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A BIHARMONIC EQUATION WITH SINGULAR NONLINEARITY

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Abstract We study the biharmonic equation $\Delta^2 u = u^{-\alpha}$, $0 < \alpha < 1$, in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, subject to Dirichlet boundary conditions. Under some suitable assumptions on Ω related to the positivity of the Green function for the biharmonic operator, we prove the existence and uniqueness of a solution.

Keywords: biharmonic operator; singular nonlinearity; Green function; integral equation

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Secondary 45G05; 47H10

1. Introduction and the main results

In this paper we study the biharmonic elliptic problem

$$\left.\begin{aligned} \Delta^2 u = u^{-\alpha}, \quad u > 0 \text{ in } \Omega,\\ u = \partial_\nu u = 0 \text{ on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ is a smooth bounded domain in the sense that we will describe in the following, ν is the exterior unit normal at $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$ is the outer normal derivative at $\partial\Omega$.

The case of the Laplace equation with singular nonlinearities that corresponds to (1.1) is well understood. More precisely, it is shown in [**5**] that the problem

$$-\Delta u = u^{-\alpha}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a unique solution for all $\alpha > 0$. Boundary behaviour and nearly optimal regularity of this solution have been investigated in [**16**], where it is proven that there exist $c_1, c_2, M > 0$ such that

$$\begin{aligned} c_1\delta(x) \leqslant u(x) \leqslant c_2\delta(x) & \quad \text{in } \Omega \quad (\text{if } 0 < \alpha < 1),\\ c_1\delta(x)\log^{1/2}\left(\frac{M}{\delta(x)}\right) \leqslant u(x) \leqslant c_2\delta(x)\log^{1/2}\left(\frac{M}{\delta(x)}\right) & \quad \text{in } \Omega \quad (\text{if } \alpha = 1),\\ c_1\delta^{2/(1+\alpha)}(x) \leqslant u(x) \leqslant c_2\delta^{2/(1+\alpha)}(x) & \quad \text{in } \Omega \quad (\text{if } 0 < \alpha < 1). \end{aligned}$$

For a comprehensive account of second-order elliptic equations involving singular nonlinearities, we refer the reader to [**8**].

We denote by $G(\cdot\,,\cdot)$ the Green function associated with the biharmonic operator Δ^2 subject to Dirichlet boundary conditions, that is, for all $y \in \Omega$, $G(\cdot\,,y)$ satisfies in the distributional sense:

$$\begin{aligned} \Delta^2 G(\cdot\,,y) &= \delta_y(\cdot) && \text{in } \Omega, \\ G(\cdot\,,y) &= \partial_\nu G(\cdot\,,y) = 0 && \text{on } \partial\Omega. \end{aligned}$$

The study of the Green function for the biharmonic equation dates back to Boggio [**1**] in 1901. He proved that the Green function is positive in any ball of $\mathbb{R}^n$. Boggio [**2**] and Hadamard [**17**] conjectured that this fact should be true at least in any smooth convex domain of $\mathbb{R}^n$.

Since the late 1940s, various counter-examples have been constructed that disprove the Boggio–Hadamard conjecture. For instance, if a domain in $\mathbb{R}^2$ has a right angle, then the associated Green function fails to be everywhere positive [**3**]. A similar result holds for thin ellipses: Garabedian [**7**] found that in an ellipse in $\mathbb{R}^2$ with the ratio of the half axes being approximately 2, the Green function for the biharmonic operator changes sign (for an elementary proof, see also [**21**]). In turn, if the ellipse is close to a ball in the plane, Grunau and Sweers [**12**] proved that the Green function is positive. Recently, Grunau and Sweers [**13**–**15**] and Grunau and Robert [**10**] provided interesting characterizations of the regions where the Green function is negative. They also obtained that if a domain is sufficiently close to the unit ball in a suitable $C^{4,\gamma}$-sense, then the biharmonic Green function under Dirichlet boundary condition is positive.

It is worth noting here that the positivity property of the Green function for the biharmonic operator is a special feature of the prescribed boundary condition. Indeed, if, instead of the Dirichlet boundary condition, one assumes the Navier boundary condition (that is, $u = \Delta u = 0$ on $\partial\Omega$), then a straightforward application of the second-order comparison principle yields the positivity of the Green function. However, even under Navier conditions there is in general no positivity result for the Green function when the biharmonic operator is perturbed (see, for example, [**4**, **18**]).

In this paper we assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded domain that satisfies the following assumptions:

(A1) the boundary $\partial\Omega$ is of class C^{16} if $n = 2$ and of class C^{12} if $n \geqslant 3$;

(A2) the Green function $G(\cdot\,,\cdot)$ is positive.

The assumption (A1) on the regularity of $\partial\Omega$ dates back to Krasovskiĭ [**19**] and is taken from [**6**], where sharp upper bounds for the Green function are obtained. The need for condition (A2) will become clearer once we specify what it is understood by a solution of (1.1). We say that u is a solution of (1.1) if

$$u \in C(\bar{\Omega}), \quad u > 0 \text{ in } \Omega,$$

and u satisfies the integral equation

$$u(x) = \int_{\Omega} G(x,y) u^{-\alpha}(y)\,\mathrm{d}y \quad \text{for all } x \in \Omega. \tag{1.2}$$

The restriction $0 < \alpha < 1$ is needed in order to make the integral in (1.2) finite. It will appear several times in the proofs in the following sections. Note also that condition (A2) implies the standard maximum principle for the biharmonic operator in Ω.

Our main result concerning (1.1) is the following.

Theorem 1.1. *Assume that $0 < \alpha < 1$ and conditions (A1), (A2) hold. Then, the problem (1.1) has a unique solution u and there exist $c_1, c_2 > 0$ such that*

$$c_1 \delta^2(x) \leqslant u(x) \leqslant c_2 \delta^2(x) \quad \text{in } \Omega, \tag{1.3}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Moreover, $u \in C^2(\bar{\Omega})$ and if $0 < \alpha < \frac{1}{2}$, then $u \in C^3(\bar{\Omega})$.

The existence of a solution will be obtained by means of the Schauder fixed-point theorem. Towards this aim, we employ the sharp estimates for the Green function given in [**6**]. The uniqueness relies heavily on the boundary estimate (1.3), which is obtained by using the behaviour of the Green function (see Proposition 2.2).

We leave open the case $\alpha \geqslant 1$. We believe that in this case the study of (1.1) is more delicate: the solution will have a different boundary behaviour (this has already been seen in the case of the corresponding Laplace equation) and a weaker regularity in $\bar{\Omega}$.

The remaining part of the paper is organized as follows. In § 2 we derive some preliminary results concerning (1.1). Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

In this section we collect some useful results regarding problem (1.1). The first result in this sense is due to Dall'Acqua and Sweers [**6**, Theorem 12, Lemma C.2] and provides upper bounds for the Green function of the biharmonic operator subject to Dirichlet boundary conditions.

Proposition 2.1 (Dall'Acqua and Sweers [6]). *Let k be an n-dimensional multi-index. Then, there exists a positive constant c depending on Ω and k such that, for any $x, y \in \Omega$, we have the following.*

(i) *For $|k| \geqslant 2$:*

(a) *if $n > 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c|x-y|^{4-n-|k|} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2;$$

(b) *if $n = 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c \log\left(2 + \frac{\delta(y)}{|x-y|}\right) \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2;$$

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(c) *if $n < 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c\delta(y)^{4-n-|k|} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{n+|k|-2}.$$

(ii) *For $|k| < 2$:*

(1) *if $n > 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c|x-y|^{4-n-|k|} \min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^{2-|k|} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{2};$$

(2) *if $n = 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c \log\left(2 + \frac{\delta(y)}{|x-y|}\right) \min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^{2-|k|} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{2};$$

(3) *if $2(2-|k|) \leqslant n < 4 - |k|$, then*

$$|D_x^k G(x,y)| \leqslant c\delta(y)^{4-n-|k|} \min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^{2-|k|} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{n+|k|-2};$$

(4) *if $n < 2(2-|k|)$, then*

$$\begin{aligned}|D_x^k G(x,y)| &\leqslant c\delta^{2-|k|-n/2}(x)\delta^{2-n/2}(y)\\ &\qquad\times \min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^{n/2} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{n/2}.\end{aligned}$$

Let φ_1 be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. It is well known that φ_1 has constant sign in Ω, so by a suitable normalization we may assume $\varphi_1 > 0$ in Ω. Therefore, φ_1 satisfies

$$\left.\begin{aligned} -\Delta\varphi_1 = \lambda_1\varphi_1, \quad \varphi_1 > 0 \quad &\text{in } \Omega,\\ \varphi_1 = 0 \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta)$. By the Hopf Maximum Principle [**20**] we have $\partial_\nu\varphi_1 < 0$ on $\partial\Omega$. Also, by the regularity of Ω we have $\varphi_1 \in C^4(\bar{\Omega})$ and

$$c\delta(x) \leqslant \varphi_1(x) \leqslant \frac{1}{c}\delta(x) \quad \text{in } \Omega, \tag{2.2}$$

for some $0 < c < 1$.

Proposition 2.2. *Let u be a solution of problem (1.1). Then, there exist $c_1, c_2 > 0$ such that u satisfies (1.3).*

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Proof. Let $a(x) = \varphi_1^2(x)$, $x \in \bar{\Omega}$. It is easy to see that, since $\varphi_1 \in C^4(\bar{\Omega})$, $f := \Delta^2 a$ is bounded in $\bar{\Omega}$; so, by the continuity of u there exists $m > 0$ small enough such that

$$u(x) - ma(x) = \int_\Omega G(x,y)[u^{-\alpha}(y) - mf(y)]\,\mathrm{d}y \geqslant 0 \quad \text{for all } x \in \Omega.$$

Therefore,

$$u(x) \geqslant ma(x) \geqslant c_0\delta^2(x) \quad \text{in } \Omega, \tag{2.3}$$

for some $c_0 > 0$. This proves the first part of the inequality in (1.3). For the second part, assume first $n > 4$ and let $x \in \Omega$. Using Proposition 2.1 (1), for all $y \in \Omega$ we have

$$\begin{aligned} G(x,y) &\leqslant c|x-y|^{2-n}\delta^2(x)\min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2 \\ &\leqslant c|x-y|^{2-n}\delta^2(x)\min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{2\alpha} \\ &= c|x-y|^{2-2\alpha-n}\delta^2(x)\delta^{2\alpha}(y). \end{aligned} \tag{2.4}$$

Now, from (2.3) and (2.4) we have

$$\begin{aligned} u(x) &= \int_\Omega G(x,y)u^{-\alpha}(y)\,\mathrm{d}y \\ &\leqslant c_1\int_\Omega G(x,y)\delta^{-2\alpha}(y)\,\mathrm{d}y \\ &\leqslant c_2\delta^2(x)\int_\Omega |x-y|^{2-2\alpha-n}\,\mathrm{d}y \\ &\leqslant c_2\delta^2(x)\int_{0\leqslant|x-y|\leqslant\operatorname{diam}(\Omega)} |x-y|^{2-2\alpha-n}\,\mathrm{d}y \\ &= c_2\delta^2(x)\int_0^{\operatorname{diam}(\Omega)} t^{1-2\alpha}\,\mathrm{d}t \leqslant c_3\delta^2(x). \end{aligned} \tag{2.5}$$

Now let $n = 4$. We use Proposition 2.1 (2) to derive an inequality similar to (2.4). More precisely, for all $y \in \Omega$ we have

$$\begin{aligned} G(x,y) &\leqslant c\log\left(2 + \frac{\delta(y)}{|x-y|}\right)\min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^2\min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{2\alpha} \\ &\leqslant c|x-y|^{-2-2\alpha}\delta^2(x)\delta^{2\alpha}(y)\log\left(2 + \frac{\operatorname{diam}(\Omega)}{|x-y|}\right). \end{aligned} \tag{2.6}$$

If $n = 3$, let $\beta = \max\{0, 2\alpha - \frac{1}{2}\} < \frac{3}{2}$, and by Proposition 2.1 (4) we have

$$G(x,y) \leqslant c\delta^{1/2}(x)\delta^{1/2}(y)\min\left\{1, \frac{\delta(x)}{|x-y|}\right\}^{3/2}\min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{3/2} \tag{2.7}$$

$$\begin{aligned} &\leqslant c|x-y|^{-3/2-\beta}\delta^2(x)\delta^{\beta+1/2}(y) \\ &\leqslant C|x-y|^{-3/2-\beta}\delta^2(x)\delta^{2\alpha}(y). \end{aligned} \tag{2.8}$$

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Finally, if $n = 2$, let $\beta = \max\{0, 2\alpha - 1\} < 1$, and by Proposition 2.1 (3) we have

$$\begin{aligned} G(x,y) &\leqslant c\delta(x)\delta(y)\min\left\{1, \frac{\delta(x)}{|x-y|}\right\}\min\left\{1, \frac{\delta(y)}{|x-y|}\right\} \\ &\leqslant c|x-y|^{-1}\delta^2(x)\delta(y)\min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{\beta} \\ &\leqslant c|x-y|^{-1-\beta}\delta^2(x)\delta^{1+\beta}(y) \\ &\leqslant C|x-y|^{-1-\beta}\delta^2(x)\delta^{2\alpha}(y). \end{aligned} \tag{2.9}$$

We now use the estimates (2.6)–(2.9) to derive a similar inequality to that in (2.5).

This completes the proof of Proposition 2.2. □

Proposition 2.3. *Let $0 < \alpha < 1$ and $u \in C(\bar{\Omega})$ be such that $u(x) \geqslant c_0\delta^2(x)$ in Ω for some $c_0 > 0$. Consider*

$$w(x) = \int_{\Omega} G(x,y)u^{-\alpha}(y)\,\mathrm{d}y \quad \textit{for all } x \in \bar{\Omega}.$$

Then

(i) $w \in C^2(\bar{\Omega})$;

(ii) $w \in C^3(\bar{\Omega})$ *for any* $0 < \alpha < \frac{1}{2}$.

Proof. With a proof analogous to that of Proposition 2.2 it is easy to see that v is well defined. For $0 < \varepsilon < 1$ small, define $\Omega_\varepsilon = \{x \in \bar{\Omega} \colon \delta(x) < \varepsilon\}$. Set $u_\varepsilon = \max\{u, c_0\varepsilon^2\}$ and

$$w_\varepsilon(x) = \int_{\Omega} G(x,y)u_\varepsilon^{-\alpha}(y)\,\mathrm{d}y \quad \text{for all } x \in \bar{\Omega}.$$

It is easy to see that $w_\varepsilon = w$ on $\Omega \setminus \Omega_\varepsilon$. Since $u_\varepsilon^{-\alpha}$ is bounded in $\bar{\Omega}$, by the estimates in Proposition 2.1 it follows that $w_\varepsilon \in C^3(\bar{\Omega})$ and

$$D_x^k w_\varepsilon(x) = \int_{\Omega} D_x^k G(x,y)u_\varepsilon^{-\alpha}(y)\,\mathrm{d}y \quad \text{for all } x \in \bar{\Omega}$$

for any n-dimensional multi-index k with $|k| \leqslant 3$. The proof of this fact is similar to that of [**9**, Lemma 4.1]. We employ in the following the same approach as in [**9**] to show that $w \in C^2(\bar{\Omega})$ (respectively, $w \in C^3(\bar{\Omega})$ if $0 < \alpha < \frac{1}{2}$).

First assume that $n > 4$ and let k be an n-dimensional multi-index with $|k| \leqslant 2$. Fix $\beta > 0$ such that $2\alpha < \beta < 2$.

By Proposition 2.1 (a) (if $|k| = 2$) and Proposition 2.1 (1) (if $|k| \leqslant 1$) we have

$$
\begin{aligned}
\left| D_x^k w_\varepsilon(x) - \int_\Omega D_x^k G(x,y) u^{-\alpha}(y)\,\mathrm{d}y \right| &\leqslant \int_{\Omega_\varepsilon} |D_x^k G(x,y)|(u^{-\alpha}(y) + (c_0\varepsilon^2)^{-\alpha})\,\mathrm{d}y \\
&\leqslant c_1\varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{4-|k|-n} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2 \mathrm{d}y \\
&\leqslant c_1\varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{4-|k|-n} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^\beta \mathrm{d}y \\
&\leqslant c_1\varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{4-|k|-\beta-n} \delta^\beta(y)\,\mathrm{d}y \\
&\leqslant c_1\varepsilon^{\beta-2\alpha} \int_{\Omega} |x-y|^{4-|k|-\beta-n}\,\mathrm{d}y \\
&\leqslant c_1\varepsilon^{\beta-2\alpha} \int_{0\leqslant|x-y|\leqslant\mathrm{diam}(\Omega)} |x-y|^{4-|k|-\beta-n}\,\mathrm{d}y \\
&\leqslant c_1\varepsilon^{\beta-2\alpha} \int_0^{\mathrm{diam}(\Omega)} t^{3-|k|-\beta}\,\mathrm{d}t \\
&\leqslant c_2\varepsilon^{\beta-2\alpha} \int_0^{\mathrm{diam}(\Omega)} t^{1-\beta}\,\mathrm{d}t \\
&\leqslant c_3\varepsilon^{\beta-2\alpha} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

The case $2 \leqslant n \leqslant 4$ can be analysed in the same way. For instance, if $n = 3$ and $|k| = 1$, we use Proposition 2.1 (2) to derive

$$
\begin{aligned}
&\left| D_x^k w_\varepsilon(x) - \int_\Omega D_x^k G(x,y) u^{-\alpha}(y)\,\mathrm{d}y \right| \\
&\qquad \leqslant c_1\varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} \log\left(2 + \frac{\delta(y)}{|x-y|}\right) \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2 \mathrm{d}y \\
&\qquad \leqslant c_1\varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{-\beta}\delta^\beta(y) \log\left(2 + \frac{\delta(y)}{|x-y|}\right) \mathrm{d}y \\
&\qquad \leqslant c_1\varepsilon^{\beta-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{-\beta} \log\left(2 + \frac{\mathrm{diam}(\Omega)}{|x-y|}\right) \mathrm{d}y \\
&\qquad \leqslant c_2\varepsilon^{\beta-2\alpha} \int_0^{\mathrm{diam}(\Omega)} t^{2-\beta} \log\left(2 + \frac{\mathrm{diam}(\Omega)}{t}\right) \mathrm{d}t \\
&\qquad \leqslant c_3\varepsilon^{\beta-2\alpha} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

We have obtained that

$$
D_x^k w_\varepsilon \to \int_\Omega D_x^k G(\cdot\,,y) u^{-\alpha}(y)\,\mathrm{d}y \quad \text{uniformly as } \varepsilon \to 0
$$

for any n-dimensional multi-index k with $0 \leqslant |k| \leqslant 2$. It follows that $w \in C^2(\bar{\Omega})$ and

$$D_x^k w(x) = \int_\Omega D_x^k G(x,y) u^{-\alpha}(y)\,\mathrm{d}y \quad \text{for all } x \in \bar{\Omega}$$

for any multi-index k with $0 \leqslant |k| \leqslant 2$.

To prove (ii), let k be a multi-index with $|k| = 3$ and $2\alpha < \beta < 1$. From Proposition 2.1 (a) we have

$$\begin{aligned}
\left| D_x^k w_\varepsilon(x) - \int_\Omega D_x^k G(x,y) u^{-\alpha}(y)\,\mathrm{d}y \right| &\leqslant 2(c_0\varepsilon^2)^{-\alpha} \int_{\Omega_\varepsilon} |D_x^k G(x,y)|\,\mathrm{d}y \\
&\leqslant c_1 \varepsilon^{-2\alpha} \int_{\Omega_\varepsilon} |x-y|^{1-n} \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^\beta \mathrm{d}y \\
&\leqslant c_1 \varepsilon^{\beta-2\alpha} \int_\Omega |x-y|^{1-n-\beta}\,\mathrm{d}y \\
&\leqslant c_1 \varepsilon^{\beta-2\alpha} \int_0^{\mathrm{diam}(\Omega)} t^{-\beta}\,\mathrm{d}t \\
&\leqslant c_2 \varepsilon^{\beta-2\alpha} \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

since $\beta < 1$. With the same arguments as above we find $w \in C^3(\bar{\Omega})$. This ends the proof. □

3. Proof of Theorem 1.1

Let $a(x) = \varphi_1^2(x)$, $x \in \bar{\Omega}$. Motivated by Proposition 2.2, we seek solutions u of (1.1) in the form

$$u(x) = a(x)v(x),$$

where $v \in C(\bar{\Omega})$, $v > 0$ in $\bar{\Omega}$. This leads us to the following integral equation for v:

$$v(x) = \frac{1}{a(x)} \int_\Omega \frac{G(x,y)}{a^\alpha(y)} v^{-\alpha}(y)\,\mathrm{d}y \quad \text{for all } x \in \bar{\Omega}. \tag{3.1}$$

We can now regard (3.1) as the fixed-point problem

$$\mathcal{F}(v) = v,$$

where

$$\mathcal{F}(v) = \frac{1}{a(x)} \int_\Omega \frac{G(x,y)}{a^\alpha(y)} v^{-\alpha}(y)\,\mathrm{d}y.$$

Recall that $\mathcal{F}$ is an integral operator of the form

$$\mathcal{F}(v) = \int_\Omega K(x,y) v^{-\alpha}(y)\,\mathrm{d}y,$$

where the kernel K is given by

$$K\colon \bar{\Omega}\times\Omega\to[0,\infty],\quad K(x,y)=\begin{cases}\dfrac{G(x,y)}{a(x)a^{\alpha}(y)} & \text{if } x,y\in\Omega,\\[2mm] \dfrac{\partial_\nu^2 G(x,y)}{\partial_\nu^2 a(x)a^{\alpha}(y)} & \text{if } x\in\partial\Omega,\ y\in\Omega.\end{cases}$$

Note that K is well defined since $\partial_\nu^2 a(x)=2(\partial_\nu\varphi_1(x))^2>0$ on $\partial\Omega$.

We first need the following result.

Lemma 3.1.

(i) *For any $y\in\Omega$, the function $K(\cdot\,,y)\colon\bar{\Omega}\to[0,\infty]$ is continuous.*

(ii) *The mapping*

$$\bar{\Omega}\ni x\mapsto\int_\Omega K(x,y)\,\mathrm{d}y$$

is continuous and there exists $M>1$ such that

$$\frac{1}{M}\leqslant\int_\Omega K(x,y)\,\mathrm{d}y\leqslant M\quad\text{for all } x\in\bar{\Omega}. \tag{3.2}$$

Proof. Recall first that the biharmonic Green function

$$G\colon\Omega\times\Omega\setminus\{(z,z)\colon z\in\Omega\}\to(0,\infty)$$

is continuous. Also, by the estimates in [**11**, Theorem 1] we have

$$\lim_{(x,y)\to(z,z)}G(x,y)=+\infty\quad\text{for all } z\in\Omega.$$

Hence, $G\colon\Omega\times\Omega\to(0,\infty]$ is continuous (in the extended sense). Therefore, $K(\cdot\,,y)$ is continuous (in the extended sense) in Ω. It remains to prove the continuity of $K(\cdot\,,y)$ on $\partial\Omega$. Let $\varepsilon>0$. Since $G(\cdot\,,y)\in C^4(\bar{\Omega}\setminus\{y\})$ and $a\in C^4(\bar{\Omega})$, for any $z\in\partial\Omega$ we have

$$\begin{aligned}G(z+t\nu,y)&=t^2(\tfrac12\partial_\nu^2 G(z,y)+G_1(z,t))\quad\text{as } t\nearrow 0,\\ a(z+t\nu,y)&=t^2(\tfrac12\partial_\nu^2 a(z,y)+a_1(z,t))\qquad\text{as } t\nearrow 0,\end{aligned}$$

where

$$\lim_{t\nearrow 0}G_1(z,t)=\lim_{t\nearrow 0}a_1(z,t)=0\quad\text{uniformly for } z\in\partial\Omega.$$

Hence, as $t\nearrow 0$ we have

$$\begin{aligned}|K(z+t\nu,y)-K(z,y)|&=\left|\frac{\frac12\partial_\nu^2 G(z,y)+G_1(z,t)}{\frac12\partial_\nu^2 a(z,y)+a_1(z,t)}-\frac{\partial_\nu^2 G(z,y)}{\partial_\nu^2 a(z,y)}\right|\\ &\leqslant\frac{|G_1(z,y)|\partial_\nu^2 a(z,y)+|a_1(z,t)|\,|\partial_\nu^2 G(z,y)|}{\partial_\nu^2 a(z,y)|\frac12\partial_\nu^2 a(z,y)+a_1(z,t)|}.\end{aligned}$$

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Thus, there exists $\eta_1 > 0$ such that

$$|K(z+t\nu, y) - K(z,y)| < \tfrac{1}{2}\varepsilon \quad \text{for all } z \in \partial\Omega \text{ and } -\eta_1 < t < 0. \tag{3.3}$$

Also, by the smoothness of the boundary $\partial\Omega$ there exists $\eta_2 > 0$ such that

$$|K(z,y) - K(\bar{z},y)| < \tfrac{1}{2}\varepsilon \quad \text{for all } z, \bar{z} \in \partial\Omega,\ |z-\bar{z}| < \eta_2. \tag{3.4}$$

Define $\eta = \min\{\eta_1, \eta_2\}/2$ and fix $z \in \partial\Omega$. Now let $x \in \bar{\Omega}$ be such that $|x-z| < \eta$. Also, let $\bar{x} \in \partial\Omega$ be such that $|x-\bar{x}| = \delta(x) = \operatorname{dist}(x, \partial\Omega)$. Then $|x-\bar{x}| \leqslant |x-z| < \eta$ and $|\bar{x}-z| \leqslant |x-\bar{x}| + |z-x| < 2\eta < \eta_2$, so by (3.4) we have

$$|K(\bar{x},y) - K(z,y)| < \tfrac{1}{2}\varepsilon. \tag{3.5}$$

Now, from (3.3) and (3.5) we obtain

$$|K(x,y) - K(z,y)| \leqslant |K(x,y) - K(\bar{x},y)| + |K(\bar{x},y) - K(z,y)| < \varepsilon$$

so $K(\cdot\,,y)$ is continuous at $z \in \partial\Omega$. This completes the proof of (i).

(ii) First assume that $n > 4$. Using (2.2) and Proposition 2.1 (1) we have

$$\begin{aligned} K(x,y) &\leqslant c_1 \delta^{-2}(x)\delta^{-2\alpha}(y) G(x,y) \\ &\leqslant c_2 |x-y|^{2-n} \delta^{-2\alpha}(y) \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^2 \\ &\leqslant c_2 |x-y|^{2-n} \delta^{-2\alpha}(y) \min\left\{1, \frac{\delta(y)}{|x-y|}\right\}^{2\alpha} \\ &\leqslant c_2 |x-y|^{2-2\alpha-n} \quad \text{for all } x, y \in \Omega. \end{aligned}$$

Since $0 < \alpha < 1$, the mapping $x \mapsto |x-y|^{2-2\alpha-n}$ is integrable on Ω, so by means of Lebesgue's Dominated Convergence Theorem we deduce that

$$\bar{\Omega} \ni x \mapsto \int_\Omega K(x,y)\,\mathrm{d}y$$

is continuous. This fact, combined with $K > 0$ in Ω, proves the existence of a number $M > 1$ that satisfies (3.2).

For $2 \leqslant n \leqslant 4$ we proceed similarly with different estimates (as in the proof of Proposition 2.2) to derive the same conclusion. This finishes the proof of Lemma 3.1. □

We are now ready to prove Theorem 1.1. Let $M > 1$ satisfy (3.2) and fix $0 < \varepsilon < 1$ such that

$$\varepsilon^{1-\alpha^2} \leqslant M^{-1-\alpha}. \tag{3.6}$$

Define

$$g_\varepsilon \colon \mathbb{R} \to \mathbb{R}, \quad g_\varepsilon(t) = \begin{cases} \varepsilon^{-\alpha} & \text{if } t < \varepsilon, \\ t^{-\alpha} & \text{if } t \geqslant \varepsilon, \end{cases}$$

and, for any $v \in C(\bar{\Omega})$, $v > 0$ in $\bar{\Omega}$ consider the operator

$$T_\varepsilon(v)(x) = \int_\Omega K(x, y) g_\varepsilon(v(y))\, \mathrm{d}y \quad \text{for all } x \in \bar{\Omega}.$$

If $v \in C(\bar{\Omega})$ satisfies $v > 0$ in $\bar{\Omega}$, then $g_\varepsilon(v) \leqslant \varepsilon^{-\alpha}$ in $\bar{\Omega}$ so by (3.2) we find $T_\varepsilon(v) \leqslant M\varepsilon^{-\alpha}$ in $\bar{\Omega}$. Now let

$$v_1 \equiv M^{-1-\alpha}\varepsilon^{\alpha^2}, \qquad v_2 \equiv M\varepsilon^{-\alpha}.$$

and

$$[v_1, v_2] = \{v \in C(\bar{\Omega})\colon v_1 \leqslant v \leqslant v_2\}.$$

By Lemma 3.1 it is easy to see that $T_\varepsilon([v_1, v_2]) \subseteq [v_1, v_2]$. Further, by Lemma 3.1 and the Arzela–Ascoli Theorem, it follows that

$$T_\varepsilon : [v_1, v_2] \to [v_1, v_2]$$

is compact. Hence, by Schauder's fixed-point theorem, there exists $v \in C(\bar{\Omega})$, $v_1 \leqslant v \leqslant v_2$ in $\bar{\Omega}$ such that $T_\varepsilon(v) = v$. By (3.6) it follows that $v \geqslant v_1 \geqslant \varepsilon$ in $\bar{\Omega}$, so $g_\varepsilon(v) = v^{-\alpha}$. Therefore, v satisfies (3.1), that is, $u = av$ is a solution of (1.1). Now, the boundary estimate (1.3) and the regularity of solution u follow from Propositions 2.2 and 2.3, respectively. In the following we derive the uniqueness of the solution to (1.1).

Let u_1, u_2 be two solutions of (1.1). Using Proposition 2.2 there exists $0 < c < 1$ such that

$$c\delta^2(x) \leqslant u_i(x) \leqslant \frac{1}{c}\delta^2(x) \text{ in } \Omega, \quad i = 1, 2. \tag{3.7}$$

This means that we can find a constant $C > 1$ such that $Cu_1 \geqslant u_2$ and $Cu_2 \geqslant u_1$ in Ω.

We claim that $u_1 \geqslant u_2$ in Ω. Supposing the contrary, let

$$M = \inf\{A > 1\colon Au_1 \geqslant u_2 \text{ in } \Omega\}.$$

By our assumption, we have $M > 1$. From $Mu_1 \geqslant u_2$ in Ω, it follows that

$$M^\alpha u_2(x) - u_1(x) = \int_\Omega G(x, y)[M^\alpha u_2^{-\alpha}(y) - u_1^{-\alpha}(y)]\, \mathrm{d}y \geqslant 0 \quad \text{for all } x \in \Omega,$$

and then

$$M^{\alpha^2} u_1(x) - u_2(x) = \int_\Omega G(x, y)[M^{\alpha^2} u_1^{-\alpha}(y) - u_2^{-\alpha}(y)]\, \mathrm{d}y \geqslant 0 \quad \text{for all } x \in \Omega.$$

We have thus obtained $M^{\alpha^2} u_1 \geqslant u_2$ in Ω. Since $M > 1$ and $\alpha^2 < 1$, this last inequality contradicts the minimality of M. Hence, $u_1 \geqslant u_2$ in Ω. Similarly, we deduce $u_1 \leqslant u_2$ in Ω, so $u_1 \equiv u_2$ and the uniqueness is proved. This finishes the proof of Theorem 1.1.

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