_n\} \; \varepsilon \; s(A)$. Similarly $S(A)$ is defined to be the smallest superspace having this property. If $s(A) = S(A)$, we call the space A solid. Of course, the $\ell(s,t)$ are solid. It is known [2] that for a solid space X with $X^{kk} = X$

$$(A,X) = (S(A),X) = (A^{kk}, \; X) = (s(A^a)^a \; , \; X) \; .$$

Note that $f \; \varepsilon \; H(U)$ can be identified as a sequence $\{a_n\}$ if $f(z) = \sum_0^\infty \; a_n \; z^n$. We denote $\{(n+1)^{-p} \; a_n\}$ by $I^p f$ and $\{(n+1)^p a_n\}$ by $I^{-p} f$, the fractional integral and the fractional derivative of f of order p. Also, for a space $S \subset H(U)$ we denote $\{I^p f; \; f \; \varepsilon \; S\}$ by $I^p S$ and for two sequence spaces A,B we denote $\{ \{a_n b_n\}; \; \{a_n\} \varepsilon \; A, \{b_n\} \varepsilon \; B\}$ by $A*B$. Note that

$$\{ \; (n+1)^{-p} \; \} \; \varepsilon \ell(1/p, \infty)$$

so that

$$I^p \; A \subset \ell(1/p, \infty) \; * \; A \; .$$

Throughout this paper, $1/p + 1/p' = 1 (1 \leq p \leq \infty)$ and $C(p,q,\ldots)$ denotes a positive constant depending only on $p,q,\ldots$, but its size may vary under the same notation.

2. Results

Ahern and Jevtić[1] have calculated multipliers from $A^{p,q,\alpha}$ into ℓ^s in the case $0 < p < \infty$, $q = 1,2$. They prove that if $r = \max(p,1)$ then

$$(s(A^{p,1,\alpha})^a)^a = \{ \; \{\lambda_n\}; \quad \{(k+1)^{\frac{\alpha+1}{p}} \; \lambda_k\} \varepsilon \ell(\infty, r)\}$$

$$(s(A^{p,q,\alpha})^a)^a = \{ \; \{\lambda_k\}; \; \{ \; (k+1)^{-\frac{\alpha+1}{p}} \; \lambda_k\} \varepsilon \ell(2,r)\} \;\; (q \geq 2)$$

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(Indeed, they proved these for $1 < p < \infty$ and remarked on the case $0 < p \leq 1$. See [1] Remarks.) Noting that

$$A^{p,q,\alpha} \subset (s(A^{p,q,\alpha})^a)^a ,$$

it is natural to conjecture the following:

THEOREM 1. *For* $1 \leq q \leq 2$, $A^{p,q,\alpha} \subset I^{-(\alpha+1)/p}\, \ell(q', \max(p,1))$, *where* $1/q + 1/q' = 1$.

Proof. Note that $\lambda = \{\lambda_n\} \in I^{-(\alpha+1)/p}\, \ell(q',p)$ if and only if

$$\sum_0^\infty \Big(\sum_{k \in I_n} |(k+1)^{-(\alpha+1)/p} \lambda_k|^{q'}\Big)^{p/q'} < \infty .$$

Let $f(z) = \sum_0^\infty a_n z^n \in H(U)$. Then

$$\begin{aligned} &\sum \Big(\sum_{I_n} |(k+1)^{-(\alpha+1)/p} a_k|^{q'}\Big)^{p/q'} \\ &\leq \sum 2^{-n(\alpha+1)} \Big(\sum_{I_n} |a_k|^{q'}\Big)^{p/q'} . \end{aligned}$$

Applying the result of Mateljević and Pavlović [6], this term is dominated by

$$C(p,q,\alpha) \int_0^1 \Big(\sum_0^\infty |a_k|^{q'} r^k\Big)^{p/q'} (1-r)^\alpha \, dr .$$

Since $\sum_0^\infty a_k r^{k/q'} z^k = f(r^{1/q'} z)$, the Hausdorff-Young theorem [3. Theorem 6.1] gives

$$\begin{aligned} \Big(\sum_0^\infty |a_k|^{q'} r^k\Big)^{1/q'} &= \Big(\sum_0^\infty |a_k r^{k/q'}|^{q'}\Big)^{1/q'} \\ &\leq \|f(r^{1/q'} z)\|_q \end{aligned}$$

for $1 < q < 2$. On the other hand, if we let $f_{r^{1/q'}}(z) = f(r^{1/q'} z)$,

simple calculations give

$$\begin{aligned} &\int_0^1 \|f(r^{1/q'} z)\|_q^p (1-r)^\alpha \, dr \\ &= \int_0^1 \sup_\rho M_q(\rho, f_{r^{1/q'}})^p (1-r)^\alpha \, dr \\ &= \int_0^1 M_q(r^{1/q'}, f)^p (1-r)^\alpha \, dr \end{aligned}$$

$$= q' \int_0^1 M_q(\rho,f)^p \ (1-\rho^{q'})^\alpha \ \rho^{q'-1} \ d\rho$$

$$\le (q')^{1+\alpha} \int_0^1 M_q(\rho,f)^p \ (1-\rho)^\alpha \ d\rho \ ,$$

where we used the fact that $(1-\rho^{q'}) \le q'(1-\rho)$ in the last inequality. Thus we have

$$\Sigma \ (\sum_{I_n} |(k+1)^{-\frac{\alpha+1}{p}} a_k|^{q'})^{\frac{p}{q'}} \le C(p,q,\alpha) \ ||f||^p_{p,q,\alpha} \ .$$

This process can also be applied when $p \le 1$ by the duality method aforementioned ([1] Remarks). The proof is now complete.

COROLLARY 1. *If* $0 < p \le 2$ *and* $p < q$, $1 \le q \le \infty$, *then*

(1) $$H^p \subset I^{1/q-1/p} \ \ell(q',r) \ ,$$

where $r = \max(p,1)$. *That is,* $f = \{a_n\} \in H^p$, *then*

$$\sum_n (\sum_{I_n} |(k+1)^{\frac{1}{q}-\frac{1}{p}} a_k|^{q'})^{\frac{r}{q'}} < \infty \ ,$$

with the obvious understanding when $q = 1$ *or* $q = \infty$.

Proof. First we note that it suffices to prove (1) for $1/q-1/p$ small. Indeed, if $\{a_k\} \in I^{1/q_1 -1/p} \ \ell(q_1',r)$ and $1 \le q_1 \le q_2 \ \infty$, then

$$\{(k+1)^{1/q_1-1/p} \ a_k\} \in \ell(q_1'.r) \ ;$$

Since

$$(k+1)^{1/q_2-1/q_1} \in \ell(\frac{1}{1/q_1-1/q_2}, \ \infty) = (\ell(q_1',r), \ (q_2',r)),$$

we have

$$\{(k+1)^{1/q_2-1/p} \ a_k\} = \{(k+1)^{1/q_2-1/q_1} \ (k+1)^{1/q_1-1/p} \ a_k\}$$

$$\in \ell(\frac{1}{1/q_1-1/q_2}, \ \infty) * \ell(q_1',r)$$

$$\subset \ell(q_2' \ ,r) \ .$$

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Thus, we have proven that

$$I^{1/q_1-1/p}\,\ell(q_1',r) \subset I^{1/q_2-1/p}\,\ell(q_2',r)$$

in the case that $1 \le q_1 < q_2 \le \infty$.

Now, if $0 < p < 2$, the Hardy-Littlewood theorem [3. Theorem 5.11] and Theorem 1 gives

$$H^p \subset A^{p,q,-p/q} \subset I^{1/q-1/p}\,\ell(q',r)$$

by taking $q < 2$. Finally, the remaining case when $p = 2$ is easy

$$I^{1/2-1/q}\,H^2 \subset \ell\left(\frac{1}{1/2-1/q'}, \infty\right) * \ell^2$$

$$\subset \ell(q',2) ,$$

because

$$\ell\left(\frac{1}{1/2-1/q'}, \infty\right) = (\ell^2, \ell(q',2)) .$$

Hence

$$H \subset I^{1/q-1/2}\,\ell(q', 2) .$$

Since $H^{p'} \supset (H^p)^k = (H^p, \ell^1)$ if $1 < p \le 2$, we have the dual form of (1) as follows.

COROLLARY 2.

(2) $$H^p \supset I^{1/q-1/p}\,\ell(q',p) , \quad 2 \le p < \infty , \quad 1 \le q < p.$$

(3) $$BMOA(U) \supset I^{1/q}\,\ell(q',\infty) , \quad 1 \le q < \infty$$

[where $BMOA(U)$ is the space of analytic functions on U having bounded mean oscillation].

Remarks. 1. If we set $q = 2$ in (1) and (2) then we have $H^p \subset D^p$ for $p < 2$ and $H^p \supset D^p$ for $p > 2$. (See [1], [4] for D^p). Thus Corollary 1 and Corollary 2 are stronger than [3. Theorems 6.2, 6.3] and [4. Theorems C,D].

2. The limiting case of (1) namely $H^p \subset \ell(p',p)$ $(1 \le p < 2)$ is not true (thus neither is the limiting case of (2)): If we suppose $H^p \subset \ell(p',r)$ then $S(H^p) \subset \ell(p',r)$, so

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$$\ell^\infty \subset (S(H^p),\ \ell(p',r)\ ) = (H^p,\ \ell(p',r)\ )\ .$$

But $H^2 \subset H^p \subset \ell(p',2)$ [5] gives

$$(H^2,\ell(p',r)) \supset (H^p,\ell(p',r)) \supset (\ell(p',2),\ell(p',r))\ .$$

Hence

$$(H^p,\ell(p',r)) = \begin{cases} \ell(\infty,2r/2-r) & \text{if } 1 \le r \le 2\ , \\ \ell^\infty & \text{if } 2 \le r \le \infty\ . \end{cases}$$

Therefore $\ell^\infty \subset (H^p,\ell(p',r))$ only if $2 \le r \le \infty$. That is there is no ordered pair (p',r) $r < 2$ satisfying $H^p \subset \ell(p',r)$. We can say (1) is sharp in this sense. Also, if $0 < p < 1$, then from (1) we have $H^p \subset I^{1-1/p}\ \ell(\infty,1)$, but $I^{1-1/p}\ \ell(\infty,1)$ is $(H^p)^{kk}$ ([3. Theorem 6.6] Recall that $A^k = (A,\ell^1))$.

COROLLARY 3. *Let* $1 \le q \le 2$ *and* $r = max(p,1)$. *Then*

(4) $$(A^{p,q,\alpha},\ell^s) \supset I^{\alpha+1/p}\ (\ell(q',r),\ell^s),\ 1 \le s \le \infty\ ,$$

(5) $$(A^{p,q,\alpha},\ell^s) = I^{\alpha+1/p}\ \ell(\infty,\ t)\ ,\qquad q' \le s \le \infty\ .$$

where $t = sp/p-s$ *if* $s \le p$

and $t = \infty$ *if* $s \ge p$.

Proof. (4) is obvious from Theorem 1. We prove (5). Jensen's inequality gives that $M_q(\rho,f) \le M_2(\rho,f)$ so that $A^{p,2,\alpha} \subset A^{p,q,\alpha}$, which in turn gives

$$(A^{p,q,\alpha},\ell^s) \subset (A^{p,2,\alpha},\ell^s) = (I^{-(\alpha+1)/p}\ \ell(2,r),\ell^s\ )$$
$$= I^{\alpha+1/p}\ (\ell(2,r),\ell^s)\ .$$

Combining this with (4) gives (5).

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https://doi.org/10.1017/S0004972700003129 Published online by Cambridge University Press