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A necessary and sufficient condition for Riemann's singularity theorem to hold on a Prym theta divisor

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Abstract

Let (P, Ξ) be the naturally polarized model of the Prym variety associated to the étale double cover $\pi : \tilde{C} \to C$ of smooth connected curves defined over an algebraically closed field k of characteristic $\neq 2$, where genus$(C) = g \geqslant 3$, $\mathrm{Pic}^{(2g-2)}(\tilde{C}) \supset P = \{\mathcal{L} \in \mathrm{Pic}^{(2g-2)}(\tilde{C}) : \mathrm{Nm}(\mathcal{L}) = \omega_C$ and $h^0(\tilde{C}, \mathcal{L})$ is even$\}$ is the Prym variety, and $P \supset \Xi = \{\mathcal{L} \in P : h^0(\tilde{C}, \mathcal{L}) > 0\}$ is the Prym theta divisor with its reduced scheme structure. If $\mathcal{L}$ is any point on Ξ, we prove that 'Riemann's singularity theorem holds at $\mathcal{L}$', i.e. $\mathrm{mult}_{\mathcal{L}}(\Xi) = (1/2)h^0(\tilde{C}, \mathcal{L})$, if and only if $\mathcal{L}$ cannot be expressed as $\pi^*(\mathcal{M})(B)$ where $B \geqslant 0$ is an effective divisor on $\tilde{C}$, and $\mathcal{M}$ is a line bundle on C with $h^0(C, \mathcal{M}) > (1/2)h^0(\tilde{C}, \mathcal{L})$. This completely characterizes points of Ξ where the tangent cone is the set theoretic restriction of the tangent cone of $\tilde{\Theta}$, hence also those points on Ξ where Mumford's Pfaffian equation defines the tangent cone to Ξ.

Introduction

A fundamental tool for analyzing Jacobian varieties $(J(C), \Theta(C))$ of curves C of genus g is the link between linear systems on C and the geometry of Θ provided by Riemann's singularity theorem. Points of Θ correspond to effective line bundles $\mathcal{L}$ of degree $g-1$ on C, and at such a point $\mathrm{mult}_{\mathcal{L}}(\Theta) = h^0(C, \mathcal{L})$. Thus 'Brill Noether' loci (line bundles in $\mathrm{Pic}^{g-1}(C)$ with a given number of sections), gain intrinsic meaning on Θ as sets of points of fixed multiplicity. Brill Noether homology computations then imply the existence of points of given multiplicity on Θ. This impacts the Torelli problem, since the projective tangent cone to Θ at $\mathcal{L}$ has a description by the linear system $|\mathcal{L}|$ which implies the cone contains the canonical model of C if $\mathrm{mult}_{\mathcal{L}}(\Theta) \geqslant 2$. The goal of this paper is to make the analogous multiplicity correspondence for classical Prym varieties almost as complete, with precise conditions for its failure. If (P, Ξ) is the Prym variety of an étale connected double cover $\pi : \tilde{C} \to C$ of a smooth curve C of genus g, points of Ξ are effective line bundles $\mathcal{L}$ in $\mathrm{Pic}^{2g-2}(\tilde{C})$ with $\mathrm{Nm}(\mathcal{L}) = \omega_C$ and $h^0(\tilde{C}, \mathcal{L})$ even. An equation $\tilde{\vartheta}$ for $\tilde{\Theta}$ restricts on $P \subset \mathrm{Pic}^{2g-2}(\tilde{C})$ to the square of an equation ξ for Ξ, so we expect $\mathrm{mult}_{\mathcal{L}}(\Xi) = (1/2)h^0(\tilde{C}, \mathcal{L})$, and this holds if and only if the leading term of (a Taylor series for) $\tilde{\vartheta}$ is the square of the leading term of ξ. If the equality $\mathrm{mult}_{\mathcal{L}}(\Xi) = (1/2)h^0(\tilde{C}, \mathcal{L})$ holds, we say Riemann's singularity theorem holds at $\mathcal{L}$, or more briefly 'RST' holds at $\mathcal{L}$. Precise criteria for RST to hold thus would again let one interpret Brill Noether calculations intrinsically on Ξ. Since the projective tangent cone to Ξ at $\mathcal{L}$ with $\mathrm{mult}_{\mathcal{L}}(\Xi) \geqslant 2$ contains the Prym canonical model of C when RST holds, but not necessarily when it fails, this would illuminate the open Prym Torelli problem as well.

The criterion in this paper is as follows. With notation as above, call $\mathcal{L}$ on $\Xi \subset \mathrm{Pic}^{2g-2}(\tilde{C})$ 'very exceptional' if there is a line bundle $\mathcal{M}$ on C, with $2h^0(C, \mathcal{M}) > h^0(\tilde{C}, \mathcal{L})$ and $\mathcal{L} \otimes \pi^*(\mathcal{M}^{-1})$ effective. Then RST holds at $\mathcal{L}$ if and only if $\mathcal{L}$ is not very exceptional.

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Discussion of prior work

A sufficient condition for RST to hold (injectivity of the Prym Petri map) was given by Welters in [Wel85], who checked it at the boundary of moduli and deduced that RST holds everywhere on a 'sufficiently general' Prym variety. Important though this is, the difficulty of computing the condition for any specific smooth curves left open such basic problems as the density, even the existence, of double points in the 'stable' singular locus for specific Prym varieties. That is, Brill Noether calculations imply that the locus $\{\mathcal{L} \in \Xi : h^0(\tilde{C}, \mathcal{L}) \geqslant 4\}$ is of codimension $\leqslant 6$ and non-empty if $\dim(P) \geqslant 6$, but the existence of points in this locus with multiplicity exactly 2 on Ξ does not follow without RST for Pryms. The density was settled in [SV02, Theorem 3.5, p. 245] using the case $h^0(\tilde{C}, \mathcal{L}) = 4$ of the present result.

The present result was proved in the case $h^0(\tilde{C}, \mathcal{L}) = 2$ by Mumford. His hypothesis on $\mathcal{L}$ [Mum74, Proposition, p. 343] has two natural generalizations for higher values of $h^0(\tilde{C}, \mathcal{L})$: (a) the notion of 'very exceptional' used here (condition ii in Theorem 0.1 below); and (b) his 'case 1' [Mum74, p. 344] (where he assumes only $h^0(C, \mathcal{M}) \geqslant 2$), now called 'exceptional'. Since RST can hold at 'case 1' points [SV01, Example 2.18], generalization (b) does not fit the RST problem. In [Sho84, Lemma 5.7, p. 121], Shokurov observed that Mumford's argument shows that RST fails at $\mathcal{L}$ on Ξ, if $h^0(\tilde{C}, \mathcal{L}) = 4$, $\mathcal{L} \otimes \pi^*(\mathcal{M}^{-1})$ effective and $h^0(C, \mathcal{M}) \geqslant 3$. The same argument also works for higher values of $h^0(\tilde{C}, \mathcal{L})$, (cf. Lemma 2.4 below). Thus, after we checked the converse [SV02, Remarks 3.7(ii)] in the case $h^0(\tilde{C}, \mathcal{L}) = 4$, we were led to conjecture, in general, that $\mathcal{L}$ 'very exceptional' should not only be sufficient, but also necessary for RST to fail. The following is the precise theorem proved here.

Theorem 0.1. *Given a connected étale double cover $\pi : \tilde{C} \to C$ of a smooth curve C with $g(C) \geqslant 3$, associated involution $\iota : \tilde{C} \to \tilde{C}$, principally polarized Prym variety $\Xi \subset P \subset \mathrm{Pic}^{2g-2}(\tilde{C})$, and a point $\mathcal{L}$ of Ξ, the following conditions are equivalent.*

i) *Riemann's singularity theorem fails at $\mathcal{L}$, i.e. $\mathrm{mult}_{\mathcal{L}}(\Xi) \neq (1/2)h^0(\tilde{C}, \mathcal{L})$; necessarily then $\mathrm{mult}_{\mathcal{L}}(\Xi) > (1/2)h^0(\tilde{C}, \mathcal{L})$.*

ii) *A pair of effective line bundles $(\mathcal{M}, \mathcal{N})$ exists such that $\mathcal{M}$ is in $\mathrm{Pic}(C)$, $\mathcal{N}$ is in $\mathrm{Pic}(\tilde{C})$, $\mathcal{L} \cong \pi^*(\mathcal{M}) \otimes \mathcal{N}$, and $2h^0(C, \mathcal{M}) > h^0(\tilde{C}, \mathcal{L})$.*

iii) *There is a unique pair of effective line bundles $(\mathcal{M}, \mathcal{N})$ such that $\mathcal{M}$ is in $\mathrm{Pic}(C)$, $\mathcal{N}$ is in $\mathrm{Pic}(\tilde{C})$, $\mathcal{L} \cong \pi^*(\mathcal{M}) \otimes \mathcal{N}$, $2h^0(C, \mathcal{M}) > h^0(\tilde{C}, \mathcal{L})$ and $|\mathcal{N}|$ contains a divisor D with 'no invariant part', i.e. such that $\mathrm{supp}(D) \cap \mathrm{supp}(\iota^*(D)) = \emptyset$; necessarily then $h^0(\mathcal{N}) = 1$.*

In terms of the skew symmetric pairing $\beta : H^0(\mathcal{L}) \times H^0(\mathcal{L}) \to T_0^*(P)$ (see § 2.1), these are equivalent to the following.

iv) *The polynomial $\det(\beta)$ (in terms of any basis for $H^0(\mathcal{L})$) is identically zero on $T_0(P)$.*

v) *The pairing β has an isotropic subspace $W \subset H^0(\mathcal{L})$ with $\dim(W) > (1/2)h^0(\tilde{C}, \mathcal{L})$.*

Remark. Since $\det(\beta) = (Pf(\beta))^2$, it follows that Mumford's Pfaffian equation $Pf(\beta) = 0$ is an equation for the tangent cone $C_{\mathcal{L}}(\Xi)$ if and only if condition ii does not hold.

1. Definitions and conventions

The basic reference for the theory of Prym varieties is [Mum74]. Throughout this paper, $\pi : \tilde{C} \to C$ is a connected étale double cover of a smooth curve C of genus $g \geqslant 3$ over an algebraically closed field k of characteristic $\neq 2$, (P, Ξ) is the canonically polarized Prym variety, embedded in $\mathrm{Pic}(\tilde{C})$ by $\mathrm{Pic}^{2g-2}(\tilde{C}) \supset P = \{\mathcal{L} : \mathrm{Nm}(\mathcal{L}) = \omega_C \text{ and } h^0(\mathcal{L}) \text{ is even}\}$, Nm: $\mathrm{Pic}(\tilde{C}) \to \mathrm{Pic}(C)$ is the norm map associated to π, and $\tilde{\Theta} \cdot P = 2\Xi$, where $\Xi = \{\mathcal{L} \in P : h^0(\mathcal{L}) > 0\}$ is the distinguished model of the Prym theta divisor. If η is the unique non-zero line bundle on C such that $\pi^*(\eta) = \mathcal{O}_{\tilde{C}}$, then $\eta^2 \cong \mathcal{O}_C$ and the pair (C, η) determines both $\tilde{C}$ and the double cover π, and hence also the equivalent fix point free involution $\iota : \tilde{C} \to \tilde{C}$. Writing ω for ω_C and $\tilde{\omega}$ for $\omega_{\tilde{C}}$, the cotangent space to $\mathrm{Pic}^{2g-2}(\tilde{C})$, which is isomorphic to $H^0(\tilde{C}, \tilde{\omega})$, splits under the involution into the sum of invariant and anti-invariant subspaces isomorphic, respectively, to $H^0(C, \omega)$ and $H^0(C, \omega \otimes \eta)$. We denote that a line bundle $\mathcal{M}$ is effective by writing $\mathcal{M} \geqslant 0$ and write $\mathcal{L} \geqslant \mathcal{M}$ to mean that $(\mathcal{L} - \mathcal{M}) \geqslant 0$. An effective divisor D has 'no invariant part' if and only if $\mathrm{supp}(D) \cap \mathrm{supp}(\iota^*(D)) = \emptyset$; for example, the trivial divisor is an effective divisor with no invariant part.

Next we introduce a useful sequence of definitions of pairs of line bundles $(\mathcal{M}, \mathcal{N})$ which may be associated to a point $\mathcal{L}$ of Ξ, characterized by increasingly restrictive properties.

Definition 1.1. Given $\pi : \tilde{C} \to C$, Prym variety (P, Ξ) and $\mathcal{L} \in \Xi$, we say $(\mathcal{M}, \mathcal{N})$, with $\mathcal{M}$ in $\mathrm{Pic}(C)$, $\mathcal{N}$ in $\mathrm{Pic}(\tilde{C})$, is an *effective pair* for $\mathcal{L}$ if:

i) $\mathcal{M} \geqslant 0, \mathcal{N} \geqslant 0$;

ii) $\mathcal{L} \cong \pi^*(\mathcal{M}) \otimes \mathcal{N}$.

Every point $\mathcal{L}$ of Ξ admits an effective pair, e.g. with $\mathcal{M} = \mathcal{O}_C$.

Definition 1.2. An effective pair $(\mathcal{M}, \mathcal{N})$ is called *exceptional* if $h^0(C, \mathcal{M}) \geqslant 2$. A point $\mathcal{L}$ on Ξ is an 'exceptional singularity' of Ξ if and only if $\mathcal{L}$ admits an exceptional pair. By [Mum74, pp. 342–3] an 'exceptional singularity' is always a singular point of Ξ.

Definition 1.3. An effective pair $(\mathcal{M}, \mathcal{N})$ for $\mathcal{L}$ is called a $(*)$ pair, if the following inequality holds: $(*)\ 2h^0(\mathcal{M}) + h^0(\mathcal{N}) \geqslant h^0(\mathcal{L}) + 3$. Since $h^0(\mathcal{N}) \leqslant h^0(\mathcal{L})$, every $(*)$ pair is exceptional.

Definition 1.4. An effective pair $(\mathcal{M}, \mathcal{N})$ is *very exceptional* or a *Shokurov pair* for $\mathcal{L}$ if $2h^0(\mathcal{M}) > h^0(\mathcal{L})$. $\mathcal{L}$ is a 'very exceptional', or 'Shokurov' singularity of Ξ, if and only if $\mathcal{L}$ admits a Shokurov pair. Since $h^0(\mathcal{L})$ is even, every Shokurov pair is a $(*)$ pair.

Definition 1.5. An effective (exceptional, $(*)$, etc.) pair $(\mathcal{M}, \mathcal{N})$ is *maximal* if $|\mathcal{N}|$ contains a divisor with no invariant part.

Remark. Every point $\mathcal{L}$ of Ξ has a maximal effective pair, since if $\mathcal{M}, \mathcal{N}$ satisfy Definition 1.1, then $\mathcal{N} = \mathcal{O}(B + \pi^*(A))$, where $A \geqslant 0, B \geqslant 0$ are effective divisors, and B has no invariant part. Thus $(\mathcal{M} \otimes \mathcal{O}(A), \mathcal{O}(B))$ is a maximal effective pair for $\mathcal{L}$. Since this construction cannot decrease the value of $h^0(\mathcal{M})$ it turns an exceptional pair into a maximal exceptional pair, and a Shokurov pair into a maximal Shokurov pair (and shows that non-maximal versions of these pairs are not unique). It is not clear that when this construction is applied to a $(*)$ pair whether the resulting maximal pair still satisfies $(*)$. As to uniqueness, we show that $\mathcal{L}$ has at most one maximal Shokurov pair (Lemma 5.4) and, equivalently, at most one maximal $(*)$ pair (Lemma 5.5).

2. Isotropic subspaces for the Mumford pairing

We recall the skew symmetric pairing introduced by Mumford [Mum74] and generalize [Mum74, Proposition, p. 343] to a correspondence between isotropic subspaces and certain linear series on the base curve C.

2.1 Definition of the pairing $\beta : H^0(\mathcal{L}) \times H^0(\mathcal{L}) \to H^0(C, \omega \otimes \eta)$

i) For line bundles $\mathcal{L}$ and $\mathcal{L}'$, and sections $s \in H^0(\mathcal{L})$ and $t \in H^0(\mathcal{L}')$, we use the notation $s \cdot t$ for the cup product in $H^0(\mathcal{L} \otimes \mathcal{L}')$.

ii) For $\mathcal{L} \in \Xi$ and $(s, t) \in H^0(\mathcal{L}) \times H^0(\mathcal{L})$, let $\langle s, t\rangle = s \cdot \iota^*(t) \in H^0(\tilde{\omega})$, via the composition [Mum74, p. 343, line 4], $H^0(\mathcal{L}) \times H^0(\mathcal{L}) \cong H^0(\mathcal{L}) \times H^0(\iota^*(\mathcal{L})) \cong H^0(\mathcal{L}) \times H^0(\tilde{\omega} \otimes \mathcal{L}^*) \to H^0(\tilde{\omega})$.

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iii) Then let $\beta(s,t) = (\langle s,t\rangle - \langle t,s\rangle) = s\cdot\iota^*(t) - t\cdot\iota^*(s)$, so the map $\beta : H^0(\mathcal{L})\otimes H^0(\mathcal{L}) \to H^0(\omega\otimes\eta) \cong$ {the (-1) eigenspace for ι^* acting on $H^0(\tilde{\omega})$} is skew symmetric; see [Mum74, p. 343], [Wel85, p. 673].

iv) For each z in $T_0P = H^0(\omega\otimes\eta)^*$, let $\beta_z : H^0(\mathcal{L})\times H^0(\mathcal{L}) \to k$ denote the scalar valued skew pairing taking (s,t) to $\beta(s,t)(z) = \beta_z(s,t)$.

Lemma 2.2. *Fix $\mathcal{L}\in\Xi$ and consider $\beta : H^0(\mathcal{L})\times H^0(\mathcal{L}) \to T_0^*(P)$ as in § 2.1, item iii.*

i) *Suppose $\mathcal{M}$ is a line bundle on C, $\Lambda\subset H^0(\mathcal{M})$ is a vector subspace of positive dimension ℓ defining a linear subsystem $|\Lambda|\subset|\mathcal{M}|$ (possibly with base points), B is an effective divisor on $\tilde{C}$ such that $\mathcal{L}\cong\pi^*(\mathcal{M})(B)$ and $u\in H^0(\mathcal{O}_{\tilde{C}}(B))$ is an equation for B. Then $\pi^*(\Lambda)\cdot u\subset H^0(\mathcal{L})$ is an isotropic subspace for β of dimension ℓ.*

ii) *Conversely, any isotropic subspace $W\subset H^0(\mathcal{L})$ of positive dimension ℓ has the form $\pi^*(\Lambda)\cdot u$ as in part i. Moreover, we can choose Λ and u so that the divisor $B = \operatorname{div}(u)$ has no invariant part; if this is done, then Λ, $\mathcal{M}$, and B are determined uniquely by W.*

Proof. i) Since pullback of sections $\pi^* : H^0(C,\mathcal{M}) \to H^0(\tilde{C},\pi^*(\mathcal{M}))$ is injective, and multiplication by $u : H^0(\tilde{C},\pi^*(\mathcal{M})) \to H^0(\tilde{C},\pi^*(\mathcal{M})(B))$ is injective, the map $\Lambda\to H^0(\mathcal{L}), \sigma\mapsto\pi^*(\sigma)\cdot u$ is an isomorphism from Λ onto its image $\pi^*(\Lambda)\cdot u\subset H^0(\mathcal{L})$, so $\dim(\pi^*(\Lambda)\cdot u) = \dim(\Lambda) = \ell$. Now, if $\sigma,\tau\in\Lambda$, then $\beta(\pi^*(\sigma)\cdot u, \pi^*(\tau)\cdot u) = \langle\pi^*(\sigma)\cdot u, \pi^*(\tau)\cdot u\rangle - \langle\pi^*(\tau)\cdot u, \pi^*(\sigma)\cdot u\rangle = \pi^*(\sigma)\cdot u\cdot \iota^*(\pi^*(\tau)\cdot u) - \pi^*(\tau)\cdot u\cdot\iota^*(\pi^*(\sigma)\cdot u) = \pi^*(\sigma)\cdot u\cdot\pi^*(\tau)\cdot\iota^*(u) - \pi^*(\tau)\cdot u\cdot\pi^*(\sigma)\cdot\iota^*(u) = 0$ by the commutativity of multiplication of sections.

ii) Take $s_0\neq 0$ in W and let $\Psi = W/s_0$. Then $\Psi\subset k(\tilde{C})$ is an ℓ-dimensional vector space of rational functions on $\tilde{C}$ such that $W = \Psi\cdot s_0$. Now take any $\psi\in\Psi$. Then $\psi = s/s_0$ for some $s\in W$, and since W is isotropic we have $0 = \beta(s,s_0) = \langle s,s_0\rangle - \langle s_0,s\rangle = s\cdot\iota^*(s_0) - s_0\cdot\iota^*(s) = \psi\cdot s_0\cdot\iota^*(s_0) - s_0\cdot\iota^*(\psi\cdot s_0) = \psi\cdot s_0\cdot\iota^*(s_0) - \iota^*(\psi)\cdot s_0\cdot\iota^*(s_0) = (\psi - \iota^*(\psi))\cdot s_0\cdot\iota^*(s_0)$. Since $s_0\cdot\iota^*(s_0)\neq 0$, thus $\psi - \iota^*(\psi) = 0$, so $\psi = \iota^*(\psi)$. Hence there is a unique rational function φ on C such that $\psi = \pi^*(\varphi)$. Thus there is an ℓ-dimensional vector space $\Phi\subset k(C)$ such that $\Psi = \pi^*(\Phi)$ and $W = \pi^*(\Phi)\cdot s_0$.

Now, if $\varphi_1,\ldots,\varphi_\ell$ is a basis for Φ, then D, the least upper bound (l.u.b.) of the polar divisors $(\varphi_1)_\infty,\ldots,(\varphi_\ell)_\infty$, is the smallest effective divisor on C such that $\Phi\subset L(D)$, where $L(D) = \{\varphi\in k(C)^* : \operatorname{div}(\varphi) + D \geqslant 0\}\cup\{0\}$. Set $\mathcal{M}_0 = \mathcal{O}_C(D)$, $\sigma\in H^0(C,\mathcal{M}_0)$ the tautological equation for D, and $\Lambda_0 = \Phi\cdot\sigma$. Since $\Phi\subset L(D)$ then $\Lambda_0\subset L(D)\cdot\sigma = H^0(C,\mathcal{M}_0)$ has dimension ℓ.

Now $\psi_1 = \pi^*(\varphi_1),\ldots,\psi_\ell = \pi^*(\varphi_\ell)$ is a basis for $\Psi = \pi^*(\Phi)$ and $\pi^*(D) =$ l.u.b. $\{(\psi_1)_\infty,\ldots,(\psi_\ell)_\infty\}$. Since $\psi\cdot s_0$ is a regular section of $\mathcal{L}$ for every ψ in Ψ, $(s_0)\geqslant(\psi_i)_\infty$, for $i = 1,\ldots,\ell$, hence $(s_0)\geqslant\pi^*(D)$. Thus $B_0 = (s_0) - \pi^*(D)\geqslant 0$ on $\tilde{C}$, hence $u = s_0/\pi^*(\sigma)$ is a regular section of $\mathcal{O}_{\tilde{C}}(B_0)$. Since $s_0 = \pi^*(\sigma)\cdot u$ is a non-zero section of $\mathcal{L}$, $\mathcal{L}\cong\pi^*(\mathcal{M}_0)(B_0)$, and $W = \Psi\cdot s_0 = \pi^*(\Phi)\cdot\pi^*(\sigma)\cdot u = \pi^*(\Phi\cdot\sigma)\cdot u = \pi^*(\Lambda_0)\cdot u$, as desired.

This construction gives Λ_0 and B_0 such that Λ_0 has no base divisor, rather than the desired property that B_0 has no invariant part. To get the representation in the lemma, let $B_0 = \sum_{\tilde{C}} n_p\cdot p$ be the full base divisor of the linear system $|W|$ (as in the construction above), and for each point $\bar{p}$ of C set $m_{\bar{p}} = \min\{n_p, n_{p'}\}$, where $\pi^{-1}(\bar{p}) = \{p,p'\}$ for the double cover $\pi:\tilde{C}\to C$. Then the base divisor can be written uniquely as $B_0 = \sum_{\tilde{C}} n_p\cdot p = \sum_C m_{\bar{p}}\cdot(p+p') + \sum_{\tilde{C}}(n_p - m_{\bar{p}})\cdot p = \pi^*(A) + B$, where $A = \sum_C m_{\bar{p}}\cdot\bar{p}$ and $B = \sum_{\tilde{C}}(n_p - m_{\bar{p}})\cdot p$ has no invariant part. If Λ_0 is chosen as above, τ is an equation on C for A, and v is an equation on $\tilde{C}$ for B, and we define $\Lambda = \Lambda_0\cdot\tau$, then $W = \pi^*(\Lambda)\cdot v$ is a representation of W where $\operatorname{div}(v) = B$ has no invariant part. Both Λ and B are uniquely determined by W, since the system $|W|$ determines its base locus, the invariant part

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$\pi^*(A)$ of its base locus, thus also the 'non-invariant' part B. Then $|W| - B$ is the pullback of a unique linear system $|\Lambda|$ on C and $\Lambda \subset H^0(C, \mathcal{M})$ for a unique line bundle $\mathcal{M}$. □

Lemma 2.3. *We keep the notation of Lemma 2.2.*

i) *Suppose $\mathcal{M}$ is a line bundle on C, $h^0(\mathcal{M}) \geqslant 2$ (but $|\mathcal{M}|$ is allowed to have base points), B is an effective divisor on $\tilde{C}$ such that $\mathcal{L} \cong \pi^*(\mathcal{M})(B)$ where B and $B' = \iota^*(B)$ have disjoint supports, and $u \in H^0(\mathcal{O}_{\tilde{C}}(B))$ is an equation for B. Then $\pi^*(H^0(\mathcal{M})) \cdot u \subset H^0(\mathcal{L})$ is a maximal isotropic subspace for β of dimension $h^0(\mathcal{M}) \geqslant 2$.*

ii) *Conversely, any maximal isotropic subspace $W \subset H^0(\mathcal{L})$ of dimension $\geqslant 2$ has the form $\pi^*(H^0(\mathcal{M})) \cdot u$ as in part i, where both $\mathcal{M}$ and $\operatorname{div}(u) = B$ are uniquely determined by W.*

iii) *If two isotropic subspaces V, W of $H^0(\mathcal{L})$ have non-zero intersection, then their span is isotropic.*

Proof. Part ii is immediate from Lemma 2.2, part ii. Now let $\pi^*(H^0(\mathcal{M})) \cdot u$ be as in part i. Then by Lemma 2.2, part i, we already know that $\pi^*(H^0(\mathcal{M})) \cdot u \subset H^0(\mathcal{L})$ is an isotropic subspace for β of dimension $h^0(\mathcal{M}) \geqslant 2$, so it remains to prove maximality. If a non-zero element $\pi^*(\sigma) \cdot u$ of $\pi^*(H^0(\mathcal{M})) \cdot u$ belongs to another isotropic subspace, it belongs to one of form $\pi^*(H^0(\mathcal{M}_1)) \cdot u_1$, where $\operatorname{div}(u_1)$ also has no invariant part. Then $\pi^*(\sigma) \cdot u = \pi^*(\tau) \cdot u_1$, for τ in $H^0(\mathcal{M}_1)$. Equating invariant parts of divisors of these sections, we see that $\operatorname{div}(\pi^*(\sigma)) = \operatorname{div}(\pi^*(\tau))$, so $\operatorname{div}(\sigma) = \operatorname{div}(\tau)$, hence $\mathcal{M} = \mathcal{M}_1$, $\operatorname{div}(u) = \operatorname{div}(u_1)$, hence $\pi^*(H^0(\mathcal{M})) \cdot u = \pi^*(H^0(\mathcal{M}_1)) \cdot u_1$, hence $\pi^*(H^0(\mathcal{M})) \cdot u$ is maximal isotropic. For part iii, the proof of part i shows that V, W lie in a common maximal isotropic subspace. (The case $\dim(V) \leqslant 1$ or $\dim(W) \leqslant 1$ is trivial.) □

Remark. It follows from Lemma 2.3, part iii, that if $\ker(\beta) = \{v : \beta(v, H^0(\mathcal{L})) = 0\} \neq \{0\}$, then $H^0(\mathcal{L})$ is β-isotropic, i.e. β is identically zero (since $\ker(\beta)$ lies in every maximal isotropic subspace). In particular, unlike scalar valued skew pairings, neither property iv nor v of Theorem 0.1 implies that $\ker(\beta) \neq 0$.

Lemma 2.4. *If $\mathcal{L}$ can be expressed as $\pi^*(\mathcal{M})(B)$ for $\mathcal{M} \in \operatorname{Pic}(C)$ with $B \geqslant 0$ on $\tilde{C}$, and $h^0(\mathcal{M}) > (1/2)h^0(\mathcal{L})$, then $\det(\beta)$ is identically zero on $T_0(P)$. Here $\det(\beta)$ is the polynomial defined by the determinant of a matrix for β with respect to a basis of $H^0(\mathcal{L})$.*

Proof. Note that $\det(\beta) = 0$ if for all z in $T_0(P)$, the determinant of a matrix for β_z is zero. If $\mathcal{L} \cong \pi^*(\mathcal{M})(B)$ for $\mathcal{M} \in \operatorname{Pic}(C)$ with $h^0(\mathcal{M}) > (1/2)h^0(\mathcal{L})$, and $B \geqslant 0$ on $\tilde{C}$, let $u \in H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(B))$ be a defining equation for B and consider $W = \pi^*(H^0(C, \mathcal{M})) \cdot u \subset H^0(\tilde{C}, \mathcal{L})$. Then W is β-isotropic by Lemma 2.2, part i and $\dim(W) > (1/2)\dim(H^0(\mathcal{L}))$, hence each (scalar-valued) skew-symmetric form β_z on $H^0(\mathcal{L})$ (for $z \in T_0(P)$), is degenerate. Thus $\det(\beta_z) = 0$ for every $z \in T_0(P)$. □

Producing β-isotropic subspaces of $H^0(\mathcal{L})$ when $\det(\beta) = 0$

We begin the proof of the key implication iv implies ii in Theorem 0.1. Let $\mathcal{L}$ be in Ξ and β the vector valued pairing in § 2.1, item iii, and for each z in T_0P view the scalar valued pairing β_z in § 2.1, item iv, as a linear map $\lambda_z : H^0(\mathcal{L}) \to H^0(\mathcal{L})^*$. That is, for s in $H^0(\mathcal{L})$, $\lambda_z(s) = \beta_z(s, \cdot)$ is a linear functional on $H^0(\mathcal{L})$. Then the map taking z to λ_z is a linear map $\lambda : T_0(P) \to \operatorname{Hom}_k(H^0(\mathcal{L}), H^0(\mathcal{L})^*)$. Since β is skew symmetric, $\operatorname{rank}(\lambda_z)$ is even, and since $\dim H^0(\mathcal{L})$ is even, $\dim(\ker(\lambda_z))$ is also even.

Lemma 2.5. *With notation as above let r be the maximal rank of all maps λ_z in the image of λ, and let $U \subset T_0(P)$ be the dense Zariski open set of those z such that $\operatorname{rank}(\lambda_z) = r$. If $\det(\beta) = 0$, then for all z in U, $\ker(\beta_z) = \ker(\lambda_z) \subset H^0(\mathcal{L})$ is a non-trivial isotropic subspace for β of positive even dimension $h^0(\mathcal{L}) - r \geqslant 2$.*

Proof. The dimension statements follow from the remarks above, so it suffices to prove $\ker(\beta_z)$ is an isotropic subspace. Fix z_0 in U, and denote the corresponding scalar pairing and linear map by β_0 and λ_0. Denote $\ker(\beta_0) = \ker(\lambda_0) = V \subset H^0(\mathcal{L})$ and $\operatorname{im}(\lambda_0) = Y \subset H^0(\mathcal{L})^*$. The following two Claims finish the proof. □

Claim 2.6. $V = Y^\perp$, under the identification $H^0(\mathcal{L})^{**} = H^0(\mathcal{L})$.

Proof. As usual, $Y^\perp = (\operatorname{im}(\lambda_0))^\perp = \ker(\lambda_0^*)$. Since β is skew symmetric the map λ_0 is skew symmetric in the sense that $\lambda_0^* = -\lambda_0$, so $Y^\perp = \ker(\lambda_0^*) = \ker(\lambda_0) = V$. □

Now to show V is isotropic for β, it suffices to prove the following.

Claim 2.7. $\beta(V, Y^\perp) = 0$.

Proof. Since $\lambda : T_0(P) \to \operatorname{Hom}_k(H^0(\mathcal{L}), H^0(\mathcal{L})^*)$ is a linear map, it equals its own derivative, and by the definition of U, λ maps U into the 'constant rank r locus' in $\operatorname{Hom}_k(H^0(\mathcal{L}), H^0(\mathcal{L})^*)$. Hence, for any z_0 in U, λ maps the tangent space to U at z_0, into the tangent space to the rank r locus. Since U is open and dense in $T_0(P)$, the tangent space to U at z_0 is all of $T_0(P)$. Since the tangent space at λ_0 to the rank r locus is the space of linear maps $T : H^0(\mathcal{L}) \to H^0(\mathcal{L})^*$ such that $T(\ker(\lambda_0)) \subset \operatorname{im}(\lambda_0)$, i.e. such that $T(V) \subset Y$, it follows for all z in $T_0(P)$ that $\lambda_z(V) \subset Y$. In particular, for all t in $Y^\perp$ and all s in V, $\lambda_z(s)(t) = 0$ for all z in $T_0(P)$. That is, $\beta_z(s,t) = 0$ (in k) for all s in V, all t in $Y^\perp$ and all z in $T_0(P)$. Thus $\beta(s,t) = 0$ (in $T_0^*(P)$) for all s in V and all t in $Y^\perp$, as claimed. □

Remark. By Lemma 2.5, the failure of RST provides many isotropic subspaces $\{\ker(\beta_z)\}$ of dimension $\geqslant 2$ for β, arising from (possibly many different) line bundles $\{\mathcal{M}_z\}$ on C as in Lemma 2.2. By Lemma 2.2, part ii, RST can fail only at an exceptional line bundle $\mathcal{L}$, thus giving an alternate proof of Theorem 2.1 of [SV01]. We want to apply Lemma 2.3, part iii, to combine non-trivial isotropic subspaces of $H^0(\mathcal{L})$ and deduce the existence of one very large isotropic subspace arising from one line bundle $\mathcal{M}$ on C with $2h^0(\mathcal{M}) > h^0(\mathcal{L})$.

3. The Segre inequality

For any pair $(\mathcal{M}, \mathcal{N})$ of effective line bundles, $\mathcal{M}$ in $\operatorname{Pic}(C)$, $\mathcal{N}$ in $\operatorname{Pic}(\tilde{C})$, such that $\mathcal{L} \cong \pi^*(\mathcal{M}) \otimes \mathcal{N}$, the bilinear map $H^0(\mathcal{M}) \times H^0(\mathcal{N}) \to H^0(\mathcal{L})$ taking $(\sigma, u) \mapsto \pi^*(\sigma) \cdot u$, induces a morphism on projective spaces $\gamma : \mathbb{P}H^0(\mathcal{M}) \times \mathbb{P}H^0(\mathcal{N}) \to \mathbb{P}H^0(\mathcal{L})$, since $\pi^*(\sigma) \cdot u \neq 0$, if $\sigma \neq 0$ and $u \neq 0$.

Lemma 3.1. *If $h^0(\mathcal{M}), h^0(\mathcal{N}) \geqslant 2$, and the morphism $\gamma : \mathbb{P}H^0(\mathcal{M}) \times \mathbb{P}H^0(\mathcal{N}) \to \mathbb{P}H^0(\mathcal{L})$ defined above is an injection, then $2h^0(\mathcal{M}) + 2h^0(\mathcal{N}) \leqslant h^0(\mathcal{L}) + 4$.*

Proof. Note that a map $\gamma : \mathbb{P}(A) \times \mathbb{P}(B) \to \mathbb{P}(C)$ induced by a bilinear map of vector spaces $A \times B \to C$, $(a,b) \mapsto a \cdot b$, is injective only if it embeds. That is, let $(\bar{x}, \bar{y})$ represent a non-zero tangent vector at $([v],[w])$ to $\mathbb{P}(A) \times \mathbb{P}(B)$, where (x,y) is in $A \times B$, x is determined modulo v and y is determined modulo w. If the derivative of γ takes $(\bar{x}, \bar{y})$ to $\overline{x \cdot w + v \cdot y} = \bar{0}$ modulo $v \cdot w$, i.e. if $x \cdot w + v \cdot y = a(v \cdot w)$, then $\gamma([v],[y - (aw/2)]) = \gamma([(av/2) - x],[w])$. That $(\bar{x}, \bar{y})$ is non-zero means x is not a multiple of v or y is not a multiple of w, hence γ is not injective if it does not embed.

Now the bilinear map $H^0(\mathcal{M}) \times H^0(\mathcal{N}) \to H^0(\mathcal{L})$ factors through the universal map $H^0(\mathcal{M}) \times H^0(\mathcal{N}) \to H^0(\mathcal{M}) \otimes H^0(\mathcal{N})$ followed by a linear map $\mu : H^0(\mathcal{M}) \otimes H^0(\mathcal{N}) \to H^0(\mathcal{L})$. The map γ on projective spaces thus factors via the 'Segre map' $\mathbb{P}H^0(\mathcal{M}) \times \mathbb{P}H^0(\mathcal{N}) \to \mathbb{P}(H^0(\mathcal{M}) \otimes H^0(\mathcal{N}))$ followed by the rational (not necessarily surjective) 'projection', $\mathbb{P}(H^0(\mathcal{M}) \otimes H^0(\mathcal{N})) \dashrightarrow \mathbb{P}H^0(\mathcal{L})$, with center $\mathbb{P}(V)$ where $V = \ker(\mu)$. Thus $\dim(\mathbb{P}(V)) \geqslant h^0(\mathcal{M}) \cdot h^0(\mathcal{N}) - h^0(\mathcal{L}) - 1$. (When $V = \{0\}$ put $\dim(\mathbb{P}(V)) = -1$.) If S is the 'Segre variety' which is the image of the Segre map in $\mathbb{P}(H^0(\mathcal{M}) \otimes H^0(\mathcal{N}))$, then since $\pi^*(\sigma) \cdot u \neq 0$ when neither of σ, u is zero, no point of S lies on $\mathbb{P}(V)$.

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Since γ is an embedding, the projection of S from $\mathbb{P}(V)$ is also an embedding, so $\mathbb{P}(V)$ cannot meet $\mathrm{Sec}(S)$ (the closure of the set of secants of S). Thus $\dim(\mathbb{P}(H^0(\mathcal{M}) \otimes H^0(\mathcal{N}))) > \dim(\mathbb{P}(V)) + \dim(\mathrm{Sec}(S)) \geqslant (h^0(\mathcal{M}) \cdot h^0(\mathcal{N}) - h^0(\mathcal{L}) - 1) + \dim(\mathrm{Sec}(S))$. Hence $h^0(\mathcal{M}) \cdot h^0(\mathcal{N}) - 1 > (h^0(\mathcal{M}) \cdot h^0(\mathcal{N}) - h^0(\mathcal{L}) - 1) + \dim(\mathrm{Sec}(S))$, so $h^0(\mathcal{L}) > \dim(\mathrm{Sec}(S))$.

Under the isomorphism $\mathbb{P}(H^0(\mathcal{M}) \otimes H^0(\mathcal{N})) \cong \mathbb{P}(\mathrm{Hom}(H^0(\mathcal{M})^*, H^0(\mathcal{N})))$, the points of S correspond to rank one homomorphisms, hence points of $\mathrm{Sec}(S)$ correspond to lines through the origin in the affine cone of homomorphisms $H^0(\mathcal{M})^* \to H^0(\mathcal{N})$ of rank $\leqslant 2$. This cone has dimension $2h^0(\mathcal{M}) + 2h^0(\mathcal{N}) - 4$ (cf. [Har92, Proposition 12.2, p. 151], hence $h^0(\mathcal{L}) > \dim(\mathrm{Sec}(S)) = 2h^0(\mathcal{M}) + 2h^0(\mathcal{N}) - 5$, i.e. $h^0(\mathcal{L}) + 4 \geqslant 2h^0(\mathcal{M}) + 2h^0(\mathcal{N})$. □

4. Proof that $\mathcal{L}$ admits a maximal (∗) pair if RST fails

Given $\mathcal{L}$ in Ξ, and β the pairing in § 2.1, item iii, for each $z \in T_0P$ again view (cf. Lemma 2.5) the scalar valued pairing β_z as a linear map $\lambda_z : H^0(\mathcal{L}) \to H^0(\mathcal{L})^*$, and the map $z \mapsto \lambda_z$ as a linear map $\lambda : T_0(P) \to \mathrm{Hom}_k(H^0(\mathcal{L}), H^0(\mathcal{L})^*)$. Recall RST fails at $\mathcal{L}$ if and only if for all z, $\det(\beta_z) = 0$, i.e. for every z, $\ker(\beta_z) = \ker(\lambda_z)$ has positive even dimension.

Definition 4.1. Let $c = \min\{\dim(\ker(\beta_z)),\ \text{for all } z \neq 0 \text{ in } T_0(P)\}$ be the minimum 'corank' of all the scalar pairings β_z. By the remarks above, c is even, and RST fails at $\mathcal{L}$ if and only if $c \geqslant 2$.

Assume that RST fails at $\mathcal{L}$ and let $U \subset T_0(P)$ be the dense Zariski open set of those z such that $\mathrm{corank}(\beta_z) = c$ is minimal, as in Definition 4.1. By Lemma 2.5, the kernel of every scalar pairing β_z for z in U is a β-isotropic subspace of $H^0(\mathcal{L})$. Thus by Lemma 2.2, part ii, for every z in U there exists a unique triple $(\mathcal{M}_z, \Lambda_z, B_z)$ such that $|\ker \beta_z| = \pi^*(|\Lambda_z|) + B_z$, where $\Lambda_z \subset H^0(C, \mathcal{M}_z)$, $\dim(\Lambda_z) = c \geqslant 2$, and $B_z \geqslant 0$ has no invariant part. Next we want to produce from the collection of exceptional line bundles $\{\mathcal{M}_z\}$, one distinguished line bundle $\mathcal{M}_0$. Intuitively the argument is just that the function $z \mapsto \mathcal{M}_z$, defined by $z \mapsto |\ker \beta_z| \mapsto (|\ker \beta_z| - B_z) \mapsto (\pi^*)^{-1}(|\ker \beta_z| - B_z) \mapsto$ the corresponding line bundle $\mathcal{M}_z$ on C, is a rational map from a rational variety to an abelian variety, hence constant. To approximate this intuition, we will restrict z to a set where $deg(B_z)$ is constant, then finesse the fact that it is π^* rather than its inverse which is a morphism.

Lemma 4.2. *With notation as above, on some dense open subset U_0 of U, the function $z \mapsto \mathcal{M}_z$ is constant from U_0 to* $\mathrm{Pic}(C)$.

Proof. For each z in T_0P, again view β_z as a linear map $\lambda_z : H^0(\mathcal{L}) \to H^0(\mathcal{L})^*$, where λ_z is linear in z. In some dense Zariski open subset $U_1 \subset U$, we may choose bases for all the subspaces $\ker(\lambda_z) \subset H^0(\tilde{C}, \mathcal{L})$ which vary regularly with z in U_1. If s_z is one of these basis vectors for $\ker(\lambda_z)$, the function $z \mapsto s_z$ defines a regular section s of the pullback of $\mathcal{L}$ to $U_1 \times \tilde{C}$, hence an effective divisor $\mathrm{div}(s)$ on $U_1 \times \tilde{C}$ whose restriction to each curve $\{z\} \times \tilde{C}$ is the divisor $\mathrm{div}(s_z)$ in the linear system $|\ker \lambda_z|$. Then $\mathcal{D} = \mathrm{div}(s)$ is a Cohen Macaulay subscheme of $U_1 \times \tilde{C}$ and the projection $\mathcal{D} \to U_1$ has all zero-dimensional fibers, hence the map $\mathcal{D} \to U_1$ is flat (by [Mat70, (20F), p. 151], and defines a morphism from U_1 to the Hilbert scheme $\tilde{C}^{(2g-2)}$.

Next we make a family of the base divisors of the linear systems $|\ker \lambda_z|$. If for all z in U_1, the set $s_{z,1}, \ldots, s_{z,c}$ is the basis for $\ker(\lambda_z)$ chosen as above, then on U_1 the corresponding sections $s_1, \ldots, s_c$ of the pullback of $\mathcal{L}$ to $U_1 \times \tilde{C}$ define divisors $\mathrm{div}(s_1), \ldots, \mathrm{div}(s_c)$ on $U_1 \times \tilde{C}$. We throw out of U_1 the projection of pairwise intersections of distinct irreducible components of the union of the supports of the divisors $\mathrm{div}(s_1), \ldots, \mathrm{div}(s_c)$. If $U_2 \subset U_1$ is the resulting smaller dense Zariski open subset of U, then $\gcd\{\mathrm{div}(s_1), \ldots, \mathrm{div}(s_c)\} = G$ is a divisor on $U_2 \times \tilde{C}$ whose restriction to $\{z\} \times \tilde{C}$ is the base divisor of $|\ker \lambda_z|$ for every z in U_2. Consider the union of components of the divisors $\mathrm{div}(s_i)$ and their conjugates under the involution induced by ι on $U_2 \times \tilde{C}$, and throw out the closed

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set in U_2 over which two of these distinct components have non-empty intersection. We obtain a smaller dense Zariski open subset $U_0 \subset U_2$, such that the invariant part $\mathrm{inv}(G)$ of the divisor G, restricts on each fiber $\{z\} \times \tilde{C}$ with z in U_0 to the invariant part of the base divisor of $|\ker \lambda_z|$. If $B = G - \mathrm{inv}(G)$, then for each z in U_0 the restriction B_z is the part of the base divisor of $|\ker \lambda_z|$ which is residual to the invariant part.

Then for each z in U_0 we have $|\ker \lambda_z| = \pi^*(|\Lambda_z|) + B_z$, where B_z has no invariant part, and $|\Lambda_z|$ is a uniquely determined linear series on C, as in Lemma 2.2, part ii. Since the divisors $\{B_z\}$ form a flat family over U_0, they have a common degree d and determine a morphism $U_0 \to \tilde{C}^{(d)}$. Since U_0 is irreducible and rational, the composition $U_0 \to \tilde{C}^{(d)} \to \mathrm{Pic}^d(\tilde{C})$ is constant. To see this, join any two points of U_0 by a line $\mathbb{A}^1$ in the affine space $T_0(P)$ and extend the morphism $U_0 \cap \mathbb{A}^1 \to \mathrm{Pic}^d(\tilde{C})$ to a morphism $\mathbb{P}^1 \to \mathrm{Pic}^d(\tilde{C})$. The map $\mathbb{P}^1 \to \mathrm{Pic}^d(\tilde{C})$ factors through $\mathrm{Alb}(\mathbb{P}^1) = \{0\}$ (cf. [Ser59]) and hence is constant. Thus for all z in U_0, the divisors B_z are linearly equivalent, hence the linear series $\pi^*(|\Lambda_z|)$ are all contained in the common complete series $\Gamma = |\mathcal{L}(-B_z)|$.

Now consider the inclusions $\cup(\{z\} \times \pi^*(|\Lambda_z|)) \subset U_0 \times \Gamma \subset U_0 \times \tilde{C}^{(2g-2-d)}$. Using the framing $s_1, \ldots, s_c$, the set $\cup(\{z\} \times |\ker \lambda_z|)$ in $U_0 \times \tilde{C}^{(2g-2)}$ is isomorphic to $U_0 \times \mathbb{P}^{c-1}$, and hence is connected. Then under the proper injective morphism $U_0 \times \tilde{C}^{(2g-2-d)} \to U_0 \times \tilde{C}^{(2g-2)}$ sending (z, D) to $(z, D + B_z)$, the set $\cup(\{z\} \times \pi^*(|\Lambda_z|))$ in $U_0 \times \tilde{C}^{(2g-2-d)}$, maps homeomorphically to $\cup(\{z\} \times |\ker \lambda_z|)$, and hence is also connected. Then the projection of $\cup(\{z\} \times \pi^*(|\Lambda_z|))$ into $\tilde{C}^{(2g-2-d)}$ is a connected subset $S = \cup \pi^*(|\Lambda_z|) \subset \Gamma \subset \tilde{C}^{(2g-2-d)}$ of the complete linear series $\Gamma = |\mathcal{L}(-B_z)|$. Since $\pi^* : C^{(g-1-d/2)} \to \tilde{C}^{(2g-2-d)}$ is injective, there is a unique set $R \subset C^{(g-1-d/2)}$ such that $\pi^*(R) = S$. The following Claim finishes the proof. □

Claim. *R is contained in a single complete linear series in $C^{(g-1-d/2)}$.*

Proof. Since $\pi^* : C^{(g-1-d/2)} \to \tilde{C}^{(2g-2-d)}$ is proper and injective, it is a homeomorphism onto its image so R is connected. If we consider the two maps Nm: $\tilde{C}^{(2g-2-d)} \to C^{(2g-2-d)}$ and 'multiplication by two' from $C^{(g-1-d/2)}$ to $C^{(2g-2-d)}$, then $\mathrm{Nm}(S) = \mathrm{Nm}(\pi^*(R)) = 2R$. Since $S \subset \Gamma = |\mathcal{L}(-B_z)|$ and Nm preserves linear equivalence, it follows that $2R$ belongs to a single linear series in $C^{(2g-2-d)}$. Thus R is contained in a finite disjoint union of linear series, hence in only one of them since R is connected. □

4.3 Notation

The constant value of the function $U_0 \to \mathrm{Pic}(C)$ in Lemma 4.2 is denoted by $\mathcal{M}_0$, so that $\mathcal{M}_z = \mathcal{M}_0$ for all z in $U_0 \subset U \subset T_0(P)$. Define $\mathcal{N}_0$ by setting $\mathcal{N}_0 = \mathcal{L} \otimes \pi^*(\mathcal{M}_0^{-1})$. Then for all z in U_0, $|\ker \beta_z| = \pi^*(|\Lambda_z|) + B_z$ where $\Lambda_z \subset H^0(C, \mathcal{M}_0)$, $\dim(\Lambda_z) = c \geqslant 2$, $B_z \geqslant 0$ has no invariant part and $B_z \in |\mathcal{N}_0|$.

Lemma 4.4. *If RST fails at $\mathcal{L}$ in Ξ, the pair $(\mathcal{M}_0, \mathcal{N}_0)$ defined in § 4.3 is a maximal $(*)$ pair for $\mathcal{L}$ as in Definitions 1.3 and 1.5, i.e. $\mathcal{L} \cong \pi^*(\mathcal{M}_0) \otimes \mathcal{N}_0$, $h^0(\mathcal{M}_0) \geqslant 2$, $|\mathcal{N}_0|$ contains a divisor having no invariant part, and*

$$(*) \quad 2h^0(\mathcal{M}_0) + h^0(\mathcal{N}_0) \geqslant h^0(\mathcal{L}) + 3.$$

Proof. By the discussion preceding Lemma 4.2, summarized in § 4.3, it suffices to prove the inequality $(*)$. By Lemma 2.3, for all z in U_0 and all $s \neq 0$ in $\ker(\beta_z)$, the maximal isotropic subspace containing s is $\pi^*(H^0(\mathcal{M}_0)) \cdot u_z$, where the divisor $\mathrm{div}(u_z) = B_z$. Thus all these maximal isotropic subspaces have dimension $h^0(\mathcal{M}_0) \geqslant 2$. Define the incidence variety $\mathcal{B}$ in $|\mathcal{L}| \times T_0(P)$, by $\mathcal{B} = \{([s], z) : [s] \in |\ker \beta_z|, z \in U_0\} \subset |\mathcal{L}| \times T_0(P)$, so $\mathcal{B}$ is a projective space bundle over U_0 with fiber $\cong \mathbb{P}^{c-1}$, where $c = \dim(\ker \beta_z) \geqslant 2$, hence $c - 1 \geqslant 1$. Thus $\mathcal{B}$ is irreducible. We compute the dimension of $\mathcal{B}$ in two ways using the projections $\pi_1 : \mathcal{B} \to |\mathcal{L}|$ and $\pi_2 : \mathcal{B} \to T_0(P)$. From π_2, adding

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the dimensions of the image and the fibers, $\dim(\mathcal{B}) = \dim(U_0) + \dim(\mathbb{P}(\ker \beta_z)) = p + c - 1$, where $p = \dim(T_0(P)) = g(C) - 1$.

Now consider $\pi_1 : \mathcal{B} \to |\mathcal{L}|$. If $[s]$ is in the image of π_1, the fiber $\pi_1^{-1}([s]) \cong \{z \in U_0 : s \in \ker(\beta_z)\}$ $= \{z \in U_0 : \beta(s, H^0(\mathcal{L}))(z) = 0\} = U_0 \cap (\beta(s, H^0(\mathcal{L})))^{\perp}$. To compute $\dim((\beta(s, H^0(\mathcal{L})))^{\perp})$, note that $\dim(\beta(s, H^0(\mathcal{L}))) = \dim(H^0(\mathcal{L})) - \dim(\{t : \beta(s,t) = 0\})$. By Lemma 2.3, part iii, the maximal isotropic subspace containing s equals $\{t : \beta(s,t) = 0\}$, so by the discussion above it must equal $\pi^*(H^0(\mathcal{M}_0)) \cdot u$ for some $u \neq 0$ in $H^0(\mathcal{N}_0)$. Thus $\dim(\beta(s, H^0(\mathcal{L}))) = h^0(\mathcal{L}) - h^0(\mathcal{M}_0)$, hence $\dim((\beta(s, H^0(\mathcal{L})))^{\perp}) = p - h^0(\mathcal{L}) + h^0(\mathcal{M}_0)$. Since $U_0 \subset T_0(P)$ is open and dense, $U_0 \cap (\beta(s, H^0(\mathcal{L})))^{\perp}$ (which is non-empty for s in $\pi_1(\mathcal{B})$) also has dimension $p - h^0(\mathcal{L}) + h^0(\mathcal{M}_0)$. Thus $\dim(\pi_1^{-1}([s]))$ $= p - h^0(\mathcal{L}) + h^0(\mathcal{M}_0)$, a constant independent of s in $\pi_1(\mathcal{B})$.

Since $\pi_1(\mathcal{B}) = \operatorname{image}(\pi_1 : \mathcal{B} \to |\mathcal{L}|)$ is contained in $\mathbb{P}(\pi^*(H^0(\mathcal{M}_0)) \cdot H^0(\mathcal{N}_0))$, $\dim(\pi_1(\mathcal{B})) \leqslant \dim(\mathbb{P}(\pi^*(H^0(\mathcal{M}_0)) \cdot H^0(\mathcal{N}_0))) \leqslant h^0(\mathcal{M}_0) + h^0(\mathcal{N}_0) - 2$. From the above $p + c - 1 = \dim(\mathcal{B})$ $= \dim(\pi_1(\mathcal{B})) + \dim(\text{fibers of } \pi_1 \text{ over } \pi_1(\mathcal{B})) \leqslant h^0(\mathcal{M}_0) + h^0(\mathcal{N}_0) - 2 + p - h^0(\mathcal{L}) + h^0(\mathcal{M}_0)$. Thus $3 \leqslant c + 1 \leqslant 2h^0(\mathcal{M}_0) + h^0(\mathcal{N}_0) - h^0(\mathcal{L})$, and $(*)$ holds. □

5. Proof the Shokurov condition is necessary for RST to fail

By Lemma 4.4, any $\mathcal{L}$ in Ξ at which RST fails, admits a 'maximal $(*)$ pair' $(\mathcal{M}, \mathcal{N})$. Next we show that such a pair satisfies $h^0(\mathcal{N}) = 1$, hence $(\mathcal{M}, \mathcal{N})$ is also a maximal Shokurov pair.

5.1 Notation

If p is a point of $\tilde{C}$ denote its conjugate point by $\iota(p) = p'$, and denote $\pi(p) = \pi(p') = \bar{p}$. If $\mathcal{L}$ is a line bundle in Ξ and p, q two points in $\tilde{C}$, set $\widehat{\mathcal{L}} = \mathcal{L}(p' - p + q' - q)$ and $\widehat{\mathcal{N}} = \mathcal{N}(p' - p + q' - q)$.

Lemma 5.2. *Assume $\mathcal{L}$ admits an exceptional pair $(\mathcal{M}, \mathcal{N})$ with $h^0(\mathcal{N}) \geqslant 3$, hence $h^0(\mathcal{L}) \geqslant 4$. Then there exist $p \neq q$ on $\tilde{C}$ such that, in the notation of § 5.1, $h^0(\widehat{\mathcal{L}}) = h^0(\mathcal{L}) - 2 \geqslant 2$ and $h^0(\widehat{\mathcal{N}}) = h^0(\mathcal{N}) - 2 \geqslant 1$. Then $\widehat{\mathcal{L}}$ is in Ξ and $(\mathcal{M}, \widehat{\mathcal{N}})$ is an exceptional pair for $\widehat{\mathcal{L}}$. It suffices to choose p so that p is not a base point either of $|\mathcal{L}|$ or of $|\mathcal{N}|$, and $\bar{p}$ is not a base point of $|\mathcal{M}|$, and to choose q so that $q \neq p, p'$ and q is not a base point either of $|\mathcal{L}(-p)|$ or of $|\mathcal{N}(-p)|$, and $\bar{q}$ is not a base point of $|\mathcal{M}|$.*

Proof. According to Mumford's parity result [Mum71], for any point p of $\tilde{C}$, $h^0(\mathcal{L}(p' - p)) = h^0(\mathcal{L}) \pm 1$. Assume p is not a base point of either $|\mathcal{L}|$ or $|\mathcal{N}|$ and $\bar{p} = \pi(p)$ is not a base point of $|\mathcal{M}|$. Then $h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 = h^0(\mathcal{L}(p' - p))$, since adding p' cannot increase the dimension by two. Then choose $q \neq p, p'$, with q not a base point of either $|\mathcal{L}(-p)|$ or $|\mathcal{N}(-p)|$, and $\bar{q}$ not in the base divisor of $|\mathcal{M}|$. Then, since $q \neq p'$, q is also not a base point of $|\mathcal{L}(-p + p')|$, so we have $h^0(\mathcal{L}(-p + p' - q)) = h^0(\mathcal{L}(-p + p')) - 1 = h^0(\mathcal{L}) - 2$. Then Mumford's principle applied to $h^0(\mathcal{L}(-p + p'))$ implies $h^0(\widehat{\mathcal{L}}) = h^0(\mathcal{L}(-p + p' - q + q')) = h^0(\mathcal{L}) - 2 = h^0(\mathcal{L}(-p - q))$. In particular, we have $h^0(\mathcal{L}) = h^0(\mathcal{L}(p' + q'))$, since $h^0(\mathcal{L}(p' + q')) > h^0(\mathcal{L})$ would imply that $h^0(\mathcal{L}(p' + q' - p - q))$ $> h^0(\mathcal{L}) - 2$. Hence, $|\mathcal{L}(p' + q')| = |\mathcal{L}| + p' + q'$.

Next we show $h^0(\widehat{\mathcal{N}}) = h^0(\mathcal{N}) - 2$. Since Mumford's principle does not apply directly to $\mathcal{N}$, we will bootstrap from the result for $\mathcal{L}$.

Claim. $h^0(\mathcal{N}(p' + q')) = h^0(\mathcal{N}(p')) = h^0(\mathcal{N}(q')) = h^0(\mathcal{N})$.

Proof. It suffices to show $h^0(\mathcal{N}(p' + q')) = h^0(\mathcal{N})$. Suppose not, i.e. $h^0(\mathcal{N}(p' + q')) > h^0(\mathcal{N})$ so that $p' + q'$ is not in the base divisor of $|\mathcal{N}(p' + q')|$. Since $p' \neq q'$, either p' or q' is not in the base divisor. If say p' is not, then there is a divisor F in $|\mathcal{N}(p' + q')|$ such that F does not contain p'. Since $\bar{p}$ is not in the base divisor of $|\mathcal{M}|$, we may choose a divisor D in $|\mathcal{M}|$ with $\pi^*(D)$ not containing p'. Then the divisor $\pi^*(D) + F$ belongs to $|\pi^*(\mathcal{M}) \otimes \mathcal{N}(p' + q')| = |\mathcal{L}(p' + q')| = |\mathcal{L}| + p' + q'$, but does not contain p', a contradiction. If q' is not a base point of $|\mathcal{N}(p'+q')|$, argue the same way, using the assumption that $\bar{q}$ is not a base point of $|\mathcal{M}|$. □

Now since q is not a base point of $|\mathcal{N}(-p)|$, and $q \neq p', q'$, then q is also not a base point of $|\mathcal{N}(-p+p'+q')|$. Hence, $h^0(\mathcal{N}(-p+p'+q'-q)) = h^0(\mathcal{N}(-p+p'+q')) - 1$. Then $p \neq p', q'$, and p is not a base point of $|\mathcal{N}|$ implies that p is not a base point of $|\mathcal{N}(p'+q')|$, so $h^0(\mathcal{N}(-p+p'+q')) = h^0(\mathcal{N}(p'+q')) - 1$. Hence, $h^0(\widehat{\mathcal{N}}) = h^0(\mathcal{N}(-p+p'-q+q')) = h^0(\mathcal{N}(-p+p'+q')) - 1 = h^0(\mathcal{N}(p'+q')) - 2 = h^0(\mathcal{N}) - 2$, by the claim above. □

Corollary 5.3. *If $\mathcal{L}$ in Ξ admits a maximal $(*)$ pair $(\mathcal{M}, \mathcal{N})$ with $h^0(\mathcal{N}) \geqslant 3$, then there exist distinct points p, q such that $\widehat{\mathcal{L}} = \mathcal{L}(p'-p+q'-q)$ admits a maximal $(*)$ pair $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$, with $h^0(\tilde{\mathcal{N}}) = h^0(\mathcal{N}) - 2 \geqslant 1$, and $h^0(\widehat{\mathcal{L}}) = h^0(\mathcal{L}) - 2 \geqslant 2$. In particular, $\widehat{\mathcal{L}}$ is in Ξ.*

Proof. Since by assumption $|\mathcal{N}|$ has projective dimension $\geqslant 2$ and contains a divisor with no invariant part, the dense open subset of such divisors is infinite, hence some of them contain a point p satisfying the conditions in the last sentence of Lemma 5.2. Fix such a point p. Then in the hyperplane of divisors in $|\mathcal{N}|$ which contain p, there is an infinite open set of divisors with no invariant part and also containing some point q satisfying the conditions of Lemma 5.2. Then the triple $(\mathcal{M}, \widehat{\mathcal{N}}, \widehat{\mathcal{L}})$ satisfies the conclusions of Lemma 5.2, i.e. $h^0(\widehat{\mathcal{N}}) = h^0(\mathcal{N}) - 2, h^0(\widehat{\mathcal{L}}) = h^0(\mathcal{L}) - 2$. By hypothesis $(\mathcal{M}, \mathcal{N})$ is a maximal $(*)$ pair for $\mathcal{L}$, so inequality $(*)$ holds: $2h^0(\mathcal{M}) + h^0(\mathcal{N}) \geqslant h^0(\mathcal{L}) + 3$. Hence, $(*)$ also holds for the triple $(\mathcal{M}, \widehat{\mathcal{N}}, \widehat{\mathcal{L}})$, so $(\mathcal{M}, \widehat{\mathcal{N}})$ is a $(*)$ pair for $\widehat{\mathcal{L}}$, but not necessarily maximal. We examine that next.

By the choice of the points p and q, $|\mathcal{N}(-p-q)|$ contains a divisor with no invariant part. If B is the base divisor of $|\mathcal{N}(-p-q)|$, and B contains neither p nor q, then in the dense open set of divisors of $|\mathcal{N}(-p-q)|$ with no invariant part, there is one, say D, containing neither p nor q. Then $D + p' + q'$ is a divisor in $|\widehat{\mathcal{N}}|$ with no invariant part, hence $(\mathcal{M}, \widehat{\mathcal{N}})$ is maximal and we are done. However, since $h^0(\mathcal{N}(-p-q)) = h^0(\mathcal{N}) - 2 = h^0(\mathcal{N}(-p-q+p'+q'))$, then $B + p' + q'$ is the base divisor of $|\mathcal{N}(-p-q+p'+q')| = |\widehat{\mathcal{N}}|$, so if B contains p or q, then every divisor in $|\widehat{\mathcal{N}}|$ has an invariant part. Then we modify the pair $(\mathcal{M}, \widehat{\mathcal{N}})$ as follows.

If B contains p but not q, there is a D in $|\mathcal{N}(-p-q)|$ with no invariant part, and containing p but not q. Then the invariant part of $D + p' + q'$ is $p + p'$ which lies in the base divisor of $|\widehat{\mathcal{N}}|$. We transfer this invariant part of the base locus down to $\mathcal{M}$, replacing $\mathcal{M}$ by $\tilde{\mathcal{M}} = \mathcal{M}(\bar{p})$ (where $\bar{p} = \pi(p) = \pi(p')$), and replacing $\widehat{\mathcal{N}} = \mathcal{N}(p'-p+q'-q)$ by $\tilde{\mathcal{N}} = \widehat{\mathcal{N}}(-p'-p) = \mathcal{N}(-2p-q+q')$. Then $h^0(\tilde{\mathcal{M}}) \geqslant h^0(\mathcal{M})$ and since $p + p'$ is in the base locus of $|\widehat{\mathcal{N}}|$, we have $h^0(\tilde{\mathcal{N}}) = h^0(\widehat{\mathcal{N}}) = h^0(\mathcal{N}) - 2$. Since by hypothesis $(\mathcal{M}, \mathcal{N})$ is a maximal $(*)$ pair for $\mathcal{L}$, then $2h^0(\tilde{\mathcal{M}}) \geqslant 2h^0(\mathcal{M}) \geqslant h^0(\mathcal{L}) + 3 - h^0(\mathcal{N}) = h^0(\widehat{\mathcal{L}}) + 3 - h^0(\tilde{\mathcal{N}})$, hence $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ is a maximal $(*)$ pair for $\widehat{\mathcal{L}}$, with $h^0(\tilde{\mathcal{N}}) = h^0(\mathcal{N}) - 2$, and $h^0(\widehat{\mathcal{L}}) = h^0(\mathcal{L}) - 2$. Similarly, if B contains q but not p, replace $\widehat{\mathcal{N}}$ by $\tilde{\mathcal{N}} = \widehat{\mathcal{N}}(-q'-q)$, and $\mathcal{M}$ by $\tilde{\mathcal{M}} = \mathcal{M}(\bar{q})$, and if B contains both p and q, replace $\widehat{\mathcal{N}}$ by $\tilde{\mathcal{N}} = \widehat{\mathcal{N}}(-q'-q-p'-p)$, and $\mathcal{M}$ by $\tilde{\mathcal{M}} = \mathcal{M}(\bar{p} + \bar{q})$. □

Lemma 5.4. *If $\mathcal{L}$ admits a Shokurov pair, $\mathcal{L}$ admits a unique maximal Shokurov pair $(\mathcal{M}, \mathcal{N})$, and then necessarily $h^0(\mathcal{N}) = 1$.*

Proof. Existence follows from the remark at the end of § 1. For uniqueness, let $\mathcal{L}$ be a point of Ξ, $h^0(\mathcal{L}) = 2k \geqslant 2$, and let $(\mathcal{M}, \mathcal{N})$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ be two maximal Shokurov pairs for $\mathcal{L}$, with $\mathcal{N} \cong \mathcal{O}(B)$, $\tilde{\mathcal{N}} \cong \mathcal{O}(\tilde{B})$, and B, $\tilde{B}$ effective divisors with no invariant part. Then in $|\mathcal{L}| \cong \mathbb{P}^{2k-1}$, the subspaces $\pi^*(|\mathcal{M}|) + B$ and $\pi^*(|\tilde{\mathcal{M}}|) + \tilde{B}$ both have projective dimension $\geqslant k$, hence they meet. The isotropic subspaces $\pi^*(H^0(\mathcal{M})) \cdot u$ and $\pi^*(H^0(\tilde{\mathcal{M}})) \cdot \tilde{u}$ thus contain a common non-zero section, where u and $\tilde{u}$ are equations for B and $\tilde{B}$. By Lemma 2.3, part iii, their span is isotropic, but by Lemma 2.3, part i, they are both maximal, hence they are equal. By the uniqueness statement of Lemma 2.2, part ii, $\mathcal{M} \cong \tilde{\mathcal{M}}$ and $B = \tilde{B}$, so $\mathcal{N} \cong \mathcal{O}(B) = \mathcal{O}(\tilde{B}) \cong \tilde{\mathcal{N}}$. Thus in any maximal Shokurov pair $(\mathcal{M}, \mathcal{N})$, $\mathcal{N} \cong \mathcal{O}(B)$ for a *unique* B with no invariant part, hence $h^0(\mathcal{N}) = 1$. ☐

Lemma 5.5. *If $\mathcal{L}$ in Ξ admits a maximal $(*)$ pair $(\mathcal{M}, \mathcal{N})$, then $h^0(\mathcal{N}) = 1$, hence $(\mathcal{M}, \mathcal{N})$ is a (unique) maximal Shokurov pair for $\mathcal{L}$.*

Proof. By hypothesis $h^0(\mathcal{N}) \geqslant 1$. Let $h^0(\mathcal{L}) = 2k \geqslant 2$. If $h^0(\mathcal{N}) = 2$, then $(*)$ $2h^0(\mathcal{M}) + h^0(\mathcal{N}) \geqslant h^0(\mathcal{L}) + 3$, implies $2h^0(\mathcal{M}) \geqslant h^0(\mathcal{L}) + 1$, and $2h^0(\mathcal{M}) > h^0(\mathcal{L})$, hence $(\mathcal{M}, \mathcal{N})$ is a maximal Shokurov pair for $\mathcal{L}$. By Lemma 5.4, then $h^0(\mathcal{N}) = 1$, a contradiction. If $h^0(\mathcal{N}) = 3$, $(*)$ implies $2h^0(\mathcal{M}) \geqslant h^0(\mathcal{L})$, hence $2h^0(\mathcal{M}) + 2h^0(\mathcal{N}) \geqslant h^0(\mathcal{L}) + 6$. By Lemma 3.1, the natural product map $\pi^*(|\mathcal{M}|) \times |\mathcal{N}| \to |\mathcal{L}|$, is not an injection. Since it restricts to an injection on each space of the form $\pi^*(|\mathcal{M}|) \times \{B\}$ and $h^0(\mathcal{M}) \geqslant k = (1/2)h^0(\mathcal{L})$, there are distinct divisors $B_1 \neq B_2$ in $|\mathcal{N}|$ such that the two spaces $\pi^*(|\mathcal{M}|) + B_1$ and $\pi^*(|\mathcal{M}|) + B_2$ are distinct subspaces of projective dimension $\geqslant k - 1$ which meet in $|\mathcal{L}| \cong \mathbb{P}^{2k-1}$. Then by Lemma 2.3, part iii, the corresponding distinct isotropic subspaces $\pi^*(H^0(\mathcal{M})) \cdot u_1$ and $\pi^*(H^0(\mathcal{M})) \cdot u_2$ span a strictly larger isotropic subspace W of dimension $> h^0(\mathcal{M})$. Since $\dim(W) \geqslant k+1$, $\mathbb{P}(W)$ has projective dimension $\geqslant k$ in $|\mathcal{L}|$ and meets every subspace of the form $\pi^*(|\mathcal{M}|) + B$ for B in $|\mathcal{N}|$. Then no isotropic subspace of the form $\pi^*(H^0(\mathcal{M})) \cdot u$ with $u \neq 0$ in $H^0(\mathcal{N})$ is maximal. That is, these spaces all meet non-trivially the isotropic subspace W, so Lemma 2.3, part iii, yields an isotropic subspace V containing $\pi^*(H^0(\mathcal{M})) \cdot u$, with $\dim(V) \geqslant \dim(W) > h^0(\mathcal{M}) = \dim(\pi^*(H^0(\mathcal{M})) \cdot u)$. Since there exists B in $|\mathcal{N}|$ with no invariant part, the isotropic subspace $\pi^*(H^0(\mathcal{M})) \cdot u$ is maximal by Lemma 2.3, part ii, a contradiction.

Thus for all triples $(\mathcal{M}, \mathcal{N}, \mathcal{L})$ such that $(\mathcal{M}, \mathcal{N})$ is a maximal $(*)$ pair for $\mathcal{L}$, we know either $h^0(\mathcal{N}) = 1$, or $h^0(\mathcal{N}) \geqslant 4$. If there exists such a triple $(\mathcal{M}, \mathcal{N}, \mathcal{L})$ with $h^0(\mathcal{N}) \geqslant 4$, choose one with $h^0(\mathcal{N}) > 1$ and minimal. Then we find a triple $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \widehat{\mathcal{L}})$ as in Corollary 5.3, with $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ a maximal $(*)$ pair for $\widehat{\mathcal{L}}$, and $h^0(\mathcal{N}) > h^0(\tilde{\mathcal{N}}) = h^0(\mathcal{N}) - 2 > 1$, a contradiction. ☐

Corollary 5.6. *If Riemann's singularity theorem fails at $\mathcal{L}$ in Ξ, then $\mathcal{L}$ admits a unique maximal Shokurov pair $(\mathcal{M}, \mathcal{N})$; for this pair $h^0(\mathcal{N}) = 1$.*

This is shown by Lemmas 4.4, 5.5 and 5.4.

Proof of Theorem 0.1. Since [Mum74, p. 342] an equation $\tilde{\vartheta}$ for $\tilde{\Theta}$ restricts on $P \subset \mathrm{Pic}^{2g-2}(\tilde{C})$ to the square of an equation ξ for Ξ, if $\mathcal{L}$ is on Ξ, then by the classical RST on $\tilde{\Theta}$, we would have $\mathrm{mult}_{\mathcal{L}}(\Xi) = (1/2)\,\mathrm{mult}_{\mathcal{L}}(\tilde{\Theta}) = (1/2)h^0(\tilde{C}, \mathcal{L})$ if and only if the leading term of a Taylor series for $\tilde{\vartheta}$ restricts to the square of the leading term of ξ. Since [Mum74, p. 343] the restriction to $T_{\mathcal{L}}(P)$ of this leading term for $\tilde{\vartheta}$ equals $\det(\beta)$, this holds if and only if $\det(\beta)$ is not identically zero on $T_{\mathcal{L}}(P)$. Thus, parts i and iv are equivalent in Theorem 0.1. By Lemma 2.4 (and its proof), part ii implies part v, which implies part iv. Since part i implies part iii by Corollary 5.6, and part iii implies part ii tautologically, we are done. ☐

6. Further results and open questions

The question remains: what is the multiplicity of Ξ at a point $\mathcal{L}$ where RST fails? With reference to Lemma 2.4, if $(\mathcal{M}, \mathcal{N})$ is an exceptional pair for $\mathcal{L}$, then in fact $\mathrm{mult}_{\mathcal{L}}(\Xi) \geqslant h^0(C, \mathcal{M})$. Since $\mathrm{mult}_{\mathcal{L}}(\Xi) \geqslant (1/2)h^0(\tilde{C}, \mathcal{L})$ for all $\mathcal{L}$, thus $\mathrm{mult}_{\mathcal{L}}(\Xi) \geqslant \max\{h^0(C, \mathcal{M}), (1/2)h^0(\tilde{C}, \mathcal{L})\}$ always holds. It is natural to ask if $\mathrm{mult}_{\mathcal{L}}(\Xi) = h^0(C, \mathcal{M})$ when $(\mathcal{M}, \mathcal{N})$ is a maximal Shokurov pair for $\mathcal{L}$, but we do not even know if $\mathrm{mult}_{\mathcal{L}}(\Xi) \leqslant h^0(\tilde{C}, \mathcal{L})$ in general. For example, we do not know whether a singular point $\mathcal{L}$ on Ξ with $h^0(\mathcal{L}) = 2$ is a double point, but this appears to hold if $\mathcal{L} \cong \pi^*(\mathcal{M})(B)$, $|\mathcal{M}|$ is a base point free pencil and $B \geqslant 0$ has no invariant part. The best upper bound we know for multiplicities of points on Ξ for $\dim(P) \geqslant 3$, is $\mathrm{mult}_{\mathcal{L}}(\Xi) \leqslant g(C) - 2 = \dim(P) - 1$, since by [SV96] higher multiplicities imply that (P, Ξ) is a polarized product of elliptic curves, whereas

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(P, Ξ) has at most two factors by [Mum74, Theorem, p. 344]. Another open problem is to understand the structure of the tangent cones, e.g. the rank of the quadric tangent cone at a double point.

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