4

The Geodesic X-ray Transform

In this chapter we begin the study of the geodesic X-ray transform on a compact non-trapping manifold with strictly convex boundary. We prove L^2 and Sobolev mapping properties, and discuss a reduction that allows us to convert statements about the X-ray transform to statements about transport equations on SM involving the geodesic vector field. We then prove a fundamental energy identity, known as the Pestov identity, for functions on SM. As the main result in this chapter, we prove injectivity of the geodesic X-ray transform I_0 on simple two-dimensional manifolds by using the Pestov identity. We also give an initial stability estimate for the geodesic X-ray transform (improved stability estimates will be given later). Results in higher dimensions are discussed at the end of the chapter.

4.1 The Geodesic X-ray Transform

We have already encountered the geodesic X-ray transform acting on functions $f \in C^\infty(M)$ in Definition 3.1.5. The same definition applies more generally to functions in $C^\infty(SM)$.

Definition 4.1.1 Let (M, g) be a compact non-trapping manifold with strictly convex boundary. The *geodesic X-ray transform* is the operator

$$I\colon C^\infty(SM) \to C^\infty(\partial_+ SM),$$

given by

$$If(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt, \qquad (x,v) \in \partial_+ SM.$$

The geodesic X-ray transform on $C^\infty(M)$ is denoted by

$$I_0\colon C^\infty(M) \to C^\infty(\partial_+ SM), \;\; I_0 f = I(\ell_0 f),$$

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where $\ell_0\colon C^\infty(M) \to C^\infty(SM)$ is the natural inclusion, i.e. $\ell_0 f(x,v) = f(x)$ is the pullback of functions by the projection map $\pi : SM \to M$.

Recall from Lemma 3.2.6 that $\tau|_{\partial_+ SM} \in C^\infty(\partial_+ SM)$, so indeed I maps $C^\infty(SM)$ to $C^\infty(\partial_+ SM)$. We next study the mapping properties of I on L^2-based spaces. Recall that

$$L^2(SM) = L^2(SM, d\Sigma^{2n-1}),$$
$$L^2(\partial_+ SM) = L^2(\partial_+ SM, d\Sigma^{2n-2}).$$

If $p \in C^\infty(\partial_+ SM)$ is non-negative, we also consider the weighted space $L^2_p(\partial_+ SM)$ consisting of L^2-functions on $\partial_+ SM$ with respect to the measure $p\, d\Sigma^{2n-2}$.

Proposition 4.1.2 (L^2 boundedness) *I extends to a bounded operator*

$$I\colon L^2(SM) \to L^2(\partial_+ SM).$$

Proof Since $p := \mu/\tilde{\tau}$ is in $C^\infty(\partial SM)$ and it is strictly positive by Lemma 3.2.8, it suffices to prove the lemma using the measure $p\, d\Sigma^{2n-2}$ in the target space. Take $f \in C^\infty(SM)$ and write, using Cauchy–Schwarz,

$$\begin{aligned}
\|If\|^2_{L^2_p(\partial_+ SM)} &= \int_{\partial_+ SM} \left| \int_0^{\tau(x,v)} f(\varphi_t(x,v))\, dt \right|^2 p\, d\Sigma^{2n-2} \\
&\leq \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} |f(\varphi_t(x,v))|^2\, dt \right) \tau p\, d\Sigma^{2n-2} \\
&= \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} |f(\varphi_t(x,v))|^2\, dt \right) \mu\, d\Sigma^{2n-2} \\
&= \int_{SM} |f|^2\, d\Sigma^{2n-1} = \|f\|^2_{L^2(SM)},
\end{aligned}$$

where in the last line we have used Santaló's formula from Proposition 3.6.6. □

The geodesic X-ray transform is also bounded between Sobolev spaces. The proof of the next result is given in Section 4.5.

Proposition 4.1.3 (Sobolev boundedness) *For any $k \geq 0$, the operator I extends to a bounded operator*

$$I\colon H^k(SM) \to H^k(\partial_+ SM).$$

We also have $I(H^1(SM)) \subset H^1_0(\partial_+ SM)$.

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In the literature, one often sees the statement that I extends to a bounded operator

$$I : L^2(SM) \to L^2_\mu(\partial_+ SM), \tag{4.1}$$

where $\mu(x,v) = \langle \nu(x), v\rangle$. Since $|\mu| \le 1$, this is a special case of Proposition 4.1.2. However, the L^2_μ space is a useful setting for studying I since the adjoint I^* of the operator (4.1) is readily computed by Santaló's formula. Moreover, as we will see in Chapter 8, on simple manifolds the normal operator $I_0^* I_0$ (where I_0 is I restricted to functions on M) is an elliptic pseudodifferential operator of order -1 just like in the case of the Radon transform in the plane.

We conclude this section by computing the adjoint of the operator (4.1).

Lemma 4.1.4 (The adjoints I^* and I_0^*) *The adjoint of* $I : L^2(SM) \to L^2_\mu(\partial_+ SM)$ *is the bounded operator*

$$I^* : L^2_\mu(\partial_+ SM) \to L^2(SM),$$

given for $h \in C^\infty(\partial_+ SM)$ *by* $I^* h = h^\sharp$, *where*

$$h^\sharp(x,v) := h(\varphi_{-\tau(x,-v)}(x,v)).$$

The adjoint of $I_0 : L^2(M) \to L^2_\mu(\partial_+ SM)$ *is given by*

$$I_0^* h(x) = \int_{S_x M} h^\sharp(x,v)\, dS_x(v).$$

Proof Consider $f \in C^\infty(SM)$ and $h \in C^\infty(\partial_+ SM)$, and write

$$\begin{aligned}(If, h)_{L^2_\mu(\partial_+ SM)} &= \int_{\partial_+ SM} (If)\bar{h}\mu \, d\Sigma^{2n-2} \\ &= \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v))\overline{h(x,v)}\, dt \right) \mu \, d\Sigma^{2n-2}.\end{aligned}$$

We can write the above expression as

$$(If, h)_{L^2_\mu(\partial_+ SM)} = \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v))\overline{h^\sharp(\varphi_t(x,v))}\, dt \right) \mu \, d\Sigma^{2n-2}.$$

Using Santaló's formula we derive

$$(If, h)_{L^2_\mu(\partial_+ SM)} = \int_{SM} f\overline{h^\sharp}\, d\Sigma^{2n-1} = (f, h^\sharp)_{L^2(SM)},$$

and hence $I^* h = h^\sharp$.

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Choosing $f = f(x)$ gives

$$\begin{aligned}(I_0 f, h)_{L^2_\mu(\partial_+ SM)} &= \int_M f(x) \left[\int_{S_x M} \overline{h^\sharp} \, dS_x \right] dV^n \\ &= \left(f, \int_{S_x M} h^\sharp \, dS_x \right)_{L^2(M)}.\end{aligned}$$

This gives the required formula for I_0^*. □

Exercise 4.1.5 Let $\ell_0 \colon C^\infty(M) \to C^\infty(SM)$ be the map given by $\ell_0 f = f \circ \pi$, where $\pi : SM \to M$ is the canonical projection. Show that the adjoint ℓ_0^* is given by

$$(\ell_0^* h)(x) = \int_{S_x M} h(x,v) \, dS_x(v).$$

4.2 Transport Equations

We will next show that it is possible to reduce statements about the geodesic X-ray transform to statements about transport equations on SM involving the geodesic vector field X. We first define two important notions that have already appeared before in Chapter 3.

Definition 4.2.1 (The functions u^f and $h^\sharp$) Let (M, g) be a compact non-trapping manifold with strictly convex boundary. Given any $f \in C^\infty(SM)$, define

$$u^f(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \, dt, \qquad (x,v) \in SM.$$

For any $h \in C^\infty(\partial_+ SM)$ define

$$h^\sharp(x,v) := h(\varphi_{-\tau(x,-v)}(x,v)), \qquad (x,v) \in SM.$$

It follows that u^f solves the transport equation $Xu^f = -f$ and If is given by the boundary value of u^f on $\partial_+ SM$. Moreover, $h^\sharp$ is constant along geodesics. In other words $h^\sharp$ is an *invariant function* (or first integral) with respect to the geodesic flow, i.e. $Xh^\sharp = 0$.

Lemma 4.2.2 (Properties of u^f and $h^\sharp$)

(a) For any $f \in C^\infty(SM)$, one has $u^f \in C(SM) \cap C^\infty(SM \setminus \partial_0 SM)$ and u^f is the unique solution of the equation

$$Xu^f = -f \text{ in } SM, \qquad u^f|_{\partial_- SM} = 0.$$

Moreover, $u^f|_{\partial_+ SM} = If$.

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(b) *For any* $h \in C^\infty(\partial_+ SM)$*, one has* $h^\sharp \in C(SM) \cap C^\infty(SM \setminus \partial_0 SM)$ *and* $h^\sharp$ *is the unique solution of the equation*

$$Xh^\sharp = 0 \text{ in } SM, \qquad h^\sharp|_{\partial_+ SM} = h.$$

Moreover, $h^\sharp|_{\partial_- SM} = h \circ \alpha|_{\partial_+ SM}$.

Proof The regularity properties of u^f and $h^\sharp$ follow from the regularity properties of τ given in Lemma 3.2.3. We note that for $(x, v) \in SM^{\text{int}}$,

$$\begin{aligned} Xu^f(x,v) &= \frac{d}{ds} \int_0^{\tau(\varphi_s(x,v))} f(\varphi_t(\varphi_s(x,v)))\,dt\Big|_{s=0} \\ &= \frac{d}{ds} \int_0^{\tau(x,v)-s} f(\varphi_{t+s}(x,v))\,dt\Big|_{s=0} \\ &= -f(\varphi_{\tau(x,v)}(x,v)) + \int_0^{\tau(x,v)} \frac{d}{dt} f(\varphi_t(x,v))\,dt \\ &= -f(x,v). \end{aligned}$$

Clearly $Xh^\sharp = 0$. The statements about the boundary values of u^f and $h^\sharp$ follow from the definitions of I and α and the fact that $\tau|_{\partial_- SM} = 0$. □

We note that u^f is, in general, not smooth on SM. For instance, if $f = 1$ then $u^f = \tau$ and we know from Example 3.2.1 that τ is not smooth on SM. However, if f is a function whose geodesic X-ray transform vanishes, then the following result shows that $u^f \in C^\infty(SM)$ and the somewhat annoying issue with non-smoothness disappears. The result follows from the precise regularity properties of the exit time proved in Lemma 3.2.9. We defer its proof to Chapter 5, where regularity results for transport equations will be studied in more detail.

Proposition 4.2.3 (Regularity when $If = 0$) *Let* (M, g) *be a compact non-trapping manifold with strictly convex boundary. If* $f \in C^\infty(SM)$ *satisfies* $If = 0$*, then* $u^f \in C^\infty(SM)$.

The next result characterizes functions in the kernel of the geodesic X-ray transform in terms of solutions to the transport equation $Xu = f$.

Proposition 4.2.4 *Let* $f \in C^\infty(SM)$*. The following conditions are equivalent.*

(a) $If = 0$.
(b) *There is* $u \in C^\infty(SM)$ *such that* $u|_{\partial SM} = 0$ *and* $Xu = -f$.

Proof Suppose that $If = 0$. Proposition 4.2.3 guarantees that $u = u^f \in C^\infty(SM)$, and Lemma 4.2.2 gives that $Xu = -f$.

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Conversely, given $u \in C^\infty(SM)$ with $Xu = -f$, if we integrate along the geodesic flow we obtain for $(x, v) \in \partial_+ SM$ that

$$u \circ \alpha(x, v) - u(x, v) = -\int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt = -If(x, v).$$

Hence if $u|_{\partial SM} = 0$, the above equality implies $If = 0$. □

4.3 Pestov Identity

In this section we consider the Pestov identity in two dimensions. This is the basic energy identity that has been used since the work of Muhometov (1977) in studying injectivity of ray transforms in the absence of real-analyticity or special symmetries. Pestov-type identities were also used in Pestov and Sharafutdinov (1987) to prove solenoidal injectivity of the geodesic X-ray transform for tensors of any order on simple manifolds with negative sectional curvature. These identities have often appeared in a somewhat ad hoc way. Here, following Paternain et al. (2013), we give a point of view that makes the derivation of the Pestov identity more transparent.

The easiest way to motivate the Pestov identity is to consider the injectivity of the ray transform on functions. As in Section 4.1 we let $I_0 \colon C^\infty(M) \to C^\infty(\partial_+ SM)$ be defined by $I_0 := I \circ \ell_0$, where ℓ_0 is the pullback of functions from M to SM.

The first step is to recast the injectivity problem for I_0 as a uniqueness question for the partial differential operator P on SM, where

$$P := VX.$$

This involves a standard reduction to the transport equation as we have done already in Proposition 4.2.4.

Proposition 4.3.1 *Let (M, g) be a compact oriented non-trapping surface with strictly convex boundary. The following statements are equivalent.*

(a) The ray transform $I_0 \colon C^\infty(M) \to C^\infty(\partial_+ SM)$ is injective.
(b) Any smooth solution of $Xu = -f$ in SM with $u|_{\partial SM} = 0$ and $f \in C^\infty(M)$ is identically zero.
(c) Any smooth solution of $Pu = 0$ in SM with $u|_{\partial SM} = 0$ is identically zero.

Proof (a) $\implies$ (b): Assume that I_0 is injective, and let $u \in C^\infty(SM)$ solve $Xu = -f$ in SM where $u|_{\partial SM} = 0$ and $f \in C^\infty(M)$. By Proposition 4.2.4 one has $0 = If = I_0 f$. Hence $f = 0$ by injectivity of I_0, which shows that

$Xu = 0$. Thus u is constant along geodesics, and the condition $u|_{\partial SM} = 0$ gives that $u \equiv 0$.

(b) $\implies$ (c): Let $u \in C^\infty(SM)$ solve $Pu = 0$ in SM with $u|_{\partial SM} = 0$. Since the kernel of V consists of functions on SM only depending on x, this implies that $Xu = -f$ in SM for some $f \in C^\infty(M)$. By the statement in (b) we have $u \equiv 0$.

(c) $\implies$ (a): Assume that the only smooth solution of $Pu = 0$ in SM that vanishes on ∂SM is zero. Let $f \in C^\infty(M)$ be a function with $I_0 f = 0$. Proposition 4.2.4 gives a function $u \in C^\infty(SM)$ such that $Xu = -f$ and $u|_{\partial SM} = 0$. Since f only depends on x we have $Vf = 0$, and consequently $Pu = 0$ in SM and $u|_{\partial SM} = 0$. It follows that $u = 0$ and also $f = -Xu = 0$. □

We now focus on proving uniqueness for solutions of $Pu = 0$ in SM satisfying $u|_{\partial SM} = 0$. For this it is convenient to express P in terms of its self-adjoint and skew-adjoint parts in the $L^2(SM)$ inner product as

$$P = A + iB, \quad A := \frac{P + P^*}{2}, \;\; B := \frac{P - P^*}{2i}.$$

Here the formal adjoint P^* of P is given by

$$P^* := XV.$$

The commutator formula $[X, V] = X_\perp$ in Lemma 3.5.5 shows that

$$A = \frac{VX + XV}{2}, \qquad B = -\frac{1}{2i} X_\perp.$$

Now, if $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$, we may use the integration by parts formulas in Proposition 3.5.12 (note that the boundary terms vanish since $u|_{\partial SM} = 0$) to obtain that

$$\begin{aligned} \|Pu\|^2 &= ((A + iB)u, (A + iB)u) \\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned} \tag{4.2}$$

This computation suggests to study the commutator $i[A, B]$. We note that the argument just presented is typical in the proof of L^2 Carleman estimates, see e.g. Lerner (2019).

By the definition of A and B it easily follows that $i[A, B] = \frac{1}{2}[P^*, P]$. By the commutation formulas for X, $X_\perp$, and V in Lemma 3.5.5, this commutator may be expressed as

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$$\begin{aligned}[P^*, P] &= XVVX - VXXV = VXVX + X_\perp VX - VXVX - VXX_\perp \\ &= VX_\perp X - X^2 - VXX_\perp = V[X_\perp, X] - X^2 = -X^2 + VKV. \end{aligned} \tag{4.3}$$

Consequently,

$$([P^*, P]u, u) = \|Xu\|^2 - (KVu, Vu).$$

If the curvature K is non-positive, then $[P^*, P]$ is positive semidefinite. More generally, one can try to use the other positive terms in (4.2). Note that

$$\|Au\|^2 + \|Bu\|^2 = \frac{1}{2}\big(\|Pu\|^2 + \|P^*u\|^2\big).$$

The identity (4.2) may then be expressed as

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u).$$

We have now proved a version of the Pestov identity that is suited for our purposes. The main point in this proof was that the Pestov identity boils down to a standard L^2 estimate based on separating the self-adjoint and skew-adjoint parts of P and on computing one commutator, $[P^*, P]$.

Proposition 4.3.2 (Pestov identity) *If (M, g) is a compact oriented surface with smooth boundary, then*

$$\|VXu\|^2 = \|XVu\|^2 - (KVu, Vu) + \|Xu\|^2$$

for any $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$.

4.4 Injectivity of the Geodesic X-ray Transform

We now establish the injectivity of the geodesic X-ray transform I_0 on simple surfaces.

Theorem 4.4.1 *Let (M, g) be a simple surface. Then I_0 is injective.*

In fact the proof gives a more general result, showing injectivity of I acting on functions of the form $f(x, v) = f_0(x) + \alpha_j(x)v^j$ modulo a natural kernel. In particular, this implies solenoidal injectivity of the geodesic X-ray transform on 1-tensors (see Section 6.4).

Theorem 4.4.2 *Let (M, g) be a simple surface, and let $f(x, v) = f_0(x) + \alpha|_x(v)$ where $f_0 \in C^\infty(M)$ and α is a smooth 1-form on M. If $If = 0$, then $f_0 = 0$ and $\alpha = dp$ for some $p \in C^\infty(M)$ with $p|_{\partial M} = 0$.*

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Using Proposition 4.3.1, the injectivity of I_0 is equivalent with the property that the only smooth solution of $VXu = 0$ in SM with $u|_{\partial SM} = 0$ is $u \equiv 0$. In the special case where the Gaussian curvature is non-positive, this follows immediately from the Pestov identity.

Proof of Theorem 4.4.1 in the case $K \leq 0$ If $VXu = 0$ in SM with $u|_{\partial SM} = 0$, Proposition 4.3.2 implies that

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 = 0.$$

Since $K \leq 0$, all terms on the left are non-negative and hence they all have to be zero. In particular, $\|Xu\|^2 = 0$, so $Xu = 0$ in SM showing that u is constant along geodesics. Using the boundary condition $u|_{\partial SM} = 0$, we obtain that $u \equiv 0$. □

In order to prove Theorem 4.4.1 in general, we show:

Proposition 4.4.3 *Let (M, g) be a simple surface. Then given $\psi \in C^\infty(SM)$ with $\psi|_{\partial SM} = 0$, we have*

$$\|X\psi\|^2 - (K\psi, \psi) \geq 0,$$

with equality if and only if $\psi = 0$.

Proof It is enough to prove this when ψ is real valued. Using Santaló's formula, we may write

$$\begin{aligned} \|X\psi\|^2 - (K\psi, \psi) &= \int_{SM} ((X\psi)^2 - K\psi^2)\, d\Sigma^3 \\ &= \int_{\partial_+ SM} \int_0^{\tau(x,v)} (\dot{\psi}(t)^2 - K(\gamma_{x,v}(t))\psi^2(t))\mu\, d\Sigma^2\, dt, \end{aligned} \tag{4.4}$$

where $\psi(t) = \psi_{x,v}(t) := \psi(\varphi_t(x, v))$. We wish to relate the t-integral to the index form on $\gamma_{x,v}$ (see Definition 3.7.14). In fact, if we define a normal vector field $Y(t)$ along $\gamma_{x,v}$ by

$$Y(t) = Y_{x,v}(t) := \psi(t)\dot{\gamma}_{x,v}(t)^\perp,$$

then $Y \in H_0^1(\gamma_{x,v})$ since $\psi(0) = \psi(\tau(x, v)) = 0$. Using that $D_t\dot{\gamma}_{x,v}(t)^\perp = 0$ (see (3.19)), we have

$$\mathtt{I}_{\gamma_{x,v}}(Y, Y) = \int_0^{\tau(x,v)} \left[\dot{\psi}(t)^2 - K(\gamma_{x,v}(t))\psi^2(t)\right] dt.$$

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Thus we may rewrite (4.4) as

$$\|X\psi\|^2 - (K\psi, \psi) = \int_{\partial_+ SM} \mathrm{I}_{\gamma_{x,v}}(Y_{x,v}, Y_{x,v})\mu \, d\Sigma^2.$$

The no conjugate points condition implies that $\mathrm{I}_{\gamma_{x,v}}$ is positive definite on $H_0^1(\gamma_{x,v})$ (see Proposition 3.7.15). Since $\mu \geq 0$ it follows that $\|X\psi\|^2 - (K\psi, \psi) \geq 0$. If equality holds then $Y_{x,v} \equiv 0$ for each $(x, v) \in \partial_+ SM$, which gives that $\psi \equiv 0$. □

Alternative proof of Proposition 4.4.3 By (4.4), it is enough to prove that for any fixed $(x, v) \in \partial_+ SM \setminus \partial_0 SM$, one has

$$\int_0^{\tau(x,v)} (\dot{\psi}(t)^2 - K(\gamma_{x,v}(t))\psi^2(t)) \, dt \geq 0,$$

with equality if and only if $\psi = 0$, where $\psi(t) = \psi_{x,v}(t) := \psi(\varphi_t(x, v))$. Observe that $\psi(0) = \psi(\tau(x, v)) = 0$. Since (M, g) has no conjugate points, the unique solution y to the Jacobi equation $\ddot{y} + K(\gamma_{x,v}(t))y = 0$ with $y(0) = 0$ and $\dot{y}(0) = 1$ does **not** vanish for $t \in (0, \tau]$ (otherwise one would have a Jacobi field vanishing at two points by Lemma 3.7.7). Hence we may define a function q by writing

$$\psi(t) = q(t)y(t), \text{ for } t \in (0, \tau].$$

Since $\psi(0) = y(0) = 0$ and $\dot{y}(0) = 1$, we have $\psi(t) = th(t)$, $y(t) = tr(t)$ where h and r are smooth and $r(0) = 1$. It follows that $q(t) = h(t)/r(t)$ extends smoothly to $t = 0$. Using the Jacobi equation we compute

$$(\ddot{\psi} + K\psi)\psi = q \frac{d}{dt}\big(\dot{q} y^2\big).$$

Integrating by parts and using that $y(0) = q(\tau) = 0$ (since $\psi(\tau) = 0$ and $y(\tau) \neq 0$), we derive

$$\int_0^\tau (\dot{\psi}^2 - K\psi^2) \, dt = -\int_0^\tau q \frac{d}{dt}\big(\dot{q} y^2\big) \, dt = -\big[q\dot{q} y^2\big]_0^\tau + \int_0^\tau \dot{q}^2 y^2 \, dt$$
$$= \int_0^\tau \dot{q}^2 y^2 \, dt \geq 0.$$

Equality in the last line holds if and only if q is constant. Since $q(\tau) = 0$, it follows that equality holds if and only if $\psi \equiv 0$. □

We can now combine these results to prove the injectivity of I_0.

Proof of Theorem 4.4.1 By Proposition 4.3.1 it suffices to show a vanishing result for $VXu = 0$ with $u|_{\partial SM} = 0$. Proposition 4.3.2 gives

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 = 0,$$

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and combining this with Proposition 4.4.3 (note that $Vu|_{\partial SM} = 0$), we derive $Vu = Xu = 0$ and hence $u = 0$ as desired. □

The same method also yields the more general Theorem 4.4.2.

Proof of Theorem 4.4.2 Let $f(x, v) = f_0(x) + \alpha|_x(v)$ satisfy $If = 0$, and let $u := u^f$ so that $Xu = -f$ and $u|_{\partial SM} = 0$. By Proposition 4.2.3 one has $u \in C^\infty(SM)$. We wish to use the Pestov identity and for this we need to compute VXu. In this case VXu is not identically zero, but it turns out that using the special form of f the term $\|VXu\|^2$ can be absorbed in the term $\|Xu\|^2$ in the other side of the Pestov identity.

In the special coordinates in Lemma 3.5.6, one has

$$\begin{aligned} Vf &= \partial_\theta(f_0(x) + e^{-\lambda(x)}(\alpha_1(x)\cos\theta + \alpha_2(x)\sin\theta)) \\ &= e^{-\lambda}(-\alpha_1 \sin\theta + \alpha_2\cos\theta). \end{aligned}$$

Then, using (3.9) and computing simple trigonometric integrals, we have

$$\begin{aligned} \|VXu\|^2 = \|Vf\|^2 &= \int_M \int_0^{2\pi} |-\alpha_1 \sin\theta + \alpha_2 \cos\theta|^2 \, d\theta \, dx \\ &= \pi \int_M (|\alpha_1(x)|^2 + |\alpha_2(x)|^2) \, dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|Xu\|^2 = \|f\|^2 &= \int_M \int_0^{2\pi} |e^\lambda f_0 + \alpha_1 \cos\theta + \alpha_2 \sin\theta|^2 \, d\theta \, dx \\ &= 2\pi \int_M |f_0(x)|^2 \, dV^2 + \pi \int_M (|\alpha_1(x)|^2 + |\alpha_2(x)|^2) \, dx. \end{aligned}$$

Inserting the above expressions in the Pestov identity in Proposition 4.3.2, we obtain that

$$\|XVu\|^2 - (KVu, Vu) + 2\pi \|f_0\|^2_{L^2(M)} = 0.$$

Since $\|XVu\|^2 - (KVu, Vu) \geq 0$ by Proposition 4.4.3, we must have $f_0 = 0$ and also $\|XVu\|^2 - (KVu, Vu) = 0$. Using the equality part of Proposition 4.4.3 gives $Vu = 0$. This implies that $u(x, v) = u(x)$. Writing $p(x) := -u(x) \in C^\infty(M)$ we have $p|_{\partial M} = 0$, and for any $(x, v) \in SM$ one has

$$\alpha|_x(v) = f(x, v) = -Xu(x, v) = dp|_x(v). \qquad \square$$

4.5 Stability Estimate in Non-positive Curvature

In this section we show how the Pestov identity can be used to derive a basic stability estimate for I_0 when the Gaussian curvature is non-positive, i.e. $K \leq 0$. This estimate will be generalized in Section 4.6, and in Chapter 7 we give another improvement and extend the estimate to include tensors.

Theorem 4.5.1 (Stability estimate for $K \leq 0$) *Let (M,g) be a compact non-trapping surface with strictly convex boundary and $K \leq 0$. Then*

$$\|f\|_{L^2(M)} \leq \frac{1}{\sqrt{4\pi}} \|I_0 f\|_{H^1(\partial_+ SM)},$$

for any $f \in C^\infty(M)$.

The $H^1(\partial_+ SM)$ norm appearing in the statement is precisely defined via a suitable vector field T as follows.

Definition 4.5.2 (Tangential vector field) Let (M,g) be a compact oriented surface with smooth boundary. We define the *tangential vector field* T on ∂SM acting on $w \in C^\infty(\partial SM)$ by

$$T w(x,v) = \frac{d}{dt} w(x(t), v(t))\Big|_{t=0},$$

where $x\colon (-\varepsilon,\varepsilon) \to \partial M$ is any smooth curve with $x(0) = x$ and $\dot{x}(0) = \nu(x)_\perp$, and $v(t)$ is the parallel transport of v along $x(t)$ so that $v(0) = v$.

Definition 4.5.3 (H^1 norms on ∂SM and $\partial_+ SM$) We define the $H^1(\partial SM)$ norm of w via

$$\|w\|^2_{H^1(\partial SM)} := \|w\|^2_{L^2(\partial SM)} + \|Tw\|^2_{L^2(\partial SM)} + \|Vw\|^2_{L^2(\partial SM)}.$$

Similarly, if $w \in C^\infty(\partial_+ SM)$ we define its $H^1(\partial_+ SM)$ norm as

$$\|w\|^2_{H^1(\partial_+ SM)} := \|w\|^2_{L^2(\partial_+ SM)} + \|Tw\|^2_{L^2(\partial_+ SM)} + \|Vw\|^2_{L^2(\partial_+ SM)}.$$

We state a few important facts about the vector field T. Recall the notation $\mu = \langle \nu, v\rangle$ on ∂SM.

Lemma 4.5.4 (Properties of T) *One has*

$$T = (V\mu)X + \mu X_\perp\big|_{\partial SM}.$$

In the splitting (3.12), *T is given by*

$$T = (\nu_\perp, 0).$$

The vector fields T and V form an orthonormal frame of $T(\partial SM)$ with respect to the Sasaki metric. This frame is commuting in the sense that $[T,V] = 0$, and T and V are skew-adjoint in the $L^2(\partial SM)$ inner product.

Proof Let (M,g) be contained in a closed manifold (N,g). Fix $(x_0,v_0) \in \partial SM$ and choose Riemannian normal coordinates $x = (x^1,x^2)$ near x_0 in (N,g).

Let θ be the angle between v and $\partial/\partial x_1$. This gives coordinates (x,θ) near (x_0, v_0). Note that these coordinates are not the same as the special coordinates in Lemma 3.5.6.

In the (x,θ) coordinates the curve $(x(t), v(t))$ corresponds to $(x(t),\theta(t))$, and one has

$$Tw|_{(x_0,v_0)} = \partial_{x_1} w(\nu_\perp)^1 + \partial_{x_2} w(\nu_\perp)^2 + (\partial_\theta w)\dot\theta(0).$$

Note that $\tan\theta(t) = \frac{v^2(t)}{v^1(t)}$. Differentiating in t gives

$$(1+\tan^2\theta)\dot\theta = \frac{\dot v^2 v^1 - v^2 \dot v^1}{(v^1)^2}.$$

Since $v(t)$ is parallel and the Christoffel symbols vanish at x_0, one has $\dot v^j(0) = 0$. This implies that $\dot\theta(0) = 0$ and thus

$$Tw|_{(x_0,v_0)} = \partial_{x_1} w(\nu_\perp)^1 + \partial_{x_2} w(\nu_\perp)^2.$$

Writing $\nabla_x w = (\partial_{x_1} w, \partial_{x_2} w)$, this can be rewritten in Euclidean notation as

$$Tw|_{(x_0,v_0)} = \nu_\perp \cdot \nabla_x w.$$

On the other hand, in the (x,θ) coordinates above one has

$$Xw|_{(x_0,v_0)} = v_0 \cdot \nabla_x w,$$
$$X_\perp w|_{(x_0,v_0)} = (v_0)_\perp \cdot \nabla_x w.$$

It is easy to check using the special coordinates in Lemma 3.5.6 that $V\mu = V(\langle \nu, v\rangle) = \langle \nu, v^\perp\rangle = \langle \nu_\perp, v\rangle$. Since $\mu = \langle \nu, v\rangle = \langle \nu_\perp, v_\perp\rangle$, we have

$$\begin{aligned}(V\mu)Xw + \mu X_\perp w|_{(x_0,v_0)} &= (\nu_\perp\cdot v_0)v_0\cdot\nabla_x w + (\nu_\perp\cdot(v_0)_\perp)(v_0)_\perp\cdot\nabla_x w \\ &= \nu_\perp\cdot\nabla_x w.\end{aligned}$$

This proves that $T = (V\mu)X + \mu X_\perp$ since both sides are invariantly defined.

The formula $T = (\nu_\perp, 0)$ in the splitting (3.12) also follows. Since $V = (0, v^\perp)$ in this splitting, it follows from the definition (3.14) of the Sasaki metric that T and V are orthonormal. The fact that $[T,V] = 0$ follows from the commutator formulas in Lemma 3.5.5 and the fact that $V^2\mu = -\mu$. Finally, since T and V give an orthonormal commuting frame on ∂SM they are divergence free: for T this follows from

$$\operatorname{div}(T) = \langle \nabla_T T, T\rangle + \langle \nabla_V T, V\rangle = \frac{1}{2}T\big(|T|^2\big) + \frac{1}{2}T\big(|V|^2\big) = 0,$$

since $|T| = |V| = 1$ and $\nabla_V T - \nabla_T V = [V,T] = 0$. Hence T and V are skew-adjoint. □

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The proof of Theorem 4.5.1 is also based on the Pestov identity, however instead of the condition $I_0 f = 0$ (so $u|_{\partial SM} = 0$) we will use that $u|_{\partial_+ SM} = I_0 f$. Thus we need to prove a version of the Pestov identity for functions that may not vanish on ∂SM. There will be a boundary term involving the vector field T.

Proposition 4.5.5 (Pestov identity with boundary terms) *Let (M,g) be a compact two-dimensional manifold with smooth boundary. Given any $u \in C^\infty(SM)$, one has*

$$\|VXu\|^2 = \|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 + (Tu, Vu)_{\partial SM}.$$

Proof We begin with the expression $\|VXu\|^2 - \|XVu\|^2$ and integrate by parts using Proposition 3.5.12 (note that integrating by parts with respect to V does not give any boundary terms). This yields

$$\begin{aligned}\|VXu\|^2 - \|XVu\|^2 &= (VXu, VXu) - (XVu, XVu)\\ &= -(VVXu, Xu) + (XXVu, Vu) + (XVu, \mu Vu)_{\partial SM}\\ &= ((XVVX - VXXV)u, u)\\ &\quad + (XVu, \mu Vu)_{\partial SM} + (VVXu, \mu u)_{\partial SM}.\end{aligned}$$

From (4.3) we have $XVVX - VXXV = VKV - X^2$. Integrating by parts again, we see that

$$\begin{aligned}\|VXu\|^2 - \|XVu\|^2 = \|Xu\|^2 - (KVu, Vu) + (Xu, \mu u)_{\partial SM}\\ + (XVu, \mu Vu)_{\partial SM} + (VVXu, \mu u)_{\partial SM}.\end{aligned}$$

We continue to integrate by parts with respect to V in the boundary terms. Thus

$$\begin{aligned}(VVXu, \mu u)_{\partial SM} &= -(VXu, (V\mu)u)_{\partial SM} - (VXu, \mu Vu)_{\partial SM}\\ &= (Xu, (V^2\mu)u)_{\partial SM} + (Xu, (V\mu)Vu)_{\partial SM}\\ &\quad - (VXu, \mu Vu)_{\partial SM}.\end{aligned}$$

Combining this with the other boundary terms and using the identities $[X,V] = X_\perp$ and $V^2\mu = -\mu$, we obtain that

$$\|VXu\|^2 - \|XVu\|^2 = \|Xu\|^2 - (KVu, Vu) + ((V\mu)Xu + \mu X_\perp u, Vu)_{\partial SM}.$$

Thus the boundary term is $(Tu, Vu)_{\partial SM}$ as required. □

We are now going to prove some additional regularity properties of the function τ. As in Lemma 3.1.10, consider a function $\rho \in C^\infty(N)$ in a closed extension N of M such that $\rho(x) = d(x, \partial M)$ in a neighbourhood of ∂M in

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M and such that $\rho \geq 0$ in M and $\partial M = \rho^{-1}(0)$. Clearly $\nabla\rho(x) = \nu(x)$ for $x \in \partial M$. Using ρ, we extend ν to the interior of M as $\nu(x) = \nabla\rho(x)$ for $x \in M$.

As before we let $\mu(x,v) := \langle v, \nu(x)\rangle$ for $(x,v) \in SM$, and

$$T := (V\mu)X + \mu X_\perp.$$

Note that T is now defined on all SM and agrees with the vector field T in Definition 4.5.2 on ∂SM. In fact T and V are tangent to every $\partial SM_\varepsilon = \{(x,v) \in SM : \; x \in \rho^{-1}(\varepsilon)\}$, where $M_\varepsilon = \rho^{-1}([\varepsilon,\infty))$.

Exercise 4.5.6 Prove that $[V,T] = 0$ in SM.

Lemma 4.5.7 *The functions $T\tau$ and $V\tau$ are bounded on $SM \setminus \partial_0 SM$.*

Proof We set $h(x,v,t) := \rho(\gamma_{x,v}(t))$ for $(x,v) \in SM \setminus \partial_0 SM$ and use the identity $X_\perp = [X,V]$ to compute

$$T(h(x,v,0)) = T(\rho) = (V\mu)X\rho + \mu X_\perp\rho = (V\mu)X\rho - \mu V(X\rho) = 0,$$

since $X\rho(x,v) = \mu(x,v)$. Therefore, there exists a smooth function $a(x,v,t)$ such that

$$T(h(x,v,t)) = ta(x,v,t).$$

Next we apply T to the equality $h(x,v,\tau(x,v)) = 0$ to get

$$T(h(x,v,t))|_{t=\tau(x,v)} + \frac{\partial h}{\partial t}(x,v,\tau(x,v))T\tau = 0.$$

If we write $(y,w) = (\gamma_{x,v}(\tau(x,v)), \dot\gamma_{x,v}(\tau(x,v)))$, then the identity above can be rewritten as

$$\tau(x,v)a(x,v,\tau(x,v)) + \mu(y,w)T\tau = 0.$$

If $(x,v) \in SM \setminus \partial_0 SM$, then $\mu(y,w) < 0$ and we may write

$$T\tau = -\frac{\tau(x,v)a(x,v,\tau(x,v))}{\mu(y,w)},$$

and since

$$0 \leq \frac{\tau(x,v)}{-\mu(y,w)} \leq \frac{\tau(y,-w)}{\mu(y,-w)},$$

it follows that $T\tau$ is bounded by Lemma 3.2.8. Since $V(\rho) = 0$, the proof for $V\tau$ is entirely analogous. □

The following corollary is immediate.

Corollary 4.5.8 *Let (M,g) be a compact non-trapping surface with strictly convex boundary. Given $f \in C^\infty(SM)$, the function*

$$u^f(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt$$

has Tu^f and Vu^f bounded in $SM \setminus \partial_0 SM$.

We can now prove Proposition 4.1.3:

Proof of Proposition 4.1.3 We only prove the case $k = 1$ and refer to (Sharafutdinov, 1994, Theorem 4.2.1) for the general case in any dimension $n \geq 2$. Recall the formula

$$u^f(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt.$$

Then $If = u^f|_{\partial_+ SM}$, and we have proved in Proposition 4.1.2 that $\|If\|_{L^2(\partial_+ SM)} \leq C\|f\|_{L^2(SM)}$. From Definition 3.5.3, we have

$$Vu^f(x,v) = f(\varphi_{\tau(x,v)}(x,v))V\tau(x,v) + \int_0^{\tau(x,v)} df(Z_t(x,v))\,dt,$$

where $Z_t(x,v) = \frac{d}{ds}\varphi_t(\rho_s(x,v))|_{s=0}$. By Lemma 4.5.7 we have

$$|Vu^f| \leq C\left[|f(\varphi_\tau)| + \int_0^\tau |df|_{\varphi_t}|\,dt\right].$$

As in Proposition 4.1.2, the $L^2(\partial_+ SM)$ norm of the second term is $\leq C\|f\|_{H^1(SM)}$. For the first term, we use that $\varphi_\tau|_{\partial_+ SM} = \alpha|_{\partial_+ SM}$. Then Lemma 3.3.5 and the trace theorem on SM imply that

$$\|f(\varphi_\tau)\|_{L^2(\partial_+ SM)} \leq C\|f\|_{L^2(\partial SM)} \leq C\|f\|_{H^1(SM)}.$$

Thus $\|V(If)\|_{L^2(\partial_+ SM)} \leq C\|f\|_{H^1(SM)}$. A similar argument works for $T(If)$, showing that $I\colon H^1(SM) \to H^1(\partial_+ SM)$ is bounded.

Finally, note that If vanishes on the boundary of $\partial_+ SM$ whenever $f \in C^\infty(SM)$. Thus $I(C^\infty(SM)) \subset H^1_0(\partial_+ SM)$, which implies that $I(H^1(SM)) \subset H^1_0(\partial_+ SM)$ by density. □

Proof of Theorem 4.5.1 We wish to use the Pestov identity from Proposition 4.5.5 for u^f. Since this identity was derived for smooth functions and u^f fails to be smooth at the glancing region $\partial_0 SM$, we apply the identity in SM_ε (as defined above) and to the function $u = u^f|_{SM_\varepsilon}$ for ε small. Since $K \leq 0$, $Xu^f = -f$, and $Vf = 0$, we derive

$$\|f\|^2_{L^2(SM_\varepsilon)} \leq -(Tu^f, Vu^f)_{\partial SM_\varepsilon}.$$

Letting $\varepsilon \to 0$ and using Corollary 4.5.8, we deduce (cf. Exercise 4.5.9)

$$\|f\|^2_{L^2(SM)} \leq -\big(Tu^f, Vu^f\big)_{\partial SM}. \tag{4.5}$$

Since $u^f|_{\partial_- SM} = 0$ and $I_0 f = u^f|_{\partial_+ SM} \in H^1_0(\partial_+ SM)$, we deduce

$$\|f\|^2_{L^2(SM)} \leq -(TI_0 f, VI_0 f)_{\partial_+ SM} \leq \frac{1}{2}\big(\|TI_0 f\|^2 + \|VI_0 f\|^2\big) \leq \frac{1}{2}\|I_0 f\|^2_{H^1},$$

and the theorem is proved. □

Exercise 4.5.9 Consider the vector field $N := \mu X - V(\mu)X_\perp$ and let F_t be its flow. Show that for ε small enough $F_\varepsilon \colon \partial SM \to \partial SM_\varepsilon$. Write $F_\varepsilon^* d\Sigma_\varepsilon^2 = q_\varepsilon d\Sigma^2$, where q_ε is smooth and $q_0 = 1$ since F_0 is the identity. Show that

$$\big(Tu^f, Vu^f\big)_{\partial SM_\varepsilon} = \big(q_\varepsilon\big(Tu^f \circ F_\varepsilon\big), Vu^f \circ F_\varepsilon\big)_{\partial SM}.$$

Use Corollary 4.5.8 and the dominated convergence theorem to conclude that as $\varepsilon \to 0$

$$\big(q_\varepsilon\big(Tu^f \circ F_\varepsilon\big), Vu^f \circ F_\varepsilon\big)_{\partial SM} \to \big(Tu^f, Vu^f\big)_{\partial SM}.$$

Exercise 4.5.10 Let (M, g) be a non-trapping surface with strictly convex boundary and let $f \in C^\infty(SM)$. Using the Pestov identity with boundary term and Corollary 4.5.8, show that $XVu^f \in L^2(SM)$. Using $X_\perp = [X, V]$, conclude that $X_\perp u^f \in L^2(SM)$ and thus $u^f \in H^1(SM)$.

4.6 Stability Estimate in the Simple Case

In this section we show how to upgrade the stability estimate in Theorem 4.5.1 from the case of non-positive curvature to the case of simple surfaces. A glance at the Pestov identity with boundary terms in Proposition 4.5.5 reveals that we need to find a better way to manage the 'index form' like-term $\|XVu\|^2 - (KVu, Vu)$. We shall do this by using solutions to the Riccati equation; these exist for simple surfaces as we show next.

Proposition 4.6.1 *Let (M, g) be a simple surface. There exists a smooth function $a \colon SM \to \mathbb{R}$ such that*

$$Xa + a^2 + K = 0.$$

Proof Consider M_0 a slightly larger simple surface such that its interior contains M (see Proposition 3.8.7), and let τ_0 denote the exit time function for M_0. We define a vector field at $(x, v) \in SM$ as follows:

$$\mathbf{e}(x, v) := d\varphi_{\tau_0(x,-v)}(V(\varphi_{-\tau_0(x,-v)})),$$

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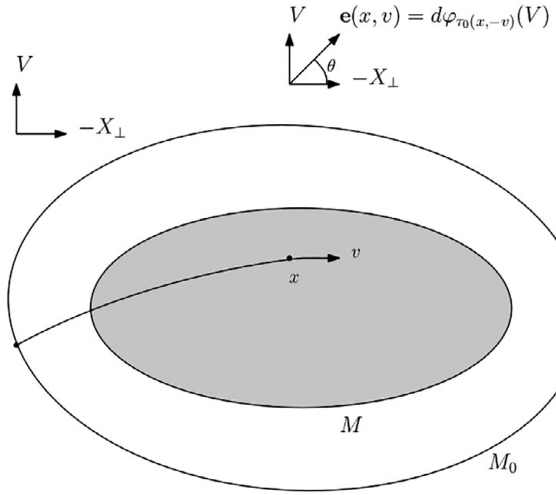

Figure 4.1 The vector field $\mathbf{e}$ and the function $a = \tan\theta$.

where φ_t is, as usual, the geodesic flow, see Figure 4.1. Since $\tau_0|_{SM}$ is smooth, the vector field $\mathbf{e}$ is also smooth. As discussed in Section 3.7.2, the geodesic flow preserves the contact plane spanned by $X_\perp$ and V and thus there are smooth functions $y, z\colon SM \to \mathbb{R}$ such that

$$\mathbf{e} = -yX_\perp + zV.$$

It was proved in Section 3.7.2 that $t \mapsto y(\varphi_t(x,v))$ solves a Jacobi equation. We can see this also as follows: note first that

$$\mathbf{e}(\varphi_t(x,v)) = d\varphi_t(\mathbf{e}(x,v)),$$

and therefore $[\mathbf{e}, X] = 0$. This implies

$$0 = [-yX_\perp + zV, X],$$

and expanding the brackets using Lemma 3.5.5 we obtain

$$-(Xz + Ky)V + (Xy - z)X_\perp = 0.$$

Hence $Xz = -Ky$ and $Xy = z$. In particular, $X^2y + Ky = 0$ and $y|_{\partial_+ SM_0} = 0$.

Since M_0 has no conjugate points, $y \neq 0$ everywhere in SM and we may define $a := z/y$. It follows that $Xa = -K - a^2$ in SM as desired. □

Exercise 4.6.2 Using the vector field

$$\mathbf{d}(x,v) := d\varphi_{-\tau_0(x,v)}(V(\varphi_{\tau_0(x,v)})),$$

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show that one can construct a smooth function b such that $Xb + b^2 + K = 0$ and $a - b \neq 0$ everywhere, where a is the solution constructed in the proof above.

Using the solution a to the Riccati equation given by Proposition 4.6.1, we will show:

Lemma 4.6.3 *Let (M, g) be a simple surface. For any $\psi \in C^\infty(SM)$ we have*

$$\|X\psi\|^2 - (K\psi, \psi) = \|X\psi - a\psi\|^2 - (\mu a\psi, \psi)_{\partial SM}.$$

Proof It is enough to consider real-valued ψ. Using that a satisfies $Xa + a^2 + K = 0$, we easily check that

$$(X\psi - a\psi)^2 = (X\psi)^2 - K\psi^2 - X(a\psi^2).$$

Integrating over SM and using Proposition 3.5.12 to derive

$$\int_{SM} X(a\psi^2)\, d\Sigma^3 = -(\mu a\psi, \psi)_{\partial SM},$$

the lemma follows. □

We now show:

Theorem 4.6.4 (Stability estimate for simple surfaces) *Let (M, g) be a simple surface. Then*

$$\|f\|_{L^2(M)} \leq C\|I_0 f\|_{H^1(\partial_+ SM)}$$

for any $f \in C^\infty(M)$, where C is a constant that only depends on (M, g).

Proof As in the proof of Theorem 4.5.1, the starting point is the Pestov identity with boundary terms given in Proposition 4.5.5. We apply it on M_ε (as defined in Section 4.5) and to the function $u = u^f|_{SM_\varepsilon}$ for ε small. Since $Xu^f = -f$ and $Vf = 0$, we derive

$$\|f\|^2_{L^2(SM_\varepsilon)} = -\left\|XVu^f\right\|^2_{L^2(SM_\varepsilon)} + \left(KVu^f, Vu^f\right)_{SM_\varepsilon} - \left(Tu^f, Vu^f\right)_{\partial SM_\varepsilon}.$$

Applying Lemma 4.6.3 for $\psi = Vu^f|_{SM_\varepsilon}$, we obtain

$$\|f\|^2_{L^2(SM_\varepsilon)} \leq -\left(Tu^f, Vu^f\right)_{\partial SM_\varepsilon} + \left(\mu a Vu^f, Vu^f\right)_{\partial SM_\varepsilon},$$

where μ is defined on SM using the extension of ν explained in Section 4.5 (for small ε it is the inward normal to M_ε). We can clearly find a constant $C > 0$ depending only on (M, g) such that

$$\left(\mu a Vu^f, Vu^f\right)_{\partial SM_\varepsilon} \leq C\|Vu^f\|^2_{L^2(\partial SM_\varepsilon)}.$$

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If we let $\varepsilon \to 0$ and use Corollary 4.5.8, we obtain

$$\|f\|^2_{L^2(SM)} \le -\big(Tu^f, Vu^f\big)_{\partial SM} + C\|Vu^f\|^2_{L^2(\partial SM)}.$$

Since $u^f|_{\partial_- SM} = 0$ and $I_0 f = u^f|_{\partial_+ SM} \in H^1_0(\partial_+ SM)$, we deduce that there is a constant C such that

$$\|f\|^2_{L^2(SM)} \le C\|I_0 f\|^2_{H^1(\partial_+ SM)},$$

and the theorem is proved. □

Exercise 4.6.5 Use the fact that u^f_- is smooth for f even (cf. Theorem 5.1.2) to give a proof of the stability estimate of Theorem 4.6.4 that does not require the approximation argument with SM_ε.

4.7 The Higher Dimensional Case

Although the results in Sections 4.3–4.6 have been stated in dimension two, they remain valid in any dimension $n \ge 2$. In this section we will give the corresponding higher dimensional results. The proofs are virtually the same as in the two-dimensional case, but the Pestov identity will take a slightly different form. We will follow the presentation in Paternain et al. (2015a), which contains further details.

Let (M, g) be a compact oriented n-dimensional manifold with $n \ge 2$. When $n = 2$ the analysis on the unit sphere bundle SM was based on the vector fields X, $X_\perp$, and V. The geodesic vector field X is well defined in any dimension (see (3.5)). We wish to find higher dimensional counterparts of $X_\perp$ and V.

Recall the splitting $TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$ in Section 3.6, where the horizontal and vertical bundles $\mathcal{H}_{(x,v)}$ and $\mathcal{V}_{(x,v)}$ are canonically identified with elements in $\{v\}^\perp \subset T_x M$. Then for any $u \in C^\infty(SM)$ we can split the gradient $\nabla_{SM} u$ with respect to the Sasaki metric G as

$$\nabla_{SM} u = ((Xu)X, \overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u).$$

The *horizontal gradient* $\overset{\mathrm{h}}{\nabla}$ and *vertical gradient* $\overset{\mathrm{v}}{\nabla}$ are operators

$$\overset{\mathrm{h}}{\nabla}, \overset{\mathrm{v}}{\nabla} \colon C^\infty(SM) \to \mathcal{Z},$$

where $\mathcal{Z} := \{Z \in C^\infty(SM, TM) \,:\, Z(x,v) \in T_x M \text{ and } Z(x,v) \perp v\}$.

We define an L^2 inner product on $\mathcal{Z}$ via

$$(Z, Z')_{L^2(SM)} = \int_{SM} \langle Z(x,v), \overline{Z'(x,v)}\rangle \, d\Sigma^{2n-1}.$$

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The *horizontal divergence* $\overset{\mathrm{h}}{\mathrm{div}}$ and *vertical divergence* $\overset{\mathrm{v}}{\mathrm{div}}$ are defined as the formal L^2 adjoints of $-\overset{\mathrm{h}}{\nabla}$ and $-\overset{\mathrm{v}}{\nabla}$, respectively. They are operators

$$\overset{\mathrm{h}}{\mathrm{div}}, \overset{\mathrm{v}}{\mathrm{div}} \colon \mathcal{Z} \to C^\infty(SM).$$

We also need to define the action of X on $\mathcal{Z}$ as

$$XZ(x,v) := D_t(Z(\varphi_t(x,v)))|_{t=0},$$

where D_t denotes the covariant derivative on M.

The operators $\overset{\mathrm{h}}{\nabla}$ and $\overset{\mathrm{v}}{\nabla}$ are the required higher dimensional analogues of $X_\perp$ and V, as indicated by the following example:

Example 4.7.1 When $n = 2$, one has $\mathcal{Z} = \{z(x,v)v^\perp \,:\, z \in C^\infty(SM)\}$. It is easy to check (see Paternain et al. (2015a, Appendix B)) that

$$\overset{\mathrm{h}}{\nabla} u(x,v) = -(X_\perp u)v^\perp,$$
$$\overset{\mathrm{v}}{\nabla} u(x,v) = (Vu)v^\perp,$$

and

$$\overset{\mathrm{h}}{\mathrm{div}}(z(x,v)v^\perp) = -X_\perp z,$$
$$\overset{\mathrm{v}}{\mathrm{div}}(z(x,v)v^\perp) = Vz.$$

The following result is the analogue of the basic commutator formulas in Lemma 3.5.5. Below, $R(x,v) \colon \{v\}^\perp \to \{v\}^\perp$ is the operator determined by the Riemann curvature tensor R via $R(x,v)w = R_x(w,v)v$.

Lemma 4.7.2 (Commutator formulas) *The following commutator formulas hold on $C^\infty(SM)$:*

$$[X, \overset{\mathrm{v}}{\nabla}] = -\overset{\mathrm{h}}{\nabla},$$
$$[X, \overset{\mathrm{h}}{\nabla}] = R\,\overset{\mathrm{v}}{\nabla},$$
$$\overset{\mathrm{h}}{\mathrm{div}}\,\overset{\mathrm{v}}{\nabla} - \overset{\mathrm{v}}{\mathrm{div}}\,\overset{\mathrm{h}}{\nabla} = (n-1)X.$$

Taking adjoints, we also have the following commutator formulas on $\mathcal{Z}$:

$$[X, \overset{\mathrm{v}}{\mathrm{div}}] = -\overset{\mathrm{h}}{\mathrm{div}},$$
$$[X, \overset{\mathrm{h}}{\mathrm{div}}] = \overset{\mathrm{v}}{\mathrm{div}} R.$$

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We also have integration by parts formulas (cf. Proposition 3.5.12):

Proposition 4.7.3 (Integration by parts) *Let* $u, w \in C^\infty(SM)$ *and* $Z \in \mathcal{Z}$. *Then*

$$(Xu, w)_{SM} = -(u, Xw)_{SM} - (\langle v, \nu\rangle u, w)_{\partial SM},$$
$$(\overset{\mathrm{h}}{\nabla} u, Z)_{SM} = -(u, \overset{\mathrm{h}}{\mathrm{div}} Z)_{SM} - (u, \langle Z, \nu\rangle)_{\partial SM},$$
$$(\overset{\mathrm{v}}{\nabla} u, Z)_{SM} = -(u, \overset{\mathrm{v}}{\mathrm{div}} Z)_{SM}.$$

The formulas above imply the higher dimensional version of the Pestov identity. The proof is the same as for $n = 2$, and we can also include boundary terms (see e.g. Ilmavirta and Paternain (2020)).

Proposition 4.7.4 (Pestov identity with boundary term) *Let* (M, g) *be a compact manifold with smooth boundary. If* $u \in C^\infty(SM)$, *then*

$$\|\overset{\mathrm{v}}{\nabla} Xu\|^2 = \|X\overset{\mathrm{v}}{\nabla} u\|^2 - (R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u) + (n-1)\|Xu\|^2 + (Tu, \overset{\mathrm{v}}{\nabla} u)_{\partial SM},$$

where $Tu := \mu\overset{\mathrm{h}}{\nabla} u - Xu\overset{\mathrm{v}}{\nabla}\mu$.

Remark 4.7.5 The identity in Proposition 4.7.4 is an ‘integrated’ form of the Pestov identity. In previous works, also ‘pointwise’ or ‘differential’ versions of this identity appear. In fact, using the commutator formulas it is easy to prove the pointwise Pestov identity

$$|\overset{\mathrm{v}}{\nabla} Xu|^2 - |X\overset{\mathrm{v}}{\nabla} u|^2 + \langle R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\rangle - (n-1)|Xu|^2$$
$$= X\left[\langle \overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\rangle\right] - \overset{\mathrm{h}}{\mathrm{div}}\left[(Xu)\overset{\mathrm{v}}{\nabla} u\right] + \overset{\mathrm{v}}{\mathrm{div}}\left[(Xu)\overset{\mathrm{h}}{\nabla} u\right]$$

for any $u \in C^\infty(SM)$. Proposition 4.7.4 could be obtained by integrating this identity over SM.

The injectivity of the X-ray transform I_0 on simple manifolds follows from the Pestov identity if we can prove that $\|X\overset{\mathrm{v}}{\nabla} u\|^2 - (R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u) \geq 0$ when $u|_{\partial SM} = 0$. This follows by using Santaló’s formula and the index form as in Proposition 4.4.3. Moreover, we have the more precise counterpart of Lemma 4.6.3, which also includes boundary terms:

Lemma 4.7.6 *Let* (M, g) *be a simple manifold. There is a smooth map* U *on* SM *so that* $U(x, v)$ *is a symmetric linear operator* $\{v\}^\perp \to \{v\}^\perp$ *solving the Riccati equation*

$$XU + U^2 + R = 0 \text{ in } SM.$$

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For any $Z \in \mathcal{Z}$ we have

$$\|XZ\|^2 - (RZ, Z) = \|XZ - UZ\|^2 - (\mu UZ, Z)_{\partial SM}.$$

The proof of this lemma is very similar to the proof of Paternain et al. (2015a, Proposition 7.1). The term XU in the Riccati equation is defined using the Leibniz rule, that is, by demanding that $X(UZ) = (XU)Z + UXZ$. The solution to the Riccati equation (cf. Paternain (1999, Chapter 2)) is obtained by enlarging (M, g) slightly and flowing the (Lagrangian) vertical subspace by the geodesic flow exactly as in the proof of Proposition 4.6.1.

We now state the injectivity result for I_0, and the more general injectivity result involving functions and 1-forms as in Theorem 4.4.2.

Theorem 4.7.7 (Injectivity of I_0) *Let (M, g) be a simple manifold, and let $f(x, v) = f_0(x) + \alpha|_x(v)$ where $f_0 \in C^\infty(M)$ and α is a smooth 1-form on M. If $If = 0$, then $f_0 = 0$ and $\alpha = dp$ for some $p \in C^\infty(M)$ with $p|_{\partial M} = 0$. In particular, I_0 is injective on $C^\infty(M)$.*

Following the argument in Section 4.6, we also obtain a stability result for I_0 in any dimension.

Theorem 4.7.8 (Stability estimate for simple manifolds) *Let (M, g) be a simple manifold. Then*

$$\|f\|_{L^2(M)} \leq C\|I_0 f\|_{H^1(\partial_+ SM)}$$

for any $f \in C^\infty(M)$, where C is a constant that only depends on (M, g).

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