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On the Coxeter Transformations for Tamari Posets

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Abstract. A relation between the anticyclic structure of the dendriform operad and the Coxeter transformations in the Grothendieck groups of the derived categories of modules over the Tamari posets is obtained.

Introduction

There are now several algebraic structures on planar binary trees. First, there is an operad, called the dendriform operad, whose structure can be described by insertion of planar binary trees. Then the free dendriform algebra on one generator is also an associative algebra and in fact a Hopf algebra, called the Hopf algebra of planar binary trees. Both the dendriform operad and the Hopf algebra of planar binary trees have been shown to be related to a family of posets on planar binary trees, called the Tamari lattices.

Until recently, it was not realized that the dendriform operad is an anticyclic operad. This fact implies the existence of a linear map of order $n+1$ on the vector space spanned by planar binary trees with $n+1$ leaves. The matrix of this endomorphism seemed similar to a matrix appearing in the study of the Hopf algebra of planar binary trees [4]. This was the starting point for this article.

The main result shows that the linear maps obtained from the anticyclic structure of the dendriform operad can alternatively be described using only the Tamari posets. More precisely, recall that for a quiver, the Coxeter transformation is the action induced on the Grothendieck group by a canonical self-equivalence, called the Auslander–Reiten translation, of the derived category of modules on the quiver. Considering Tamari posets as quivers with relations gives a family of Coxeter transformations on vector spaces spanned by planar binary trees. Our result shows that up to sign, iterating twice the Coxeter transformations recovers the anticyclic structure maps. All this should hint at a deeper relationship between the dendriform operad and derived categories for Tamari posets.

Also, this implies that the Coxeter transformation for Tamari posets is periodic. It is expected that something similar should happen for any Cambrian lattice associated to a finite Coxeter group [11, 12]. More precisely, the Coxeter transformation in the Grothendieck group of the derived category of modules on a Cambrian lattice should have order dividing $2h+2$ where h is the Coxeter number of the Coxeter group.

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Let us also note that a similar, but much simpler and less interesting, theory can be formulated relating the diassociative anticyclic operad on one hand and the family of total orders or chains on the other hand.

The article starts by recalling many observations about trees, posets, algebras, operads and quivers. The main theorem and its proof are to be found in Section 6.

1 Planar Binary Trees

Let n be a nonnegative integer. A planar binary tree of degree n is a graph which is a tree embedded in the plane that has n trivalent vertices, $n+2$ univalent vertices and a distinguished univalent vertex called the root. Such trees are sometimes called "rooted". The other univalent vertices are called the leaves. From now on, we will use "vertex" to mean "trivalent vertex". Planar binary trees are pictured with their root at the bottom and leaves at the top, see Figure 1.

Let $\mathbb{Y}(n)$ be the set of planar binary trees of degree n. It is a classical combinatorial fact that the cardinality of $\mathbb{Y}(n)$ is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $\mathbb{Y}$ be the set of all planar binary trees and $\mathbb{Y}^+$ the set of all planar binary trees except the tree $|$ with no vertex. For S in $\mathbb{Y}$, let $|S|$ be the degree of S, *i.e.*, its number of vertices. Let $\curlyvee$ be the unique tree with one vertex.

Let us define some combinatorial operations on $\mathbb{Y}$. Let S and T be in $\mathbb{Y}$. Then let $S_\curlyvee T$ be the planar binary tree obtained by grafting simultaneously S to the left leaf of $\curlyvee$ and T to the right leaf of $\curlyvee$. This tree has degree $|S|+|T|+1$.

Let S/T be the tree obtained by grafting the root of S to the leftmost leaf of T. It has degree $|S|+|T|$. Similarly let $S\backslash T$ be the tree obtained by grafting the root of T to the rightmost leaf of S. It also has degree $|S|+|T|$.

Remark that one can also define $S_\curlyvee T$ as $(S/\curlyvee)\backslash T$ or $S/(\curlyvee\backslash T)$. The tree $|$ is a two-sided unit for both $\backslash$ and $/$.

There is an obvious involution on planar binary trees, given by the left-right reversal of the plane.

2 Tamari Posets

There is a natural order relation $\leq$ on the set $\mathbb{Y}(n)$, which was introduced and studied by Tamari [2].

The order relation $\leq$ is defined as the transitive closure of some covering relations. A tree S is covered by a tree T if they differ only in some neighborhood of an edge by the replacement of the configuration ⋎ in S by the configuration ⋎ in T.

This poset is called the Tamari poset of degree n, denoted by $\mathbb{T}(n)$. It is known to be a lattice. The lattice $\mathbb{T}(3)$ is depicted in Figure 1.

The left-right symmetry of trees is an anti-automorphism of this poset, sending the minimal element to the maximal element.

The minimal element of $\mathbb{T}(n)$ will be denoted by $\widehat{0}$ and the maximal element by $\widehat{1}$.

Lemma 2.1 *For any T^1, T^2 in $\mathbb{T}(n)$, the map $(s^1, s^2) \mapsto s^1\backslash s^2$ is a bijection from the Cartesian product $[T^1, \widehat{1}] \times [T^2, \widehat{1}]$ of the intervals $[T^1, \widehat{1}]$ and $[T^2, \widehat{1}]$ to the interval $[T^1\backslash T^2, \widehat{1}]$.*

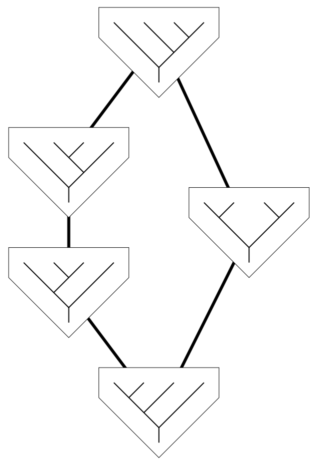

Figure 1: The Tamari poset $\mathbb{T}(3)$

Proof This is quite obvious from the definition of the partial order, as the covering relations preserve the fact that a tree can be written $s^1 \backslash s^2$. ∎

3 Dendriform Algebras

The notion of dendriform algebra was introduced by Loday [6]. Let us recall the axioms. A dendriform algebra over some field $\mathbf{k}$ is a vector space over $\mathbf{k}$ with two maps $\prec, \succ : \mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$ satisfying the following equations:

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z), \tag{1}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{2}$$

$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z. \tag{3}$$

These relations imply that the map $*$ defined by $x * y = x \prec y + x \succ y$ is associative.

There is a nice description of the free dendriform algebra on one generator in terms of planar binary trees; see [6, 7]. In particular, the underlying vector space is $\mathbf{k}\mathbb{Y}^+$. One can define the operations $\prec$ and $\succ$ on $\mathbf{k}\mathbb{Y}^+$. The product $*$ can be extended to $\mathbf{k}\mathbb{Y}$ and has an inductive definition as follows.

Proposition 3.1 *The tree* | *is a unit for* $*$. *For all* T^1, T^2, T^3, T^4 *in* $\mathbb{Y}$, *one has*

$$(T^1 \curlyvee T^2) * (T^3 \curlyvee T^4) = ((T^1 \curlyvee T^2) * T^3) \curlyvee T^4 + T^1 \curlyvee (T^2 * (T^3 \curlyvee T^4)). \tag{4}$$

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There is also a simple expression for the product $*$ in $\mathbf{k}\mathbb{Y}$ which uses the Tamari poset [8, (2)].

Proposition 3.2 *Let S and T be in $\mathbb{Y}$. One has the following relation in $\mathbf{k}\mathbb{Y}$:*

$$S * T = \sum_{S/T \leq U \leq S\backslash T} U. \tag{5}$$

We will need the following Lemma.

Lemma 3.3 *For any T^1, T^2 in $\mathbb{Y}(n)$, the $*$ product of the sum of the elements of the interval $[\widehat{0}, T^1]$ and the sum of the elements of $[\widehat{0}, T^2]$ is exactly the sum of the elements of the interval $[\widehat{0}, T^1\backslash T^2]$.*

For its proof, see for example [4, Theorems 29 and 30].

4 The Dendriform Operad

As a reference on operads and anticyclic operads, the reader may wish to consult [9, 10].

In this paper, we will only consider non-symmetric operads. A non-symmetric operad $\mathcal{P}$ in the category of vector spaces over $\mathbf{k}$ is a collection of vector spaces $\mathcal{P}(n)$ for $n \geq 1$, a collection of maps $\circ_i \colon \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ for $1 \leq i \leq n$ and a unit 1, satisfying axioms modelled after the composition of some multi-linear map at some place i inside another multi-linear map. The unit 1 plays the rôle of the identity map in the composition of multi-linear maps.

An anticyclic non-symmetric operad $\mathcal{P}$ is a non-symmetric operad together with a linear map τ on each $\mathcal{P}(n)$ such that $\tau^{n+1} = \mathrm{Id}$ and the following relations hold for $a \in \mathcal{P}(n)$ and $b \in \mathcal{P}(m)$:

$$\tau(1) = -1, \tag{6}$$

$$\tau(a \circ_n b) = -\tau(b) \circ_1 \tau(a), \tag{7}$$

$$\tau(a \circ_i b) = \tau(a) \circ_{i+1} b \quad \text{if } 1 \leq i < n. \tag{8}$$

Let us now define the dendriform operad $\mathcal{Y}$. For all $n \geq 1$, the space $\mathcal{Y}(n)$ is the vector space $\mathbf{k}\mathbb{Y}(n)$ spanned by the set of planar binary trees of degree n. The composition maps $\circ_i$ can be described using shuffles of trees; see [6, Proposition 5.11]. The unit of the operad $\mathcal{Y}$ is the unique tree with one vertex, denoted by Y. The operad $\mathcal{Y}$ is generated by two elements $\prec$ and $\succ$ with relations corresponding to formulas (1), (2), (3). These two elements should be seen as the two elements of $\mathbb{Y}(2)$, namely $\prec$ is the tree Y and $\succ$ is the tree Y.

Some of the combinatorial operations and products defined before can be restated using the composition maps of the operad $\mathcal{Y}$.

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Proposition 4.1 *For all T^1, T^2 in $\mathbb{Y}^+$, one has the following relations:*

$$T^1 * T^2 = ((\YLeft + \YRight) \circ_2 T^2) \circ_1 T^1,$$
$$T^1 \backslash T^2 = T^1 \circ_{n_1} (\YRight \circ_2 T^2),$$
$$T^1 / T^2 = T^2 \circ_1 (\YLeft \circ_1 T^1),$$

where n_1 is the degree of T^1.

The following theorem was proved in an equivalent form in [1, Theorem 4.1].

Theorem 4.2 *There exists a unique structure of anticyclic non-symmetric operad on $\mathcal{Y}$ such that*

$$\tau(\YRight) = \YLeft \quad \textit{and} \quad \tau(\YLeft) = -(\YRight + \YLeft).$$

The main aim of the present article is to gain some understanding of the induced cyclic actions on $\mathcal{Y}(n)$.

5 Quivers

5.1 Quiver with Relations from a Poset

Recall that a quiver Q is a set of vertices V and a set of arrows A with two maps from A to V giving the source and target of each arrow.

Then a module M over Q is a collection $(M_v)_{v \in V}$ of vector spaces M_v and a set of maps $f_{v,w}$ from M_v to M_w for each arrow in A with source v and target w. Modules over a quiver Q form an Abelian category, denoted by $\mathrm{mod}(Q)$.

One can restrict this category by imposing further conditions on the composition of the maps $f_{v,w}$. For example, if $\mathbb{P}$ is a finite poset, one can define a quiver $Q_{\mathbb{P}}$ with vertices the elements of $\mathbb{P}$ and arrows the covering relations of $\mathbb{P}$. That is to say, there is an arrow from v to w in $Q_{\mathbb{P}}$ if and only if $v \leq w$ in $\mathbb{P}$ and there is no element u in $\mathbb{P}$ such that $v < u < w$.

Then one can consider the category $\mathrm{mod}(\mathbb{P})$ of modules over the quiver $Q_{\mathbb{P}}$ such that for any pair $v \leq w$ in $\mathbb{P}$ and any two sequences of arrows $v = u_0 \to u_1 \to u_2 \to \cdots \to u_k = w$, $v = u'_0 \to u'_1 \to u'_2 \to \cdots \to u'_\ell = w$ in $Q_{\mathbb{P}}$, one has the relation

$$f_{u_0,u_1} f_{u_1,u_2} \cdots f_{u_{k-1},u_k} = f_{u'_0,u'_1} f_{u'_1,u'_2} \cdots f_{u'_{\ell-1},u'_\ell},$$

where composition of maps is denoted by concatenation. Then the category $\mathrm{mod}(\mathbb{P})$ is also an Abelian category. As $\mathbb{P}$ is assumed finite, this Abelian category is known to have finite cohomological dimension.

5.2 Derived Category and Coxeter Transformation

Let $\mathcal{D}\,\mathrm{mod}(\mathbb{P})$ be the bounded derived category of $\mathrm{mod}(\mathbb{P})$.

This derived category has a canonical self-equivalence which is called the Auslander–Reiten translation; see [3, 5]. It is known that this functor induces an endomorphism of the Grothendieck group K_0 of the derived category. This map is called the Coxeter transformation. This Grothendieck group has a natural basis indexed by the elements of $\mathbb{P}$, corresponding to the images of simple modules of $\mathrm{mod}(\mathbb{P})$ in the derived category.

We will denote by θ the Coxeter transformation in the Grothendieck group of the derived category $\mathcal{D}\,\mathrm{mod}(\mathbb{P})$.

Let L be the matrix defined by $L_{v,w} = 1$ if and only if $v \leq w$ in $\mathbb{P}$. Then the following result is known.

Proposition 5.1 *The matrix of the Coxeter transformation θ in the natural basis of K_0 is given by $-L(L^t)^{-1}$.*

Remark that θ is clearly an invertible map.

From now on, this construction will be used for the Tamari posets $\mathbb{T}(n)$. In particular, θ denotes the Coxeter transformation for some Tamari poset $\mathbb{T}(n)$, where n should be clear from the context. As the underlying set of $\mathbb{T}(n)$ is $\mathbb{Y}(n)$, the action of θ on $K_0(\mathcal{D}\,\mathrm{mod}(\mathbb{T}(n)))$ can be interpreted as an action on $\mathcal{Y}(n)$.

6 Periodicity Theorem

Here is the main result relating the anticyclic structure of the dendriform operad and the derived categories of modules on the Tamari lattices.

Theorem 6.1 *On the vector space $\mathcal{Y}(n)$, one has the relation*

$$\tau = (-1)^n \theta^2. \tag{9}$$

The proof of this theorem is given in the next section. Before this proof, let us state a consequence.

Corollary 6.2 *The Coxeter transformation θ in the Grothendieck group of the derived category $\mathcal{D}\,\mathrm{mod}(\mathbb{T}(n))$ of modules on the Tamari lattice $\mathbb{T}(n)$ satisfies $\theta^{2n+2} = \mathrm{Id}$.*

Proof As part of the anticyclic structure on $\mathcal{Y}$, it is known that $\tau^{n+1} = \mathrm{Id}$ on $\mathcal{Y}(n)$. ■

6.1 Proof of Theorem 6.1

The strategy of the proof is to find some inductive characterization of the map τ and then to prove that the map $(-1)^n\theta^2$ satisfies the same induction.

Proposition 6.3 *The collection of maps τ is uniquely defined by the following equations, for all T, T^1, T^2 in $\mathbb{Y}^+$.*

$$\tau(\curlyvee) = -\curlyvee,$$

$$\tau(T^1 \backslash T^2) = \tau(T^1)/\tau(T^2),$$

$$\tau(T/\curlyvee) = -\curlyvee * T.$$

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Proof The fact that $\tau(\mathrm{Y}) = -\mathrm{Y}$ follows from the definition of an anticyclic operad.

Let us first prove that τ satisfies these equations, using the axioms of anticyclic operad and the known action of τ on $\Yleft$ and $\Yright$. One has

$$\tau(T/\mathrm{Y}) = \tau(\Yleft \circ_1 T) = \tau(\Yleft) \circ_2 T = -(\Yleft + \Yright) \circ_2 T = -\mathrm{Y} * T.$$

Let n_1 be the degree of T^1. One also has

$$\tau(T^1 \backslash T^2) = \tau(T^1 \circ_{n_1} (\Yleft \circ_2 T^2)) = (\tau(T^2) \circ_1 \Yright) \circ_1 \tau(T^1) = \tau(T^1)/\tau(T^2).$$

The proof of uniqueness is an easy induction on degree. Any tree T in $\mathbb{Y}^+$ which is not Y can either be written $T^1 \backslash T^2$ for some trees in $\mathbb{Y}^+$ of smaller degrees, or has the shape T'/Y for some tree T' of smaller degree. This allows to define τ by induction. ■

Let us now prove some properties of θ and deduce from them properties of θ^2.

Proposition 6.4 *The collection of maps θ satisfy the following relations, for all T^1, T^2 in $\mathbb{Y}$.*

$$\begin{aligned}
\theta(|) &= -|, \\
\theta(\mathrm{Y}) &= -\mathrm{Y}, \\
\theta(T^1 \backslash T^2) &= -\theta(T^1) * \theta(T^2), \\
\theta(T^1 * T^2) &= -\theta(T^1)/\theta(T^2), \\
\theta^{-1}(T^1/T^2) &= -\theta^{-1}(T^1) * \theta^{-1}(T^2), \\
\theta^{-1}(T^1 * T^2) &= -\theta^{-1}(T^1) \backslash \theta^{-1}(T^2).
\end{aligned}$$

Proof It is clear that $\theta(|) = -|$ and $\theta(\mathrm{Y}) = -\mathrm{Y}$. The equations for θ^{-1} are obvious consequences of the equations for θ. It is enough to prove one of the equations for θ, as they are related by conjugation by the left-right symmetry of trees. Let us prove the first one. By the definition of $-\theta$ from Proposition 5.1, it is the composite of the matrices L and $(L^t)^{-1}$. By Lemma 2.1, the action of L^t preserves the $\backslash$ product. Hence this is also true for its inverse. By Lemma 3.3, the action of L maps the $\backslash$ product to the $*$ product. Hence $-\theta$ maps the $\backslash$ product to the $*$ product. This proves the proposition. ■

Remark that the conditions in Proposition 6.4 in fact uniquely determine the collection of maps θ. We will not need that fact.

Corollary 6.5 *For all T^1, T^2 in $\mathbb{Y}$ of degree n_1, n_2, one has the following relation*

$$(-1)^n \theta^2(T^1 \backslash T^2) = (-1)^{n_1} \theta^2(T^1)/(-1)^{n_2} \theta^2(T^2),$$

where $n = n_1 + n_2$ is the degree of T^1/T^2.

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We need another property of θ.

Proposition 6.6 *For all T in $\mathbb{Y}$ of degree n, one has*

$$\theta(T/\curlyvee) = (-1)^n \curlyvee\backslash\theta^{-1}(T) \quad \textit{and} \quad \theta^{-1}(\curlyvee\backslash T) = (-1)^n\theta(T)/\curlyvee.$$

Proof The proof is by induction on the degree of T. It is enough to prove one of the equations as they are obviously equivalent. The proposition is clearly true for small degrees. Assume that T can be written $T^1 \curlyvee T^2$ with T^1 of degree n_1 and T^2 of degree n_2 in $\mathbb{Y}$ with $n_1 + n_2 + 1 = n$. Then one has $T * \curlyvee = T/\curlyvee + (T^1/\curlyvee)\backslash(T^2 * \curlyvee)$. Hence one gets on the one hand,

$$\theta(T/\curlyvee) = \theta(T * \curlyvee) - \theta((T^1/\curlyvee)\backslash(T^2 * \curlyvee)).$$

Then using twice Proposition 6.4, this becomes

$$\theta(T)/\curlyvee + \theta(T^1/\curlyvee) * \theta(T^2 * \curlyvee).$$

Using again Proposition 6.4 and the fact that $T = (T^1/\curlyvee)\backslash T^2$, this is

$$-(\theta(T^1/\curlyvee) * \theta(T^2))/\curlyvee + \theta(T^1/\curlyvee) * (\theta(T^2)/\curlyvee).$$

Then using the induction hypothesis on T^1, one gets

$$(-1)^{n_1+1}((\curlyvee\backslash\theta^{-1}(T^1)) * \theta(T^2))/\curlyvee + (-1)^{n_1}(\curlyvee\backslash\theta^{-1}(T^1)) * (\theta(T^2)/\curlyvee).$$

On the other hand, using the fact that $T = T^1/(\curlyvee\backslash T^2)$ and Proposition 6.4, one has

$$(-1)^n \curlyvee\backslash\theta^{-1}(T) = (-1)^{n_1+n_2} \curlyvee\backslash(\theta^{-1}(T^1) * \theta^{-1}(\curlyvee\backslash T^2)).$$

Using the induction hypothesis for T^2, one gets

$$(-1)^{n_1} \curlyvee\backslash(\theta^{-1}(T^1) * (\theta(T^2)/\curlyvee)).$$

Then using Proposition 3.1 for $a = \theta^{-1}(T^1)$ and $b = \theta(T^2)$, the induction step is done. ■

Corollary 6.7 *For all T in $\mathbb{Y}$ of degree n, one has*

$$(-1)^{n+1}\theta^2(T/\curlyvee) = -\curlyvee * T.$$

From Corollaries 6.5 and 6.7 and by Proposition 6.3, one gets a proof of Theorem 6.1.

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