YET ANOTHER PROOF OF THE MINIMAX THEOREM

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(received December 23, 1957)

Introduction. There are so many proofs of this theorem in the literature, that an excuse is necessary before exhibiting another. Such may be found by examining the proof given below for the following: it uses no matrices, almost no topology and makes little use of the geometry of convex sets; it applies equally well to the case where only one of the pure strategy spaces is finite; also there is no assumption that the payoff function is bounded. Thus it can provide a short route to the more general forms of the theorem.

Affine functions on convex sets. A real function f defined on a convex subset K of a real linear space is called affine if

$$f(\lambda x_1+(1-\lambda)x_2) = \lambda f(x_1)+(1-\lambda)\, f(x_2)\ , \text{ for } o \leq \lambda \leq 1 \text{ and } x_1, x_2 \in K.$$

The proof of the minimax theorem is contained essentially in a lemma concerning a finite family of such functions. If F is a family of functions, then the family of means $\sum^n{}_{i=1} \lambda_i f_i$, where $f_i \in F$, $\lambda_i \geqslant o$, $i=1,\ldots,n$, $\sum_{i=1}^n \lambda_i = 1$ is the convex family generated by F and will be denoted by F^*.

LEMMA[†]. If F is a finite family of real affine functions defined on a convex subset K of a real linear space, then

$$\sup_{x \in K} \min_{f \in F^*} f(x) = \min_{f \in F^*} \sup_{x \in K} f(x).$$

Proof. The inequality

$$\sup_{x \in K} \min_{f \in F^*} f(x) \leq \inf_{f \in F^*} \sup_{x \in K} f(x)$$

is a simple consequence of the definitions of sup. and inf.

Put

$$(1) \qquad \sup_{x \in K} \min_{f \in F} f(x) = v.$$

Can. Math. Bull., vol. 1, no. 2, May 1958

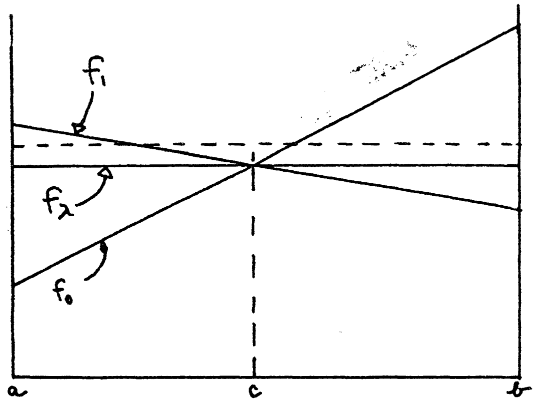
https://doi.org/10.4153/CMB-1958-011-x Published online by Cambridge University Press

Since $\min_{f\varepsilon F} f(x) = \min_{f\varepsilon F^*} f(x)$, and $v > -\infty$, it is only necessary to show for a finite v that,

(2) for some $g \,\varepsilon\, F^*$ and all $x \,\varepsilon\, K$, $g(x) \leq v$.

We begin with the case where $F = \{f_o, f_1\}$ consists of two functions. If for $0 < \lambda < 1$, we write $f_\lambda = (1-\lambda)f_o + \lambda f_1$, then $F^* = \{f_\lambda : 0 \leq \lambda \leq 1\}$. For each $x \,\varepsilon\, K$ we put $\Lambda(x) = \{\lambda : f_\lambda(x) \leq v\}$ Because of (1) we have $f_o(x) \leq v$ or $f_1(x) \leq v$. Since $f_\lambda(x)$ is an affine function of λ, each $\Lambda(x)$ is a non-vacuous closed sub-interval of $[0, 1]$ containing at least one end point. To prove (2) we need only show that there is some λ common to all $\Lambda(x)$. This is achieved by showing that if $0 \,\varepsilon\, \Lambda(a) \subset [0, 1)$ and $1 \,\varepsilon\, \Lambda(b) \subset (0, 1]$, then $\Lambda(a)$ meets $\Lambda(b)$. This corresponds to the case where $f_o(a) \leq v < f_1(a)$ and $f_1(b) \leq v < f_o(b)$. The figure shows that there is obviously a $\lambda \,\varepsilon\, \Lambda(a) \cap \Lambda(b)$, i.e. $f_\lambda(a) = f_\lambda(b) \leq v$.

In fact, if $(1-\lambda)\lambda^{-1} = [f_1(a) - f_1(b)]\,[f_o(b) - f_o(a)]^{-1}$ then $f_\lambda(a) = f_\lambda(b)$. Since f_λ is affine, it is constant on the segment in K joining a to b, and for some point c on this segment, $f_\lambda(a) = f_\lambda(b) = f_\lambda(c) = f_o(c) = f_1(c) \leq v$.



We now extent the result (2) to the case where $F = \{f_o, f_1, \ldots, f_n\}$ is a family of more than two functions. Put $K_o = \{x : f_o(x) > v\}$, where v is defined in (1) with the new F. Since f_o is affine, the set K_o is a convex subset of K. If K_o is vacuous (2) is trivial; if it is not, put $F_0 = \{f_1, f_2, \ldots, f_n\}$, then by (1),

$$\sup_{x \,\varepsilon\, K_o} \min_{f \,\varepsilon\, F_o} f(x) \leq v.$$

Assuming that the theorem is true for n functions (on K_o), there is an $h \,\varepsilon\, F_o^*$, such that on K_o, $h(x) \leq v$. Now on K, $\min(f_o(x), h(x)) \leq v$. Thus if $F_1 = \{f_o, h\}$ we have by the case for two functions, that for some $g \,\varepsilon\, F_1^*$, $g(x) \leq v$ on K. However $F_1^* \subset F^*$, so that (2) is established in the general case.

Dual games. We say that a dual game (X, Y, f) consists of a real function (the payoff) defined on the product $X \times Y$ of two sets (pure strategy spaces). It is extended to $X^* \times Y^*$, where X^* and Y^* (mixed strategy spaces) are convex subsets of real linear spaces in which X and Y, respectively, are embedded. The set X^* is the class of all sums $\sum_{i=1}^{n} \mu_i x_i$, $x_i \in X$, $\mu_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \mu_i = 1$, and similarly for Y^*, while f is extended to $X^* \times Y^*$ so that it is an affine function of each variable. If we put $f_y(x) = f(x, y)$, then $\{f_y : y \in Y\}$ is a family of affine functions defined on the convex set X^*.

THEOREM Any dual game (X, Y, f) in which Y is finite satisfies the equality

$$\sup_{x \in X^*} \min_{y \in Y^*} f(x, y) = \min_{y \in Y^*} \sup_{x \in X^*} f(x, y).$$

Thus the game has a value, and the player on Y has an optimal strategy.

Proof. Apply the lemma to the family $F = \{f_y : y \in Y\}$.

Other proofs. This form of the theorem was first established by Wald [5] . For a survey of many proofs see Kuhn [3] . To these should be added the proofs of Dantzig [1] , Peck and Dulmage [4] , Fan [2] and Zieba [6] .

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† A similar lemma may be established for a family of concave functions. The proof is a little more difficult.