ON BESSEL AND GRÜSS INEQUALITIES
FOR ORTHORNORMAL FAMILIES IN INNER PRODUCT SPACES

S.S. DRAGOMIR

A new reverse of Bessel's inequality for orthornormal families in real or complex inner product spaces is obtained. Applications for some Grüss type results are also provided.

1. INTRODUCTION

In the recent paper [2], the following refinement of the Grüss inequality has been proved:

THEOREM 1. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\phi, \Phi, \gamma, \Gamma$ are real or complex numbers and x, y are vectors in H such that either

$$(1.1) \qquad \mathrm{Re}\langle \Phi e - x, x - \phi e \rangle \geqslant 0 \quad and \quad \mathrm{Re}\langle \Gamma e - y, y - \gamma e \rangle \geqslant 0$$

or, equivalently,

$$(1.2) \qquad \left\| x - \frac{\phi + \Phi}{2} e \right\| \leqslant \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leqslant \frac{1}{2} |\Gamma - \gamma|$$

hold, then we have the following refinement of the Grüss inequality

$$
\begin{aligned}
(1.3) \quad &\big| \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \big| \\
&\leqslant \frac{1}{4} |\Phi - \phi| \, |\Gamma - \gamma| - \big[\mathrm{Re}\langle \Phi e - x, x - \phi e \rangle \big]^{1/2} \big[\mathrm{Re}\langle \Gamma e - y, y - \gamma e \rangle \big]^{1/2} \\
&\leqslant \frac{1}{4} |\Phi - \phi| \, |\Gamma - \gamma| \,.
\end{aligned}
$$

The constant $1/4$ is best possible in both inequalities.

Note that the inequality between the first and last term in (1.3) was first established in [1].

A generalisation of the above result for finite families of orthornormal vectors has been pointed out in [3].

THEOREM 2. *Let $\{e_i\}_{i \in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), $i \in F$ and $x, y \in H$. If*

$$(1.4) \qquad \operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geqslant 0,$$

$$\operatorname{Re}\left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \geqslant 0,$$

or, equivalently,

$$(1.5) \qquad \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leqslant \frac{1}{2}\left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2},$$

$$\left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leqslant \frac{1}{2}\left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2},$$

hold, then we have the inequalities

$$(1.6) \quad \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|$$

$$\leqslant \frac{1}{4}\left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}$$

$$- \left[\operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \right]^{1/2} \left[\operatorname{Re}\left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \right]^{1/2}$$

$$\leqslant \frac{1}{4}\left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}.$$

The constant $1/4$ is best possible.

REMARK 1. We note that the inequality between the first term and the last term for real inner products under the assumption (1.4) has been proved by Ujević in [4].

The following corollary provides a reverse for the well known Bessel's inequality in real or complex inner product spaces (see also [3]).

COROLLARY 1. *With the above assumptions for $\{e_i\}_{i \in I}$, F, ϕ_i, Φ_i and x, one has the inequalities*

$$(1.7) \qquad 0 \leqslant \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2,$$

with $1/4$ as the best possible constant in both inequalities.

The main aim of this paper is to point out a different reverse for the Bessel and Grüss inequalities stated above. Some related results are also outlined.

2. A REVERSE OF BESSEL'S INEQUALITY

The following lemma holds.

LEMMA 1. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\lambda_i \in \mathbb{K}$, $i \in F$, $r > 0$ and $x \in H$. If*

$$(2.1) \qquad \left\| x - \sum_{i\in F}\lambda_i e_i \right\| \leqslant r,$$

then we have the inequality

$$(2.2) \qquad 0 \leqslant \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2 \leqslant r^2 - \sum_{i\in F}|\lambda_i - \langle x, e_i\rangle|^2.$$

PROOF: Consider

$$\begin{aligned}
I_1 &:= \left\| x - \sum_{i\in F}\lambda_i e_i \right\|^2 = \left\langle x - \sum_{i\in F}\lambda_i e_i, x - \sum_{j\in F}\lambda_j e_j \right\rangle \\
&= \|x\|^2 - \sum_{i\in F}\lambda_i\overline{\langle x, e_i\rangle} - \sum_{i\in F}\overline{\lambda_i}\langle x, e_i\rangle + \sum_{i\in F}\sum_{j\in F}\lambda_i\overline{\lambda_j}\langle e_i, e_j\rangle \\
&= \|x\|^2 - \sum_{i\in F}\lambda_i\overline{\langle x, e_i\rangle} - \sum_{i\in F}\overline{\lambda_i}\langle x, e_i\rangle + \sum_{i\in F}|\lambda_i|^2
\end{aligned}$$

and

$$\begin{aligned}
I_2 &:= \sum_{i\in F}|\lambda_i - \langle x, e_i\rangle|^2 = \sum_{i\in F}\left(\lambda_i - \langle x, e_i\rangle\right)\left(\overline{\lambda_i} - \overline{\langle x, e_i\rangle}\right) \\
&= \sum_{i\in F}\left[|\lambda_i|^2 + |\langle x, e_i\rangle|^2 - \overline{\lambda_i}\langle x, e_i\rangle - \lambda_i\overline{\langle x, e_i\rangle}\right] \\
&= \sum_{i\in F}|\lambda_i|^2 + \sum_{i\in F}|\langle x, e_i\rangle|^2 - \sum_{i\in F}\overline{\lambda_i}\langle x, e_i\rangle - \sum_{i\in F}\lambda_i\overline{\langle x, e_i\rangle}.
\end{aligned}$$

If we subtract I_2 from I_1 we deduce an identity that is interesting in its own right

$$(2.3) \qquad \left\| x - \sum_{i\in F}\lambda_i e_i \right\|^2 - \sum_{i\in F}|\lambda_i - \langle x, e_i\rangle|^2 = \|x\|^2 - \sum_{i\in F}|\langle x, e_i\rangle|^2,$$

from which we easily deduce (2.2).   ☐

The following reverse of Bessel's inequality holds.

THEOREM 3. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I, ϕ_i, Φ_i, $i \in I$ real or complex numbers. Assume a $x \in H$. If either*

(i) $\operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geqslant 0$; or, equivalently,

(ii) $\left\| x - \sum_{i \in F} (\phi_i + \Phi_i)/2 e_i \right\| \leqslant \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \Big/ 2$;

holds, then the following reverse of Bessel's inequality

$$(2.4) \qquad 0 \leqslant \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2,$$

is valid.

The constant $1/4$ is best possible in both inequalities.

PROOF: Firstly, we observe that for $y, a, A \in H$, the following are equivalent

$$(2.5) \qquad\qquad \operatorname{Re}\langle A - y, y - a \rangle \geqslant 0$$

and

$$(2.6) \qquad\qquad \left\| y - \frac{a + A}{2} \right\| \leqslant \frac{1}{2} \|A - a\|.$$

Now, for $a = \sum_{i \in F} \phi_i e_i$, $A = \sum_{i \in F} \Phi_i e_i$, we have

$$\|A - a\| = \left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\| = \left(\left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\|^2 \right)^{1/2}$$

$$= \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \|e_i\|^2 \right)^{1/2} = \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2},$$

giving, for $y = x$, the desired equivalence.

Now, if we apply Lemma 1 for $\lambda_i = (\phi_i + \Phi_i)/2$ and

$$r := \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2},$$

we deduce the first inequality in (2.4).

Let us prove that $1/4$ is best possible in the second inequality in (2.4). Assume that there is a $c > 0$ such that

$$(2.7) \qquad 0 \leqslant \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leqslant c \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2,$$

provided that ϕ_i, Φ_i, x and F satisfy (i) and (ii).

We choose $F = \{1\}$, $e_1 = e = \left(1/\sqrt{2}, (1/\sqrt{2})\right) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\Phi_1 = \Phi = m > 0$, $\phi_1 = \phi = -m$, $H = \mathbb{R}^2$ to get from (2.7) that

$$(2.8) \qquad 0 \leqslant x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2}$$
$$\leqslant 4cm^2 - \frac{(x_1 + x_2)^2}{2},$$

provided

$$(2.9) \qquad 0 \leqslant \langle me - x, x + me \rangle$$
$$= \left(\frac{m}{\sqrt{2}} - x_1\right)\left(x_1 + \frac{m}{\sqrt{2}}\right) + \left(\frac{m}{\sqrt{2}} - x_2\right)\left(x_2 + \frac{m}{\sqrt{2}}\right).$$

From (2.8) we get

$$(2.10) \qquad x_1^2 + x_2^2 \leqslant 4cm^2$$

provided (2.9) holds.

If we choose $x_1 = m/\sqrt{2}$, $x_2 = -m/\sqrt{2}$, then (2.9) is satisfied and by (2.10) we get $m^2 \leqslant 4cm^2$, giving $c \geqslant 1/4$. ∎

REMARK 2.   If $F = \{1\}$, $e_1 = 1$, $\|e\| = 1$ and for $\phi, \Phi \in \mathbb{K}$ and $x \in H$ one has either

$$(2.11) \qquad \operatorname{Re}\langle \Phi e - x, x - \phi e \rangle \geqslant 0$$

or, equivalently,

$$(2.12) \qquad \left\| x - \frac{\phi + \Phi}{2} e \right\| \leqslant \frac{1}{2}|\Phi - \phi|,$$

then

$$(2.13) \qquad 0 \leqslant \|x\|^2 - |\langle x, e \rangle|^2$$
$$\leqslant \frac{1}{4}|\Phi - \phi|^2 - \left| \frac{\phi + \Phi}{2} - \langle x, e \rangle \right|^2 \leqslant \frac{1}{4}|\Phi - \phi|^2.$$

The constant $1/4$ is best possible in both inequalities.

REMARK 3.   It is important to compare the bounds provided by Corollary 1 and Theorem 3.

For this purpose, consider

$$B_1(x, e, \phi, \Phi) := \frac{1}{4}(\Phi - \phi)^2 - \langle \Phi e - x, x - \phi e \rangle$$

and

$$B_2(x, e, \phi, \Phi) := \frac{1}{4}(\Phi - \phi)^2 - \left(\frac{\phi + \Phi}{2} - \langle x, e \rangle \right)^2,$$

where H is a real inner product, $e \in H$, $\|e\| = 1$, $x \in H$, $\phi, \Phi \in \mathbb{R}$ with $\langle \Phi e - x, x - \phi e \rangle \geqslant 0$, or equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leqslant \frac{1}{2} |\Phi - \phi|.$$

If we choose $\phi = -1$, $\Phi = 1$, then we have

$$B_1(x, e) = 1 - \langle e - x, x + e \rangle = 1 - (\|e\|^2 - \|x\|^2) = \|x\|^2,$$
$$B_2(x, e) = 1 - \langle x, e \rangle^2,$$

provided $\|x\| \leqslant 1$.

Choose $x = ke$, with $0 < k \leqslant 1$. Then we get

$$B_1(k) = k^2, \quad B_2(k) = 1 - k^2,$$

which shows that $B_1(k) > B_2(k)$ if $0 < k < \sqrt{2}/2$ and $B_1(k) < B_2(k)$ if $\sqrt{2}/2 < k \leqslant 1$.

We may state the following proposition.

PROPOSITION 1. *Let $\{e_i\}_{i \in I}$ be a family of orthornormal vectors in H, F a finite part of I, ϕ_i, $\Phi_i \in \mathbb{K}$ $(i \in F)$. If $x \in H$ either satisfies (i), or, equivalently, (ii) of Theorem 3, then the upper bounds*

$$B_1(x, e, \phi, \Phi, F) := \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle,$$

$$B_2(x, e, \phi, \Phi, F) := \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2,$$

for the Bessel's difference $B_s(x, \mathbf{e}, F) := \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$, cannot be compared in general.

3. A REFINEMENT OF THE GRÜSS INEQUALITY

The following result holds.

THEOREM 4. *Let $\{e_i\}_{i \in I}$ be a family of orthornormal vectors in H, F a finite part of I, $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ and $x, y \in H$. If either*

$$(3.1) \qquad \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geqslant 0,$$

$$\operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \geqslant 0,$$

or, equivalently,

$$(3.2) \qquad \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leqslant \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2},$$

$$\left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leqslant \frac{1}{2} \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2},$$

hold, then we have the inequalities

$$
(3.3) \qquad 0 \leqslant \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|
$$

$$
\leqslant \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}
$$

$$
- \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right|
$$

$$
\leqslant \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}.
$$

The constant 1/4 is best possible.

PROOF: Using Schwarz's inequality in the inner product space $(H, \langle \cdot, \cdot \rangle)$ one has

$$
(3.4) \quad \left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \leqslant \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2
$$

and since a simple calculation shows that

$$
\left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle
$$

and

$$
\left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2
$$

for any $x, y \in H$, then by (3.4) and by the reverse of Bessel's inequality in Theorem 3, we have

$$
(3.5) \quad \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \leqslant \left(\|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \left(\|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)
$$

$$
\leqslant \left[\frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right|^2 \right]
$$

$$
\times \left[\frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2 - \sum_{i \in F} \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right|^2 \right]
$$

$$
:= K.
$$

Recall Aczél's inequality for real numbers, that is, we recall that

$$
(3.6) \qquad \left(a^2 - \sum_{i \in F} a_i^2 \right) \left(b^2 - \sum_{i \in F} b_i^2 \right) \leqslant \left(ab - \sum_{i \in F} a_i b_i \right)^2,
$$

provided that $a, b, a_i, b_i > 0$, $i \in F$, and either $a^2 - \sum_{i \in F} a_i^2 > 0$ or $b^2 - \sum_{i \in F} b_i^2 > 0$. So we may state that

$$(3.7) \quad K \leqslant \left[\frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2} \right.$$
$$\left. - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2.$$

Using (3.5) and (3.7) we conclude that

$$(3.8) \quad \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \leqslant \left[\frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \cdot \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2} \right.$$
$$\left. - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2.$$

Taking the square root in (3.8) and taking into account that the quantity in the last square brackets is nonnegative (use for example (2.4) and the Cauchy–Bunyakovsky–Schwarz's inequality for real numbers), we deduce the second inequality in (3.3).

The fact that $1/4$ is the best possible constant follows by Theorem 3 and we omit the details. $\square$

The following corollary may be stated.

COROLLARY 2$'$. *Let $e \in H$, $\|e\| = 1$, $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ such that either*

$$(3.9) \qquad \operatorname{Re}\langle \Phi e - x, x - \phi e \rangle \geqslant 0 \quad and \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e \rangle \geqslant 0$$

or, equivalently,

$$(3.10) \qquad \left\| x - \frac{\phi + \Phi}{2} e \right\| \leqslant \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leqslant \frac{1}{2} |\Gamma - \gamma|.$$

Then we have the following refinement of Grüss' inequality

$$(3.11) \qquad 0 \leqslant |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|$$
$$\leqslant \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| - \left| \frac{\phi + \Phi}{2} - \langle x, e \rangle \right| \left| \frac{\gamma + \Gamma}{2} - \langle y, e \rangle \right|$$
$$\leqslant \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

The constant $1/4$ is best possible in both inequalities.

4. SOME COMPANION INEQUALITIES

The following companion of the Grüss inequality also holds.

THEOREM 5. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and $\phi_i, \Phi_i \in \mathbb{K}$, $i \in F$, $x, y \in H$ and $\lambda \in (0,1)$, such that either*

$$(4.1) \qquad \operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - \big(\lambda x + (1-\lambda)y\big), \lambda x + (1-\lambda)y - \sum_{i\in F} \phi_i e_i \right\rangle \geqslant 0$$

or, equivalently,

$$(4.2) \qquad \left\| \lambda x + (1-\lambda)y - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leqslant \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{1/2},$$

holds. Then we have the inequality

$$(4.3)$$
$$\operatorname{Re}\left[\langle x, y \rangle - \sum_{i\in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \leqslant \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} |\Phi_i - \phi_i|^2$$
$$- \frac{1}{4} \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1-\lambda)y, e_i \rangle \right|^2$$
$$\leqslant \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} |\Phi_i - \phi_i|^2.$$

The constant 1/16 is the best possible constant in (4.3) in the sense that it cannot be replaced by a smaller constant.

PROOF: We know that for any $z, u \in H$, one has

$$\operatorname{Re}\langle z, u \rangle \leqslant \frac{1}{4} \|z + u\|^2.$$

Then for any $a, b \in H$ and $\lambda \in (0,1)$ one has

$$(4.4) \qquad \operatorname{Re}\langle a, b \rangle \leqslant \frac{1}{4\lambda(1-\lambda)} \big\| \lambda a + (1-\lambda)b \big\|^2.$$

Since

$$\langle x, y \rangle - \sum_{i\in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i\in F} \langle x, e_i \rangle e_i, y - \sum_{i\in F} \langle y, e_i \rangle e_i \right\rangle,$$

for any $x, y \in H$, then, by (4.4), we get

$$(4.5) \qquad \operatorname{Re}\left[\langle x, y \rangle - \sum_{i\in F} \langle x, e_i \rangle \langle e_i, y \rangle \right]$$
$$= \operatorname{Re}\left[\left\langle x - \sum_{i\in F} \langle x, e_i \rangle e_i, y - \sum_{i\in F} \langle y, e_i \rangle e_i \right\rangle \right]$$
$$\leqslant \frac{1}{4\lambda(1-\lambda)} \left\| \lambda \Big(x - \sum_{i\in F} \langle x, e_i \rangle e_i \Big) + (1-\lambda)\Big(y - \sum_{i\in F} \langle y, e_i \rangle e_i \Big) \right\|^2$$

$$= \frac{1}{4\lambda(1-\lambda)} \left\| \lambda x + (1-\lambda)y - \sum_{i \in F} \langle \lambda x + (1-\lambda)y, e_i \rangle e_i \right\|^2$$

$$= \frac{1}{4\lambda(1-\lambda)} \left[\left\| \lambda x + (1-\lambda)y \right\|^2 - \sum_{i \in F} \left| \langle \lambda x + (1-\lambda)y, e_i \rangle \right|^2 \right].$$

If we apply the reverse of Bessel's inequality in Theorem 3 for $\lambda x + (1-\lambda)y$, we may state that

$$(4.6) \qquad \left\| \lambda x + (1-\lambda)y \right\|^2 - \sum_{i \in F} \left| \langle \lambda x + (1-\lambda)y, e_i \rangle \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1-\lambda)y, e_i \rangle \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $1/16$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x = y$, then it becomes (i) of Theorem 3, implying for $\lambda = 1/2$ the inequality (2.4), for which, we have shown that $1/4$ is the best constant. $\qquad\square$

REMARK 4.   In practical applications we may use only the inequality between the first and the last term in (4.3).

REMARK 5.   If in Theorem 5, we choose $\lambda = 1/2$, then we get

$$(4.7) \qquad \operatorname{Re} \left[\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right]$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \left\langle \frac{x+y}{2}, e_i \right\rangle \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2,$$

provided

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geqslant 0$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leqslant \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2}.$$

We note that (4.7) is a refinement of the corresponding result in [3].

COROLLARY 3. *With the assumptions of Theorem 5 and if*

$$(4.8) \qquad \mathrm{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \big(\lambda x \pm (1-\lambda)y\big), \lambda x \pm (1-\lambda)y - \sum_{i\in F}\phi_i e_i \right\rangle \geqslant 0$$

or, equivalently

$$(4.9) \qquad \left\| \lambda x \pm (1-\lambda)y - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i \right\| \leqslant \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{1/2},$$

then we have the inequality

$$(4.10) \qquad \left| \mathrm{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \right| \leqslant \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

The constant $1/16$ is best possible in (4.10).

REMARK 6. If H is a real inner product space and $m_i, M_i \in \mathbb{R}$ with the property

$$(4.11) \qquad \left\langle \sum_{i\in F}M_i e_i - \big(\lambda x \pm (1-\lambda)y\big), \lambda x \pm (1-\lambda)y - \sum_{i\in F}m_i e_i \right\rangle \geqslant 0$$

or, equivalently,

$$(4.12) \qquad \left\| \lambda x \pm (1-\lambda)y - \sum_{i\in F}\frac{M_i+m_i}{2}\cdot e_i \right\| \leqslant \frac{1}{2}\left[\sum_{i\in F}(M_i-m_i)^2\right]^{1/2},$$

then we have the Grüss type inequality

$$(4.13) \qquad \left| \langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle \right| \leqslant \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}(M_i-m_i)^2.$$

5. INTEGRAL INEQUALITIES

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a σ−algebra of subsets and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Let $\rho \geqslant 0$ be a μ−measurable function on Ω. Denote by $L^2_\rho(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω such that

$$(5.1) \qquad \int_\Omega \rho(s)|f(s)|^2\,d\mu(s) < \infty.$$

Consider the family $\{f_i\}_{i\in I}$ of functions in $L^2_\rho(\Omega,\mathbb{K})$ with the properties that

$$(5.2) \qquad \int_\Omega \rho(s)f_i(s)\overline{f_j}(s)d\mu(s) = \delta_{ij}, \quad i,j \in I,$$

where δ_{ij} is 0 if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2_\rho(\Omega,\mathbb{K})$.

The following proposition holds.

PROPOSITION 2. *Let* $\{f_i\}_{i \in I}$ *be an orthornormal family of functions in* $L^2_\rho(\Omega, \mathbb{K})$, *F a finite subset of* I, $\phi_i, \Phi_i \in \mathbb{K}$ $(i \in F)$ *and* $f \in L^2_\rho(\Omega, \mathbb{K})$, *so that either*

$$(5.3) \qquad \int_\Omega \rho(s) \, \mathrm{Re}\left[\left(\sum_{i \in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \, \overline{f_i}(s)\right)\right] d\mu(s) \geqslant 0$$

or, equivalently,

$$(5.4) \qquad \int_\Omega \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot f_i(s) \right|^2 d\mu(s) \leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2$$

hold. Then we have the inequality

$$(5.5) \qquad 0 \leqslant \int_\Omega \rho(s) |f(s)|^2 d\mu(s) - \sum_{i \in F} \left| \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \, .$$

The constant $1/4$ *is best possible in both inequalities.*

The proof follows by Theorem 3 applied for the Hilbert space $L^2_\rho(\Omega, \mathbb{K})$ and the orthornormal family $\{f_i\}_{i \in I}$.

The following Grüss type inequality also holds.

PROPOSITION 3. *Let* $\{f_i\}_{i \in I}$ *and F be as in Proposition 2. If* $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ $(i \in F)$ *and* $f, g \in L^2_\rho(\Omega, \mathbb{K})$ *so that either*

$$(5.6) \qquad \int_\Omega \rho(s) \, \mathrm{Re}\left[\left(\sum_{i \in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \, \overline{f_i}(s)\right)\right] d\mu(s) \geqslant 0,$$

$$\int_\Omega \rho(s) \, \mathrm{Re}\left[\left(\sum_{i \in F} \Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i \in F} \overline{\gamma_i} \, \overline{f_i}(s)\right)\right] d\mu(s) \geqslant 0,$$

or, equivalently,

$$(5.7) \qquad \int_\Omega \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) \leqslant \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \, ,$$

$$\int_\Omega \rho(s) \left| g(s) - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} f_i(s) \right|^2 d\mu(s) \leqslant \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \, ,$$

hold, then we have the inequalities

$$(5.8) \quad \left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \sum_{i \in F} \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \int_\Omega \rho(s) f_i(s) \overline{g(s)} d\mu(s) \right|$$

$$\leqslant \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}$$

$$- \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \int_\Omega \rho(s) g(s) \overline{f_i}(s) d\mu(s) \right|$$

$$\leqslant \frac{1}{4} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \left(\sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{1/2}.$$

The constant $1/4$ is the best possible.

The proof follows by Theorem 4 and we omit the details.

REMARK 7. Similar results may be stated if one applies the inequalities in Section 4. We omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (5.5) or for the Grüss type inequality (5.8) to hold.

COROLLARY 4. *Let $\{f_i\}_{i \in I}$ be an orthornormal family of functions in the real Hilbert space $L^2_\rho(\Omega)$, F a finite part of I, $M_i, m_i \in \mathbb{R}$ $(i \in F)$ and $f \in L^2_\rho(\Omega)$ so that*

$$(5.9) \quad \sum_{i \in F} m_i f_i(s) \leqslant f(s) \leqslant \sum_{i \in F} M_i f_i(s) \quad \text{for } \mu - \text{ almost everywhere } s \in \Omega,$$

then we have the inequalities

$$(5.10) \quad 0 \leqslant \int_\Omega \rho(s) f^2(s) d\mu(s) - \sum_{i \in F} \left[\int_\Omega \rho(s) f(s) f_i(s) d\mu(s) \right]^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2 - \sum_{i \in F} \left[\frac{M_i + m_i}{2} - \int_\Omega \rho(s) f(s) f_i(s) d\mu(s) \right]^2$$

$$\leqslant \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2.$$

The constant $1/4$ is best possible.

COROLLARY 5. *Let $\{f_i\}_{i \in I}$ and F be as in Corollary 4. If $M_i, m_i, N_i, n_i \in \mathbb{R}$ $(i \in F)$ and $f, g \in L^2_\rho(\Omega)$ such that*

$$(5.11) \quad \sum_{i \in F} m_i f_i(s) \leqslant f(s) \leqslant \sum_{i \in F} M_i f_i(s)$$

and

$$\sum_{i \in F} n_i f_i(s) \leqslant g(s) \leqslant \sum_{i \in F} N_i f_i(s) \quad \text{for } \mu - \text{ almost everywhere } s \in \Omega,$$

then we have the inequalities

$$\left| \int_\Omega \rho(s)f(s)g(s)d\mu(s) - \sum_{i\in F} \int_\Omega \rho(s)f(s)f_i(s)d\mu(s) \int_\Omega \rho(s)g(s)f_i(s)d\mu(s) \right|$$

$$\leqslant \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{1/2}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{1/2} - \sum_{i\in F}\left|\frac{M_i+m_i}{2} - \int_\Omega \rho(s)f(s)f_i(s)d\mu(s)\right|$$

$$\times \left|\frac{N_i+n_i}{2} - \int_\Omega \rho(s)g(s)f_i(s)d\mu(s)\right|$$

$$\leqslant \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{1/2}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{1/2}.$$

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School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428
MCMC 8001 Victoria
Australia
e-mail   sever.dragomir@vu.edu.au
urladdrhttp://rgmia.vu.edu.au/SSDragomirWeb.html