TAUBERIAN THEOREMS FOR $[J, f(x)]$ TRANSFORMATIONS

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1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ $(s_n = a_0 + \cdots + a_n, n \geqq 0)$ be a series of real or complex numbers. Denote by $\{t_n^{(1)}\}$ and $\{t_n^{(2)}\}$

(1.1) $$t_n^{(j)} = \sum_{k=0}^{\infty} a_{nk}^{(j)} s_k \quad n \geqq 0 \ (j = 1, 2)$$

two linear transforms T_1 and T_2 of $\{s_n\}$. Estimates of the form

(1.2) $$\limsup_{\lambda \to \infty} \left| t_{n(\lambda)}^{(1)} - t_{m(\lambda)}^{(2)} \right| \leqq C \cdot \limsup_{n \to \infty} \left| d_n \right|$$

for sequences $\{s_n\}$ satisfying

(1.3) $$\limsup_{n \to \infty} \left| d_n \right| < \infty$$

where $\{d_n\}$ is a certain fixed linear transform of the sequence $\{a_n\}$ $(n \geqq 0)$ and $n(\lambda) \to \infty$, $m(\lambda) \to \infty$ $(\lambda \uparrow \infty)$ depend on the transforms T_1, T_2 and $\{d_n\}$, were considered for the first time by Hadwiger [2]. The smallest value of C satisfying (1.2) for all sequences $\{s_n\}$ satisfying (1.3) is known as the Tauberian constant associated with the pair of transforms T_1, T_2 and $\{d_n\}$.

In §2 we get the explicit expression of the Tauberian constant associated with two transforms $\{t_n^{(1)}\}$, $\{t_n^{(2)}\}$, one a Hausdorff sequence-to-sequence transform and the other a $[J, f(x)]$ series-to-function transform; $\{d_n\}$ being the Cesàro transform of order α $(0 \leqq \alpha \leqq 1)$ of the sequence $\{na_n\}$. This generalizes the work of [7].

As an application we derive from Theorem 2.1 of §2 Littlewood's Tauberian theorem for the Abel transformation (Theorem 2.3), by using Tauberian constants.

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2. Tauberian constants and Littlewood's theorem

Given an infinite sequence $\{\mu_n\}$ $(n \geqq 0)$ the Hausdorff transform $\{t_n\}$ generated by the sequence $\{\mu_n\}$ of a sequence $\{s_n\}$ is defined (see [3]) by

$$t_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k) s_k, \qquad n \geqq 0,$$

where $\Delta^{p+1}\mu_r = \Delta^p\mu_r - \Delta^p\mu_{r+1}$ $(p \geqq 0)$, $\Delta^0\mu_r = \mu_r$. A Hausdorff transformation generated by a sequence $\{\mu_n\}$ is regular if, and only if, there exists a function $\gamma(t)$ satisfying

(2.1) $\gamma(t)$ is normalized and of bounded variation in $[0,1]$, $\gamma(0) = \gamma(0+) = 0$ and $\gamma(1) = 1$,

and $\mu_n = \int_0^1 t^n d\gamma(t)$ for $n \geqq 0$. Thus the regular Hausdorff transform $\{H_n(\gamma)\}$ of a sequence $\{s_n\}$ may be defined by

$$H_n(\gamma) = \int_0^1 \sum_{k=0}^{n} P_{nk}(t) s_k \, d\gamma(t), \qquad n \geqq 0,$$

where $\gamma(t)$ satisfies (2.1) and

$$P_{nk}(t) = \begin{cases} \binom{n}{k} t^k (1-t)^{n-k} & \text{if } 0 \leqq k \leqq n \\ 0 & \text{if } k > n \end{cases} \tag{2.2}$$

(the function $P_{nk}(t)$ defined here is the function $P_{nk}(t)$ defined by [9, (2.3)] for $\lambda_n = n$).

For a series $\sum_{n=0}^{\infty} a_n$ and a fixed $\alpha \geqq 0$, define the sequence $\{a_n^{(\alpha)}\}$ as the Hausdorff transform of $\{na_n\}$ with $\gamma(t) = \psi_\alpha(t)$, where $\psi_\alpha(t) = 1 - (1-t)^\alpha$ if $0 \leqq t \leqq 1$ and $\alpha > 0$, $\psi_0(t) = 0$ if $0 \leqq t < 1$, $\psi_0(1) = 1$. That is $\{a_n^{(\alpha)}\}$ is the Cesàro transform of order α of the sequence $\{na_n\}$ and we have the explicit expression

$$a_n^{(\alpha)} = \frac{\alpha\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(n-k+1)} \cdot ka_k \text{ for } n \geqq 0 \text{ and } \alpha > 0,$$

$$a_n^{(0)} = na_n \qquad \text{for } n \geqq 0.$$

The regular series-to-function $[J, f(x)]$-transform of a series $\sum_{k=0}^{\infty} a_k$ is defined in [8, §5] by

$$J_x(\beta) \equiv \sum_{k=0}^{\infty} a_k \sum_{m=k}^{\infty} (-x)^m f^{(m)}(x)/m!, \qquad x > 0, \tag{2.3}$$

where

(2.4) $$f(x) = \int_0^1 t^x d\beta(t), \qquad x > 0,$$

and $\beta(t)$ is a function satisfying

(2.5) $\beta(t)$ is of bounded variation in $[0,1]$, $\beta(0) = \beta(0+) = 0$, $\beta(1-0) = \beta(1) = 1$.

LEMMA 2.1. *Suppose $\beta(t)$ satisfies (2.5). If for some α, $0 \leqq \alpha \leqq 1$,*

(2.6) $$\int_1^\infty |\beta(e^{-t})| t^{\alpha-1} dt < \infty$$

and $a_n^{(\alpha)} = O(1)$ as $n \to \infty$, then $J_x(\beta)$ exists for $x > 0$ and

$$J_x(\beta) = a_0 + \sum_{p=1}^\infty a_p \int_0^\infty P^*_{x,p-1}(u) x \beta(e^{-u}) du$$

$$= a_0 + \int_0^\infty \frac{\beta(e^{-u})}{u} \sum_{p=1}^\infty p a_p P^*_{xp}(u) du$$

where for $u, x > 0$, $k > 0$ $P^*_{xk}(u) = e^{-xu}(xu)^k/k!$ and $P^*_{x,0}(u) = e^{-xu}$.

PROOF. It is well-known that $a_n^{(\alpha)} = O(1)(n \to \infty)$ implies $na_n = O(n^\alpha)$ or $|a_n| \leqq Mn^{\alpha-1}$ for $n \geqq 1$. Now for $0 < \alpha < 1$ (for $\alpha = 0$ and $\alpha = 1$ the proof is even simpler) we have

$$\sum_{k=0}^\infty \left| a_k \right| \left| \sum_{m=k}^\infty (-x)^m f^{(m)}(x)/m! \right| = |a_0| + \sum_{k=1}^\infty |a_k| \left| \sum_{m=k}^\infty \int_0^\infty P^*_{xm}(u) d\beta(e^{-u}) \right|$$

$$= |a_0| + \sum_{k=1}^\infty |a_k| \left| \int_0^\infty \left\{ \sum_{m=k}^\infty P^*_{xm}(u) \right\} d\beta(e^{-u}) \right|$$

(and integrating by parts)

$$= |a_0| + \sum_{k=1}^\infty |a_k| \left| \int_0^\infty \beta(e^{-u}) \left(\frac{d}{du} \sum_{m=k}^\infty P^*_{xm}(u) du \right| \right.$$

(and by term-by-term differentiation)

$$\leqq |a_0| + \sum_{k=1}^\infty |ka_k| \left| \int_0^\infty P^*_{xk}(u) |\beta(e^{-u})| u^{-1} du \right.$$

$$\leqq |a_0| + M \int_0^\infty |\beta(e^{-u})| u^{-1} \sum_{k=1}^\infty P^*_{xk}(u) k^\alpha du$$

$$= |a_0| + Mx^\alpha \int_0^\infty |\beta(e^{-u})| u^{\alpha-1} e^{-xu} \sum_{k=1}^\infty \frac{(xu)^{k/q}}{\{k!\}^{1/q}} \frac{(xu)^{(k-1)/p}}{\{(k-1)!\}^{1/q}}$$

(and by Holder's inequality with $p = 1/\alpha > 1$ and $q = 1/(1-\alpha)$, so that $1/p + 1/q = 1$)

$$\leqq |a_0| + Mx^\alpha \int_0^\infty |\beta(e^{-u})| u^{\alpha-1} e^{-xu} \left(\sum_{k=1}^\infty \frac{(xu)^k}{k!}\right)^{1/q} \left(\sum_{k=1}^\infty \frac{(xu)^{k-1}}{(k-1)!}\right)^{1/p} du$$

$$\leqq |a_0| + Mx^\alpha \int_0^\infty |\beta(e^{-u})| u^{\alpha-1} du$$

$$< \infty.$$

Thus by (2.3) $J_x(\beta)$ exists for $x > 0$ and by the same argument (2.7) is true too.

THEOREM 2.1. *Let α be a fixed number $0 \leqq \alpha < 1$. Suppose $\{H_n(\gamma)\}$ and $J_x(\beta)$ are regular transformations and*

$$\int_0^1 \frac{|\gamma(t)|}{t} dt < \infty, \int_0^1 \frac{|1-\beta(e^{-u})|}{u} du < \infty, \int_1^\infty \frac{|\beta(e^{-u})|}{u^{1-\alpha}} du < \infty.$$

For a sequence $\{s_n\}$ satisfying $a_n^{(\alpha)} = O(1) (n \to \infty)$ we have for each q, $0 < q < \infty$, and any pair of functions $n(\lambda) \to \infty$, $x(\lambda) \to \infty$, $n(\lambda)/x(\lambda) \to q$ $(\lambda \to \infty)$

$$\limsup_{\lambda \to \infty} |H_{n(\lambda)}(\gamma) - J_{x(\lambda)}(\beta)| \leqq G_q^{(\alpha)} \cdot \limsup_{n \to \infty} |a_n^{(\alpha)}| \tag{2.7}$$

where

$$G_q^{(\alpha)} = H_q^{(\alpha)} + \int_q^\infty t^\alpha \left| d_t \left[\int_t^\infty (u-t)^{-\alpha} \frac{\beta(e^{-u})}{u} du \right] \right|, \tag{2.8}$$

$H_q^{(\alpha)}$ is, for $0 < \alpha < 1$, the total variation on $0 \leqq x \leqq q$ of

$$- \int_{x/q}^1 t^\alpha d_t \left[\int_t^1 (u-t)^{-\alpha} \frac{\gamma(u)}{u} du \right] + \int_0^{x/q} t^\alpha d_t \left[\int_1^\infty (v-t)^{-\alpha} \frac{dv}{v} \right]$$

$$- x^\alpha \int_x^\infty (v-x)^{-\alpha} \frac{\beta(e^{-v})}{v} dv - \alpha \int_0^x u^{\alpha-1} du \left[\int_u^\infty (w-u)^{-\alpha} \frac{1-\beta(e^{-w})}{w} dw \right]$$

and

$$H_q^{(0)} = \int_0^q \frac{|1-\beta(e^{-x}) - \gamma(x/q)|}{x} dx.$$

The constant $G_q^{(\alpha)}$ is the best in the following sense. There is a real sequence $\{s_n\}$ satisfying $a_n^{(\alpha)} = O(1)$ and such that both members of inequality (2.7) are equal.

Theorem 2.1 for $\alpha = 0$ and $\gamma(t) = 0$ $(0 \leqq t < 1)$, $\gamma(1) = 1$, is Theorem 5.1 of [8].

THEOREM 2.2. *Suppose* $(H_n(\gamma))$ *and* $J_x(\beta)$ *are regular transformations such that* $\gamma(t)$ *is continuous for* $0 \leqq t < 1$ *and*

$$\int_0^1 \frac{|\gamma(t)|}{t} dt < \infty, \int_0^1 \frac{|1-\beta(e^{-t})|}{t} dt < \infty, \int_1^\infty |\beta(e^{-u})| \, du < \infty.$$

For a sequence $\{s_n\}$ *satisfying* $a_n^{(1)} = O(1)$ *as* $n \to \infty$ *we have for each* $q, 0 < q < \infty$, *and any pair of functions* $n(\lambda) \to \infty$, $x(\lambda) \to \infty$, $n(\lambda)/x(\lambda) \to q$ $(\lambda \to \infty)$

$$\limsup_{\lambda\to\infty} |H_{n(\lambda)}(\gamma) - J_{x(\lambda)}(\beta)| \leqq G_q^{(1)} \cdot \limsup_{n\to\infty} |a_n^{(1)}| \tag{2.9}$$

where

$$G_q^{(1)} = \int_0^q u \left| d\left(\frac{\gamma_1(u)-1+\beta(e^{-u})}{u}\right)\right| + \int_q^\infty u \left| d\left(\frac{\beta(e^{-u})}{u}\right)\right| + |\gamma(1)-\gamma(1-0)|$$

if, either: (I) $\beta(t)$ *is continuous at* $t = 1/e$ *and* $\gamma(t)$ *is absolutely continuous in each interval* $[\delta, 1-\delta]$ $(0 < \delta < \frac{1}{2})$ *or* (II) $\beta(t)$ *is continuous in some interval* $[e^{-1}-\varepsilon, 1]$ *for some* $0 < \varepsilon < e^{-1}$, $\beta(t)$ *is absolutely continuous in each interval* $[e^{-1}+\delta, 1-\delta]$ $\left(0 < \delta < \frac{1-e^{-1}}{2}\right)$ *and* $\gamma(t)$ *is continuous in* $[0,1]$. $G_q^{(1)}$ *is the best in the sense of Theorem* 2.1.

Theorem 2.2 for $\gamma(t)=0$ $(0 \leqq t < 1)$, $\gamma(1)=1$ and $\beta(t)$ continuous in $[0,1]$ is Theorem 5.2 of [8]. The assumption $\int_1^\infty|\beta(e^{-t})|\,dt < \infty$ should be added to the assumptions of [8, Theorem 5.2].

THEOREM 2.3 (Littlewood). *If* $\{s_n\}$ $(n \geqq 0)$ *is summable Abel to* s, *that is*

$$\lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} s_n x^n = s, \text{ and } na_n = O(1) \ (n\to\infty), \text{ then } \lim_{n\to\infty} s_n = s.$$

PROOF. The $A^{(r)}$–transform $(r \geqq 0)$ of a sequence $\{s_n\}$ $(n \geqq 0)$ is defined by

$$A^{(r)}(x) \equiv \left(\frac{x}{x+1}\right)^r \sum_{m=0}^{\infty} s_m \binom{m+r}{m} \left(\frac{x}{x+1}\right)^m$$

and it is the $J_x(\psi_r(t))$-transform (see [5, Theorem 8.3] and [6, Example 4]). It is proved in [10, p. 503] that Abel's summability of $\{s_n\}$ to s and $na_n = O(1)$ imply the $A^{(r)}$-summability for each positive integer r. Now by Theorem 2.1 for $\alpha=0$, $\gamma(t)=0$ $(0 \leqq t < 1)$, $\gamma(1)=1$, $\beta(t) \equiv 1-(1-t)^r$, r a positive integer and $q = \log r$ we get

$$\limsup_{n\to\infty} |s_n - s| = \limsup_{n\to\infty} |s_n - J_{n/\log r}(\psi_r(t))|$$
$$\leqq G_{\log r}^{(0)} \cdot \limsup_{n\to\infty} |na_n|$$

where (for $0 < \lambda < 1$)

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$$G^{(0)}_{\log r} = \left\{\int_0^1 + \int_1^{\lambda \log r} \int_{\lambda \log r}^{\log r}\right\} \frac{(1-e^{-u})^r}{u} du + \int_{\log r}^{\infty} \frac{1-(1-e^{-u})^r}{u} du$$
$$\equiv I_1 + I_2 + I_3 + I_4.$$

By uniform convergence to zero as $r \to \infty$ of the integrand we get $\lim_{r\to\infty} I_1 = 0$. We have for each $0 < \lambda < 1$ $I_2 \leqq (1-r^{-\lambda})^r \log(\lambda \log r) \to 0$ $(r \to \infty)$ and

$$I_3 \leqq \int_{\lambda \log r}^{\log r} \frac{du}{u} = \log \frac{1}{\lambda}$$

which may be made small by choosing λ near 1. By the inequality $1 - x^\alpha \leqq \alpha(1-x)$ for $\alpha \geqq 1$ and $0 \leqq x \leqq 1$ we get

$$\int_{\log r}^{\infty} \frac{1-(1-e^{-u})^r}{u} du \leqq r \int_{\log r}^{\infty} \frac{e^{-u}}{u} du \leqq \frac{r}{\log r} \int_{\log r}^{\infty} e^{-u} du = \frac{1}{\log r} \to 0$$
$$(r \to \infty).$$

Hence $\lim_{r\to\infty} G^{(0)}_{\log r} = 0$. Thus $\limsup_{n\to\infty} |s_n - s| = 0$ or $\lim_{n\to\infty} s_n = s$.

3. Proof of Theorems 2.1 and 2.2

In the proof of Theorems 2.1 and 2.2 we use the following results.

LEMMA 3.1. *If $f(u)$ is a complex and bounded function in $[0,1]$ continuous at the point $u = x$ $(0 \leqq x \leqq 1)$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} P_{nk}(x) f\left(\frac{k}{n}\right) = f(x). \tag{3.1}$$

$$\sum_{k=0}^{n} P_{nk}(u) = 1 \text{ for } 0 \leqq u \leqq 1 \text{ and } n = 1,2,3,\dots. \tag{3.2}$$

For a proof see [13, p. 47, Theorem 2.8.2].

LEMMA 3.2. *Suppose that $f(u)$ is bounded in every finite interval $0 \leqq u \leqq R$, $R > 0$, and $f(u) = O(u^\delta)$ for some $\delta > 0$ as $u \to \infty$. If $f(u)$ is continuous at a point $u = \zeta$, then*

$$\lim_{x\to\infty} \sum_{k=0}^{\infty} P^*_{xk}(\zeta) f\left(\frac{k}{x}\right) = f(\zeta).$$

For a proof see [15].

LEMMA 3.3. *For $p = 0,1,2,\dots$ and $x, u > 0$ we have*

$$\frac{d}{du} \sum_{k=p}^{\infty} P^*_{xk}(u) = \frac{p}{u} P^*_{xp}(u) \text{ and } \sum_{k=p}^{\infty} \frac{1}{k+1} [P^*_{xk}(u)u]' = P^*_{xp}(u).$$

The proof is immediate.

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LEMMA 3.4. *Suppose $\beta(t)$ satisfies (2.5) and (2.6) for some α, $0 \leqq \alpha \leqq 1$. If a sequence $\{s_n\}$ satisfies $a_n^{(\alpha)} = O(1)$ $(n \to \infty)$, then for $x > 0$*

$$J_x(\beta) = \begin{cases} a_0 + \dfrac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{\infty} a_k^{(\alpha)} \int_0^{\infty} \frac{\beta(e^{-t})}{t} dt \int_0^t (t-u)^{-\alpha} \frac{d}{du} \{u^\alpha P_{xk}^*(u)\} du \\ \qquad\qquad \text{if } 0 \leqq \alpha < 1 \\ a_0 + \displaystyle\sum_{k=1}^{\infty} a_k^{(1)} \int_0^{\infty} \frac{d}{du} [P_{xk}^*(u)u] \frac{\beta(e^{-u})}{u} du \qquad \text{if } \alpha = 1 \end{cases}$$

(3.3)

$$= \begin{cases} a_0 + \dfrac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{\infty} a_k^{(\alpha)} \int_0^{\infty} P_{xk}^*(t) d_t \\ \qquad \left\{ \displaystyle\int_t^{\infty} u^\alpha d_u \left[\int_u^{\infty} (v-u)^{-\alpha} \frac{\beta(e^{-v})}{v} dv \right] \right\} \\ \qquad\qquad \text{if } 0 \leqq \alpha < 1 \\ a_0 + \displaystyle\sum_{k=1}^{\infty} a_k^{(1)} \int_0^{\infty} P_{xk}^*(v) d \left\{ \int_v^{\infty} u d \left(\frac{\beta(e^{-u})}{u} \right) \right\} \qquad \text{if } a = 1. \end{cases}$$

PROOF. Assume $0 \leqq \alpha < 1$. We have for $p \geqq 0$

$$pa_p = \sum_{k=0}^{p} \binom{p-k-\alpha-1}{p-k} \binom{k+\alpha}{k} a_k^{(\alpha)}.$$

By Lemma 2.1 we get for $x > 0$

$$J_x(\beta) = a_0 + \int_0^{\infty} \frac{\beta(e^{-u})}{u} \sum_{p=1}^{\infty} \left\{ \sum_{k=0}^{p} \binom{p-k-\alpha-1}{p-k} \binom{k+\alpha}{k} a_k^{(\alpha)} \right\} P_{xp}^*(u) du,$$

and, as is shown in the proof of Lemma 2.1, the last integral exists with $\beta(e^{-u})$ replaced by $|\beta(e^{-u})|$. Now we show that the order of double summation in the above integral may be changed. For fixed $x, u > 0$ we have for $M > xu$, since $a_0^{(\alpha)} = 0$,

$$\sum_{p=1}^{N} \left\{ \sum_{k=0}^{p} \binom{p-k-\alpha-1}{p-k} \binom{k+\alpha}{k} a_k^{(\alpha)} \right\} P_{xp}^*(u)$$

$$= \sum_{k=0}^{N} a_k^{(\alpha)} \binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u)$$

$$- \sum_{k=0}^{N} a_k^{(\alpha)} \binom{k+\alpha}{k} \sum_{p=N+1}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u).$$

Now for $0 \leqq k \leqq N$ we have

$$\left|\binom{p-k-\alpha-1}{p-k}\right| = \left|\left(1-\frac{\alpha+1}{1}\right)\cdots\left(1-\frac{\alpha+1}{p-k}\right)\right| \leqq 1.$$

Hence, since $|a_k^{(\alpha)}| \leqq M < \infty$ for $k \geqq 0$,

$$\sum_{k=0}^{N} \left| a_k^{(\alpha)} \binom{k+\alpha}{k} \sum_{p=N+1}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u) \right| \leqq M \left\{ \sum_{k=0}^{N} \binom{k+\alpha}{k} \right\} \sum_{p=N+1}^{\infty} P_{xp}^*(u)$$

$$= M \binom{N+\alpha+1}{N} P_{x,N+1}^*(u) \sum_{p=N+1}^{\infty} \left(\frac{xu}{N}\right)^{p-N-1}$$

$$= M \binom{N+\alpha+1}{N} P_{x,N+1}^*(u) \left(1-\frac{xu}{N}\right)^{-1}$$

(and by Stirling's formula)

$$\sim M_1 \, N^{\alpha+1} \frac{(xu)^{N+1}}{(N+1)!} e^{-xu}$$

$$\to 0 \qquad (N \to \infty).$$

Letting $N \to +\infty$ we see that for $x, u > 0$ we have

$$\sum_{p=1}^{\infty} \left\{ \sum_{k=0}^{p} \binom{p-k-\alpha-1}{p-k} \binom{k+\alpha}{k} a_k^{(\alpha)} \right\} P_{xp}^*(u)$$

$$= \sum_{k=0}^{\infty} a_k^{(\alpha)} \binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u).$$

Thus we get

$$J_x(\beta) = a_0 + \int_0^\infty \frac{\beta(e^{-u})}{u} \sum_{k=1}^{\infty} a_k^{(\alpha)} \binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u) du.$$

The integral on the right hand side exists, by the remark at the beginning of the proof, if $\beta(e^{-u})$ is replaced by $|\beta(e^{-u})|$. We prove now that this integral is absolutely convergent so that the order of integration and summation may be changed. Choosing $a_0 = 0$, $a_p = 1/p (p > 0)$ in the last integral we see that

$$\int_0^\infty \frac{|\beta(e^{-u})|}{u} \sum_{k=1}^{\infty} \binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P_{xp}^*(u) du$$

converges. Since $\binom{p-k-\alpha-1}{p-k} < 0$ for $p > k$, $\binom{p-k-\alpha-1}{p-k} = 1 > 0$ for $p = k$ and $\left(\text{by } \binom{k+\alpha}{k} \sim \frac{k^\alpha}{\Gamma(\alpha+1)} (k \to \infty)\right)$

$$\int_0^\infty \frac{|\beta(e^{-u})|}{u} \sum_{k=1}^{\infty} \binom{k+\alpha}{k} P_{xk}^*(u) du$$

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$$\leqq M_1 \int_0^{\infty} \frac{|\beta(e^{-u})|}{u} \sum_{k=1}^{\infty} k^{\alpha} P^*_{xk}(u) du$$

(and as is shown in the proof of Lemma 2.1)

$$\leqq M_1 \int_0^{\infty} \frac{|\beta(e^{-u})|}{u^{1-\alpha}} du < +\infty$$

we see, because

$$\sum_{p=k}^{\infty} \left| \binom{p-k-\alpha-1}{p-k} \right| P^*_{xk}(u) = 2P^*_{kx}(u) - \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P^*_{xp}(u)$$

that the above integral is absolutely convergent. Hence we may change the order of summation and integration in the last expression for $J_x(\beta)$ and we get

$$J_x(\beta) = a_0 + \sum_{k=1}^{\infty} a_k^{(\alpha)} \int_0^{\infty} \frac{\beta(e^{-u})}{u} \binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P^*_{xp}(u) du.$$

Now we have

$$\binom{k+\alpha}{k} \sum_{p=k}^{\infty} \binom{p-k-\alpha-1}{p-k} P^*_{xp}(u) = P^*_{xk}(u) \sum_{p=0}^{\infty} (-1)^p \frac{(xu)^p}{p!} \binom{k+\alpha}{k+p}$$

$$= P^*_{xk}(u) \sum_{p=0}^{\infty} \frac{(xu)^p}{p!} \sum_{r=0}^{p} (-1)^r \binom{p}{r} \binom{k+r+\alpha}{k+r}$$

and (by looking on it as a Cauchy product) we get

$$= \frac{(xu)^k}{k!} \sum_{p=0}^{\infty} (-1)^p \binom{k+p+\alpha}{k+p} \frac{(xu)^p}{p!}$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \frac{x^k}{k!} \sum_{p=0}^{\infty} (-1)^p \frac{x^p}{p!} (k+p+\alpha) \int_0^u (u-v)^{-\alpha} v^{k+p+\alpha-1} dv$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \frac{x^k}{k!} \int_0^u (u-v)^{-\alpha} \left[\sum_{p=0}^{\infty} (-1)^p \frac{x^p v^{k+p+\alpha}}{p!} \right]' dv$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^u (u-v)^{-\alpha} [P^*_{xk}(v) v^{\alpha}]' dv.$$

Substituting this expression in the last form of $J_x(\beta)$ we get

$$J_x(\beta) = a_0 + \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \sum_{k=1}^{\infty} a_k^{(\alpha)} \int_0^{\infty} \frac{\beta(e^{-u})}{u} du \int_0^u (u-v)^{-\alpha} \frac{d}{dv} [P^*_{xk}(v) v^{\alpha}] dv.$$

By Lemma 3.9 we get now the second form of $J_x(\beta)$ in the statement of our lemma. This proves our lemma for $0 \leqq \alpha < 1$. For $\alpha = 1$ the proof is similar and much simpler. We use now the identity $(p+1)[P^*_{xp}(u) - P^*_{x.p+1}(u)] = \dfrac{d}{du}[P^*_{xp}(u)u]$.

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LEMMA 3.5. *Suppose $G(t)$ is absolutely continuous in $[0,1]$. Then for each pair of functions $n \equiv n(\lambda) \to +\infty$, $x \equiv x(\lambda) \to +\infty$, $n(\lambda)/x(\lambda) \to q$, $(\lambda \to +\infty)$, $0 < q < \infty$, we have*

$$\lim_{\lambda \to \infty} \sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} dG(u) \right| = 0. \tag{3.4}$$

PROOF. First we prove the lemma for $G(u) = u^a$ where a is a positive integer. We have by (2.2)

$$\begin{aligned}\sum_{k=n+1}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} u^{a-1} du \right| &= \sum_{k=n+1}^{\infty} \left| \int_0^1 P^*_{xk}(qu) u^{a-1} du \right| \\ &= \sum_{k=n+1}^{\infty} \int_0^1 P^*_{xk}(qu) u^{a-1} du \\ &= q^{-a} \int_0^q v^{a-1} \sum_{k=n+1}^{\infty} P^*_{xk}(v) dv.\end{aligned}$$

Given $\delta > 0$, $0 < \delta < q$, define $f(t) \equiv f_\delta(t)$ by $f(t) = 1$ for $t > q - \delta$ and $f(t) = 0$ for $0 \leqq t \leqq q - \delta$. Now for $\lambda > \lambda(\delta)$, $q - \delta < n/x = n(\lambda)/x(\lambda)$. Hence for $\lambda > \lambda(\delta)$ we have, since $k \geqq n + 1$ implies $k/x > q - \delta$

$$\begin{aligned} q^{-a} \int_0^q v^{a-1} \sum_{k=n+1}^{\infty} P^*_{xk}(v) dv &\leqq q^{-a} \int_0^q v^{a-1} \sum_{\substack{k \\ q-\delta < k/x}} P^*_{xk}(v) dv \\ &= q^{-a} \int_0^q v^{a-1} \sum_{k=0}^{\infty} P^*_{xk}(v) f_\delta\left(\frac{k}{x}\right) dv \end{aligned}$$

(and by Lemma 3.2, since $\lambda \to \infty$ implies $x \equiv x(\lambda) \to +\infty$, and Lebesgue's dominated convergence theorem)

$$\begin{aligned} &\to q^{-a} \int_0^q v^{a-1} f_\delta(v) dv \qquad (\lambda \to +\infty) \\ &= q^{-a} \int_{q-\delta}^q v^{a-1} dv. \end{aligned}$$

Letting $\delta \downarrow 0$ we get

$$\lim_{\lambda \to \infty} \sum_{k=n+1}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} u^{a-1} du \right| = 0. \tag{3.5}$$

We have

$$\sum_{k=0}^{n} \int_1^{\infty} P^*_{xk}(qu) u^{a-1} du \leq x^{-(a-1)} q^{-a} \int_q^{\infty} \sum_{k=0}^{n} \frac{(k+a-1)!}{k!} P^*_{x,k+a-1}(u) du$$

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$$\leqq \frac{(n+a-1)!}{n!} x^{-(a-1)} q^{-a} \int_q^\infty \sum_{k=0}^{n+a-1} P^*_{xk}(u)du$$

$$= \frac{(n+a-1)!}{n!} x^{-(a-1)} q^{-a} \left\{ (n+a)/x - \int_0^q \sum_{k=0}^{n+a-1} P^*_{xk}(u)du \right\}$$

$$\to \frac{1}{q} \left\{ q - \int_0^q 1\,du \right\} \qquad (\lambda \to \infty)$$

$$= 0$$

by Lebesgue's bounded convergence theorem, the argument used in proving (3.5) and Lemma 3.2 applied once to the function $f_1(t) = 1$ $(0 \leqq t \leqq q - \delta,\ 0 < \delta < q)$ $f_1(t) = 0$ $(t > q - \delta)$ and next to the function $f_2(t) = 1$ $(0 \leqq t \leqq q + \delta)$, $f_2(t) = 0$ $(t > q + \delta)$ and then letting $\delta \downarrow 0$. Hence we have

$$I \equiv \lim_{\lambda \to \infty} \sum_{k=0}^{n} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} u^{a-1} du \right|$$

$$= \lim_{\lambda \to \infty} \sum_{k=0}^{n} \left| \int_0^1 P_{nk}(u) u^{a-1} du - \int_0^\infty P^*_{xk}(qu) u^{a-1} du + \int_1^\infty P^*_{xk}(qu) u^{a-1} du \right|$$

$$= \lim_{\lambda \to \infty} \sum_{k=0}^{n} \left| \int_0^1 P_{nk}(u) u^{a-1} du - \int_0^\infty P^*_{xk}(qu) u^{a-1} du \right|.$$

Now we have for

$$0 \leqq k \leqq n \int_0^1 P_{nk}(u) u^{a-1} du - \int_0^\infty P^*_{xk}(qu) u^{a-1} du = \frac{(k+a-1)!}{k!} \left[\frac{n!}{(n+a)!} - (xq)^{-a} \right].$$

Hence the sign of

$$\int_0^1 P_{nk}(u) u^{a-1} du - \int_0^\infty P^*_{xk}(qu) u^{a-1} du$$

is constant for $0 \leqq k \leqq n$ and we get

$$(3.6) \qquad I = \lim_{\lambda \to \infty} \left| \sum_{k=0}^{n} \int_0^1 P_{nk}(u) u^{a-1} du - \sum_{k=0}^{n} \int_0^\infty P^*_{xk}(qu) u^{a-1} du) \right|$$

$$= \lim_{\lambda \to \infty} \left| \int_0^1 u^{a-1} du - \int_0^1 u^{a-1} \sum_{k=0}^{n} P^*_{xk}(qu) du \right|$$

$$= \left| \int_0^1 u^{a-1} du - \int_0^1 u^{a-1} du \right|$$

$$= 0$$

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by Lebesgue's bounded convergence theorem, the argument used in proving (3.5) and Lemma 3.2 applied once to the function $f_1(t)$ and next to the function $f_2(t)$ and then letting $\delta \downarrow 0$. By (3.5) and (3.6) our lemma is true for $G(u) = u^a$, $a \geqq 1$. Hence the lemma is also true if $G(u)$ is the integral of a polynomial. Now we prove the lemma for an arbitrary absolutely continuous function $G(u)$ in $[0,1]$. We have $G(t) = \int_0^t g(u)du$ for $0 \leqq t \leqq 1$ where $g(u) \in L_1[0,1]$. Given $\varepsilon > 0$ there is a polynomial $f(u)$ such that $\int_0^1 |g(u) - f(u)|\,du < \varepsilon/4$, since the polynomials are dense in $L_1[0,1]$. Also, by the first part of the proof for $\lambda > \lambda(\varepsilon)$ and the polynomial $f(u)$ we have

$$\sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} P(u) du \right| < \frac{\varepsilon}{2}.$$

Now for $\lambda > \lambda(\varepsilon)$

$$\begin{aligned}
\sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} dG(u) \right| &= \sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} g(u) du \right| \\
&= \sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} \{g(u) - f(u) + f(u)\} du \right| \\
&\leqq \sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} f(u) du \right| \\
&\quad + \sum_{k=0}^{\infty} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} \{g(u) - f(u)\} du \right| \\
&\leqq \frac{\varepsilon}{2} + \int_0^1 |g(u) - f(u)| \left\{ \sum_{k=0}^{\infty} P_{nk}(u) + \sum_{k=0}^{\infty} P^*_{xk}(qu) \right\} du
\end{aligned}$$

$$\left(\text{and since } \sum_{k=0}^{\infty} P_{nk}(u) = \sum_{k=0}^{\infty} P^*_{xk}(qu) \equiv 1\right)$$

$$\begin{aligned}
&= \frac{\varepsilon}{2} + 2\int_0^1 |g(u) - f(u)| du \\
&< \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} \\
&= \varepsilon.
\end{aligned}$$

This proves the lemma if $G(t)$ is the integral of a function in $L_1[0,1]$, or equivalently, if $G(t)$ is absolutely continuous in $[0,1]$.

LEMMA 3.6. *Suppose $\beta(t)$ is of bounded variation in $[0,1]$ and $\beta(0) = \beta(0+) = 0$. Then for each α, $0 < \alpha < 1$, the function $K(t) \equiv \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \beta(e^{-t/u}) du$ is of bounded variation in $[0, \infty)$ and continuous in $(0, \infty]$. If in addition $\beta(t)$ is continuous at $t = 1$, then $K(t)$ is continuous at $t = 0$.*

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PROOF. For any subdivision $0=t_0 < t_1 < t_2 \dots t_n$ we have

$$\sum_{k=0}^{n-1} |K(t_{k+1}) - K(t_k)| \leqq \left(\int_0^1 |d\beta(v)|\right) \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du.$$

Hence $K(t)$ is of bounded variation in $[0, \infty)$. By Lebesgue's dominated convergence theorem we get for $0 < t_0 < \infty$

$$\begin{aligned}\lim_{t \to t_0} K(t) &= \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \left(\lim_{t \to t_0} \beta(e^{-t/u})\right) du \\ &= \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \beta(e^{-t_0/u}) du \\ &= K(t_0)\end{aligned}$$

since $\beta(v)$ is continuous almost everywhere in $[0,1]$. Similarly we get $\lim_{t \uparrow \infty} K(t) = 0$. If $\beta(v)$ is continuous at $v = 1$ we get $\lim_{t \downarrow 0} K(t) = K(0)$.

LEMMA 3.7. *For a function $\beta(t)$ of bounded variation in $[0,1]$, a real number α, $0 < \alpha < 1$, any two positive functions $x(\lambda) \to \infty$, $n(\lambda) \to \infty$, $n(\lambda)/x(\lambda) \to q$ $(\lambda \to \infty)$, $0 < q < \infty$, and any number A, $q < A < \infty$, we have*

$$\lim_{\lambda \to \infty} \sum_{k=0}^{n} \left| \int_A^\infty P_{xk}^*(t) \frac{\alpha}{t} \left(\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} [1 - \beta(e^{-t/u})] du \right) dt \right| = 0.$$

PROOF. For $0 \leqq k \leqq n$ we have

$$\begin{aligned}I_k &\equiv \left| \int_A^\infty P_{xk}^*(t) \frac{\alpha}{t} \left(\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} [1 - \beta(e^{-t/u})] du \right) dt \right| \\ &\leqq \left(1 + \sup_{0 \leqq v \leqq 1} |\beta(v)|\right) \int_A^\infty P_{xk}^*(t) \frac{\alpha}{t} \left(\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du \right) dt \\ &\equiv M \int_A^\infty \frac{P_{xk}^*(t)}{t} dt.\end{aligned}$$

Hence

$$\sum_{k=0}^{n} I_k \leqq M \sum_{k=0}^{n} \int_A^\infty P_{xk}^*(t) t^{-1} dt \to 0 \; (\lambda \to \infty)$$

by [8, (5.17), (5.18), (5.19)].

LEMMA 3.8. *For a function $\beta(t)$ bounded and L-integrable in $[0,1]$ satisfying*

$$\int_0^1 \frac{|1 - \beta(e^{-u})|}{u} du < \infty$$

and a real number α, $0 < \alpha < 1$, the function

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$$M(t) \equiv t^{-1} \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \left| 1 - \beta(e^{-t/u}) \right| du$$

is Lebesgue-integrable in each interval $[0, A]$, $A > 0$.

LEMMA 3.9. *Suppose* $\beta(t)$ *is of bounded variation in* $[0,1]$, $\beta(0) = \beta(0+) = 0$ *and* $\dfrac{\beta(e^{-v})}{v}$ *is Lebesgue-integrable over* $[1, \infty)$. *Then for each* α, $0 \leqq \alpha < 1$, *the function*

$$N(x) \equiv \int_x^\infty t^\alpha d_t \left[\int_t^\infty (v-t)^{-\alpha} \frac{\beta(e^{-v})}{v} dv \right]$$

is continuous and of bounded variation in each interval $[\varepsilon, \infty)$, $\varepsilon > 0$.

PROOF. By Fubini's theorem and Lemma 3.6 we get $K(t)/t \in L[\varepsilon, \infty]$ for each $\varepsilon > 0$. By changing variables we get

$$\int_t^\infty (v-t)^{-\alpha} \frac{\beta(e^{-v})}{v} dv = t^{-\alpha} K(t).$$

Using these two results we get by integration by parts and by using Lemma 3.6

$$-N(x) = K(x) + \alpha \int_x^\infty \frac{K(t)}{t} dt.$$

The proof follows now by Lemma 3.6.

PROOF OF THEOREM 2.1. First we establish (3.14) which is the main step in the proof. For this end properties of $\gamma_{nk}^{(\alpha)}$ and $\beta_{xk}^{(\alpha)}$ defined below are needed. For $0 < \alpha < 1$ and $k > 0$ we have by [9, (5.3)] for $\lambda_n = n$

$$(3.8)\quad \begin{aligned} &\int_0^1 \frac{1-\gamma(t)}{t} dt \int_0^t (t-u)^{-\alpha} \frac{d}{du} [P_{nk}(u) u^\alpha] du \\ &\quad = -\int_0^1 P_{nk}(u) d_u \left\{ \int_u^1 t^\alpha d_t \left[\int_t^1 (v-t)^{-\alpha} \frac{\gamma(v)}{v} dv \right] \right\} \\ &\qquad + \int_0^1 P_{nk}(u)(1-u)^{-\alpha} u^{\alpha-1} du \\ &\qquad - \int_0^1 P_{nk}(u) \left(\frac{\alpha}{u} \int_0^u (1-v)^{-\alpha} v^{\alpha-1} dv \right) du \\ &\qquad + \int_0^1 \frac{P_{nk}(u)}{u} \left(\alpha \int_0^1 (1-v)^{-\alpha} v^{\alpha-1} dv \right) du \\ &\quad = \gamma_{nk}^{(\alpha)}. \end{aligned}$$

For $\alpha = 0$ we have by [9, (5.3)]

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(3.9)
$$\begin{aligned}\int_0^1 \frac{1-\gamma(t)}{t}dt \int_0^1 (t-u)^{-\alpha} \frac{d}{du}[P_{nk}(u)u^\alpha]du \\ = -\int_0^1 P_{nk}(v)d_v \left\{\int_v^1 t^\alpha d_t \left[\int_t^1 (u-t)^{-\alpha} \frac{\gamma(u)}{u}du\right]\right\} \\ + \int_0^1 \frac{P_{nk}(v)}{v}dv \\ \equiv \gamma_{nk}^{(0)}.\end{aligned}$$

By [9, (4.5)] we have for $\lambda_k = k$

(3.10)
$$\int_0^1 \frac{P_{nk}(v)}{v}dv = \frac{1}{k}.$$

For $A > 0$, and in particular for $A > q$, $k > 0$ and $0 < \alpha < 1$ we have

(3.11)
$$\begin{aligned}\int_0^\infty P_{xk}^*(t)d_t \left\{\int_t^\infty u^\alpha d_u \left[\int_u^\infty (v-u)^{-\alpha} \frac{\beta(e^{-v})}{v}dv\right]\right\} \\ = -\int_0^A P_{xk}^*(t) \frac{\alpha}{t} \left\{\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}[1-\beta(e^{-t/u})]du\right\} dt \\ - \int_A^\infty P_{xk}^*(t) \frac{\alpha}{t} \left\{\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}[1-\beta(e^{-t/u})]du\right\} dt \\ - \int_0^\infty P_{xk}^*(t)d_t \left\{\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}\beta(e^{-t/u})du\right\} \\ + \int_0^\infty \frac{P_{xk}^*(t)}{t} \left\{\alpha\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}\, du\right\} dt \\ \equiv \beta_{xk}^{(\alpha)}.\end{aligned}$$

For $\alpha = 0$, $k > 0$ and each A, $q < A < \infty$, we have

(3.12)
$$\begin{aligned}\int_0^\infty P_{xk}^*(t)d_t \left\{\int_t^\infty u^\alpha d_u \left[\int_u^\infty (v-u)^{-\alpha} \frac{\beta(e^{-v})}{v}dv\right]\right\} \\ = -\int_0^A P_{xk}^*(t) \frac{1-\beta(e^{-t})}{t}dt - \int_A^\infty \frac{P_{xk}^*(t)}{t}[1-\beta(e^{-t})]dt \\ + \int_0^\infty \frac{P_{xk}^*(t)}{t}dt \\ = \beta_{xk}^{(0)}.\end{aligned}$$

For $k > 0$ we have

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$$\int_0^\infty \frac{P^*_{xk}(v)}{v}dv = \frac{1}{k}. \tag{3.13}$$

By [9, Lemma 5.4], Lemma 3.4, (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) we get for $0 \leqq \alpha < 1$

$$\begin{aligned} H_n(\gamma) - J_x(\beta) &= \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \\ &\cdot \left\{ \sum_{k=1}^{n} a_k^{(\alpha)}(\gamma_{nk}^{(\alpha)} - \beta_{xk}^{(\alpha)}) - \sum_{k=n+1}^{\infty} a_k^{(\alpha)} \beta_{xk}^{(\alpha)} \right\}. \end{aligned} \tag{3.14}$$

To complete the proof of our theorem it is sufficient, by Agnew's theorem (see [8]) to show that we have

$$\lim_{\lambda\to\infty} \{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^{-1} \left\{ \sum_{k=1}^{n(\lambda)} \left| \gamma_{n(\lambda),k}^{(\alpha)} - \beta_{x(\lambda),k}^{(\alpha)} \right| + \sum_{k=n(\lambda)+1}^{\infty} \left| \beta_{x(\lambda),k}^{(\alpha)} \right| \right\} = G_q^{(\alpha)}$$

and $\lim_{\lambda\to\infty}[\gamma_{n(\lambda),k}^{(\alpha)} - \beta_{x(\lambda),k}^{(\alpha)}] = 0$ for each $k \geqq 1$. By [8, Theorem 2.1 and Remark (2.2)], Lemma 3.1, Lemma 3.9, we get by applying to (3.11) an obvious modification of (5.13) and (5.16) of [8],

$$\lim_{\lambda\to\infty} \sum_{k=n+1}^{\infty} \left| \beta_{x(\lambda),k}^{(\alpha)} \right| = \int_q^\infty t^\alpha \left| d_t \left[\int_t^\infty (u-t)^{-\alpha} \frac{\beta(e^{-u})}{u} du \right] \right|. \tag{3.15}$$

By the second conclusion of [9, Theorem 3.1 for assumption (III)], Lemmas 3.5, (3.6), (3.7) and (3.8) and [9, (5.8), (5.10)] we get for $0 < \alpha < 1$

$$\begin{aligned} &\lim_{\lambda\to\infty} \sum_{k=1}^{n} \left| \gamma_{n(\lambda),k}^{(\alpha)} - \beta_{x(\lambda),k}^{(\alpha)} \right| \\ &= \int_0^q \Bigg| d_x \Bigg\{ - \int_{x/q}^1 t^\alpha d_t \left[\int_t^1 (u-t)^{-\alpha} \frac{\gamma(u)}{u} du \right] \\ &\quad + \int_0^{x/q} t^\alpha d_t \left[\int_1^\infty (v-t)^{-\alpha} \frac{dv}{v} \right] \\ &\quad - x^\alpha \int_x^\infty (v-x)^{-\alpha} \frac{\beta(e^{-v})}{v} dv - \alpha \int_0^x u^{\alpha-1} \left[\int_u^\infty (w-u)^{-\alpha} \frac{1-\beta(e^{-w})}{w} dw \right] \Bigg\} \Bigg| \\ &= H_q^{(\alpha)}, \end{aligned} \tag{3.16}$$

and for $\alpha = 0$

$$\lim_{\lambda\to\infty} \sum_{k=1}^{n} \left| \gamma_{n(\lambda),k}^{(0)} - \beta_{n(\lambda),k}^{(0)} \right| = \int_0^q \frac{|1 - \beta(e^{-u}) - (u/q)|}{u} du = H_q^{(0)}. \tag{3.17}$$

It is easy to see that we have

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$$\lim_{\lambda\to\infty} \left| \gamma^{(\alpha)}_{x(\lambda),k} - \beta^{(\alpha)}_{x(\lambda),k} \right| = 0 \text{ for } k = 1, 2, \ldots . \tag{3.18}$$

The proof follows now by the remark after (3.14), by (3.15), (3.16), (3.17) and (3.18).

LEMMA 3.10. *If $\beta(t)$ is of bounded variation in $[0,1]$ and $\beta(e^{-t}) \in L_1[1, \infty]$ and $x(\lambda) \to \infty$, $n(\lambda) \to \infty$, $n(\lambda)/x(\lambda) \to q$ $(0 < q < \infty)$, then for each A, $q < A < \infty$, we have*

$$\lim_{\lambda\to\infty} \sum_{k=1}^{n} \int_A^\infty P^*_{xk}(t) d\left[\int_0^t u d\left(\frac{1-\beta(e^{-u})}{u} \right) \right] = 0.$$

PROOF. We have for $A > q$

$$\begin{aligned} &\sum_{k=1}^{n} \int_A^\infty P^*_{xk}(t) d\left[\int_0^t u d\left(\frac{1-\beta(e^{-u})}{u} \right) \right] \\ &\quad = - \sum_{k=1}^{n} \int_A^\infty P^*_{xk}(t) \frac{1-\beta(e^{-t})}{t} dt - \sum_{k=1}^{n} \int_A^\infty P^*_{xk}(t) d\beta(e^{-t}) \\ &\quad \equiv I^{(1)}_\lambda + I^{(2)}_\lambda. \end{aligned}$$

As in the proof of Lemma 3.7 we have

$$|I^{(1)}_\lambda| \leqq K \sum_{k=1}^{n} \int_A^\infty \frac{P^*_{xk}(t)}{t} dt \to 0 \qquad (\lambda \uparrow \infty).$$

Integrating by parts we get

$$\begin{aligned} I^{(2)}_\lambda &= \beta(e^{-A}) \sum_{k=0}^{n} P^*_{xk}(A) + \int_A^\infty \beta(e^{-u}) \frac{d}{du} \sum_{k=0}^{n} P^*_{xk}(u) du \\ &\quad + x \int_A^\infty \beta(e^{-u}) e^{-xu} du - e^{-Ax} \beta(e^{-A}) \\ &= \beta(e^{-A}) \sum_{k=0}^{n} P^*_{xk}(A) - e^{-Ax}\beta(e^{-A}) + x \int_A^\infty \beta(e^{-t}) e^{-xt} dt - x \int_A^\infty \beta(e^{-t}) P^*_{xn}(t) dt \\ &\equiv I^{(21)}_\lambda + I^{(22)}_\lambda + I^{(23)}_\lambda + I^{(24)}_\lambda. \end{aligned}$$

We have by Lemma 3.2

$$\lim_{\lambda\to\infty} I^{(21)}_\lambda = 0 \text{ and } \lim_{\lambda\to\infty} I^{(22)}_\lambda = 0.$$

We have

$$|I^{(23)}_\lambda| \leqq K_1 x \int_A^\infty e^{-xt} dt = K_1 \int_{Ax}^\infty e^{-u} du \to 0 \qquad (\lambda \to \infty).$$

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Since $P^*_{xn}(t)$ is a decreasing function in t for $t \geqq n/x$ we get

$$|I_\lambda^{(24)}| \leqq x P^*_{xn}(A) \int_A^\infty |\beta(e^{-t})|\, dt$$

(and by Stirling's formula for $n!$)

$$\to 0 \qquad (\lambda \to \infty).$$

This completes the proof.

PROOF OF THEOREM 2.2. We have by [9, Lemmas 5.4 and (5.5) (for $\lambda_n = n$)] and by Lemma 3.4

$$\begin{aligned} H_n(\gamma) - J_x(\beta) = \sum_{k=1}^{n} a_k^{(1)} \Big\{ &- \int_0^1 P_{nk}(t) d\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] + \frac{1}{k} \\ &- \int_0^\infty P^*_{xk}(t) d\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right]\Big\} \\ &- \sum_{k=n+1}^{\infty} a_k^{(1)} \int_0^\infty P^*_{xk}(t) d\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right]. \end{aligned}$$

By (3.13) we get (for $q < A < \infty$)

$$\int_0^\infty P^*_{xk}(t) d\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right] + \left\{\int_0^A + \int_A^\infty\right\} P^*_{xk}(t) d_t\left[\int_0^t u d\left(\frac{1-\beta(e^{-u})}{u}\right)\right] + \frac{1}{k}.$$

Hence

$$\begin{aligned} H_n(\gamma) - J_x(\beta) = \sum_{k=1}^{n} a_k^{(1)} \Big\{ &- \int_0^1 P_{nk}(t) d_t\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] \\ &- \left(\int_0^A + \int_A^\infty\right) P^*_{xk}(t) d_t\left[\int_0^t u d\left(\frac{\beta(e^{-u})}{u}\right)\right]\Big\} \\ &- \sum_{k=n+1}^{\infty} a_k^{(1)} \int_0^\infty P^*_{xk}(t) d\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right]. \end{aligned}$$

To complete the proof of our theorem it is sufficient, by Agnew's Theorem (see [8]) to show that we have

$$\begin{aligned} \lim_{\lambda\to\infty} \Big\{ \sum_{k=1}^{\infty} \Big| &- \int_0^1 P_{nk}(t) d_t\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] \\ &- \left(\int_0^A + \int_A^\infty\right) P^*_{xk}(t) d_t\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right]\Big| \\ &+ \sum_{k=n+1}^{\infty} \Big| \int_0^\infty P^*_{xk}(t) d_t\left[\int_t^\infty u d\left(\frac{\beta(e^{-u})}{u}\right)\right]\Big| = G_q^{(1)} \end{aligned}$$

and

$$\lim_{\lambda\to\infty}\left\{-\int_0^1 P_{nk}(t)d_t\left[\int_t^1 ud\left(\frac{\gamma(u)}{u}\right)\right]\right.$$
$$\left.-\left(\int_0^A+\int_A^\infty\right)P^*_{xk}(t)d_t\left[\int_t^\infty ud\left(\frac{\beta(e^{-u})}{u}\right)\right]\right\}=0,$$

for each $k\geqq 1$.

Repeating the argument used in [8] to prove that in [8, (8.10)] we have

$$\lim_{\substack{\lambda\to\infty\\ m(\lambda)/x(\lambda)\to q}}\sum_{k=m+1}^{\infty}\frac{1}{k}\,|D_k(x)|=\int_q^\infty\frac{|\beta(e^{-u})|}{u}du$$

we get here

$$\lim_{\lambda\to\infty}\sum_{k=n(\lambda)+1}^{\infty}\left|\int_0^\infty P^*_{xk}(t)d\left[\int_t^\infty ud\left(\frac{\beta(e^{-u})}{u}\right)\right]\right|=\int_q^\infty u\left|d\left(\frac{\beta(e^{-u})}{u}\right)\right|.$$

Write $\gamma(t)=\gamma_1(t)+\gamma_2(t)$ where $\gamma_1(t)=\gamma(t)$ $(0\leqq t<1,\ \gamma_1(1)=\gamma(1-0)$. Note that $P_{nk}(1)=0$ for $0\leqq k<n$. The proof follows now by Lemma 3.10, and by repeating the argument used in the proof of [9, Theorem 2.2] and by using the fact that for $\lambda>\Lambda$

$$\int_0^A P^*_{xn}(t)d\left[\int_0^t ud\left(\frac{1-\beta(e^{-u})}{u}\right)\right]$$
$$\leqq\left(\int_0^A\left|d\left[\int_0^t ud\left(\frac{1-\beta(e^{-u})}{u}\right)\right]\right|\right)\cdot\max_{0\leqq t\leqq A}P^*_{xn}(t)=K_2\cdot\frac{n^n}{n!}e^{-n}$$

(by Stirling's formula) $\sim K_3\cdot n^{-\frac{1}{2}}\to 0 \qquad (\lambda\to\infty)$.

References

[1] R. P. Agnew, 'Abel transforms anad partial sums of Tauberian series', *Ann. of Math.* 50 (1949), 110–117.

[2] H. Hadwiger, 'Über ein Distanztheorem bei der A-Limitierung', *Comment. Math.* 16 (1944), 209–214.

[3] G. H. Hardy, *Divergent series* (Oxford University Press, 1949).

[4] E. W. Hobson, *The theory of functions of a real variable*, Vol. 1 (Dover, New York, 1957), 545.

[5] A. Jakimovski (Amir), 'Some relations between the methods of summability of Abel, Borel Cesàro, Holder and Hausdorff,' *J. d'Analyse Math.*, 3 (1953/4), 346–381.

[6] A. Jakimovski, 'The sequence-to-function analogues to Hausdorff transformations', *Bull. Res. Council Israel* 8F (1960), 135–154.

[7] A. Jakimovski, 'Tauberian constants for the Abel and Cesàro transformations', *Proc. Amer. Math. Soc.* 14 (1963), 228–238.

[8] A. Jakimovski and D. Leviatan, 'A property of approximation operators and applications to Tauberian constants', *Math. Z.* 102 (1967), 177–204.
[9] A. Jakimovski and A. Livne, 'Approximation operators and Tauberian constants', *Israel. J. Math.* 7 (1969), 263–292.
[10] K. Knopp, *Theory and applications of infinite series* (Blackie and Son, London and Glasgow, 1944).
[11] D Leviaten, 'Tauberian constants for generalized Hausdorff transformations', *J. London Math. Soc.* 43 (1968), 308–314.
[12] D. Leviatan, 'Some Tauberian theorems for quasi-Hausdorff transforms', *Math. Z.* 108 (1969), 213–222.
[13] G. G. Lorentz, *Bernstein polynomials* (Toronto University Press, 1953).
[14] A. Meir, 'Limit-distance of Hausdorff transforms of Tauberian series', *J. London Math. Soc.* 40 (1965), 295–302.
[15] O. Szasz, 'Generalization of S. Bernstein's polynomials to the infinite interval', *Collected mathematical works* (University of Cincinnati, 1955), 1401–1407.
[16] D. V. Widder, *The Laplace transform* (Princeton University Press, 1946).

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