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THE BRAIDED MONOIDAL STRUCTURES ON THE CATEGORY OF VECTOR SPACES GRADED BY THE KLEIN GROUP

D. BULACU[1,2], S. CAENEPEEL[2] AND B. TORRECILLAS[3]

[1] *Faculty of Mathematics and Informatics, University of Bucharest, Str. Academiei 14, 010014 Bucharest 1, Romania* (daniel.bulacu@fmi.unibuc.ro)
[2] *Faculty of Engineering, Vrije Universiteit Brussel, 1050 Brussels, Belgium* (scaenepe@vub.ac.be)
[3] *Departamento Álgebra y Análisis Matemático, Universidad de Almería, Ctra. Sacramento S/N, La Cañada de San Urbano, 04120 Almería, Spain* (btorreci@ual.es)

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Abstract Let k be a field, let $k^* = k \setminus \{0\}$ and let C_2 be a cyclic group of order 2. We compute all of the braided monoidal structures on the category of k-vector spaces graded by the Klein group $C_2 \times C_2$. For the monoidal structures we compute the explicit form of the 3-cocycles on $C_2 \times C_2$ with coefficients in k^*, while, for the braided monoidal structures, we compute the explicit form of the abelian 3-cocycles on $C_2 \times C_2$ with coefficients in k^*. In particular, this will allow us to produce examples of quasi-Hopf algebras and weak braided Hopf algebras with underlying vector space $k[C_2 \times C_2]$.

Keywords: braided monoidal category; Klein group; group cohomology; quasi-Hopf algebra

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1. Introduction

For a field k and a group G it is well known that the category of G-graded vector spaces Vect^G is a monoidal category. Now consider monoidal structures on Vect^G with the same tensor product, unit object k, and the natural unit constraints, but with different associativity constraints. These monoidal structures are in bijective correspondence with the normalized 3-cocycles on G with coefficients in $k^* = k \setminus \{0\}$. In the case where G is abelian, the braided monoidal structures on Vect^G are in bijective correspondence with *abelian* 3-cocycles. These are normalized 3-cocycles together with a so-called R-matrix. Isomorphism classes of braided monoidal structures are then classified by the cohomology group $H^3_{\mathrm{ab}}(G, k^*)$, which is isomorphic to the group of quadratic forms $\mathrm{QF}(G, k^*)$ by a result of Eilenberg and Mac Lane [**10**–**13**].

Associative algebras in Vect^G with one of these monoidal structures are usually not associative in the usual sense. Notable examples include Cayley–Dickson algebras and

Clifford algebras: in [**1**, **3**], Albuquerque and Majid show that they are associative algebras in a suitable symmetric monoidal category of graded vector spaces. Other examples are given by superalgebras. These are algebras in Vect^{C_2} or, more generally, associative algebras in the category of anyonic vector spaces Vect^{C_n} with a suitable monoidal structure.

In the case where G is a cyclic group, the classification of braided monoidal structures on Vect^G is presented in [**15**, **16**]. The monoidal structures on $\mathrm{Vect}^{\mathbb{Z}}$ are all trivial, and the braided monoidal structures are given by the R-matrices $\mathcal{R}_\alpha\colon \mathbb{Z}\times\mathbb{Z}\to k^*$, $\mathcal{R}_\alpha(x,y)=\alpha^{xy}$, with $\alpha\in k^*$. For a finite cyclic group, we have $H^3(\mathbb{Z}_n,k^*)\cong\mu_n(k)$, the group of nth roots of 1. The cohomology class corresponding to $q\in\mu_n(k)$ is represented by the normalized 3-cocycle

$$\phi_q(x,y,z)=\begin{cases}1 & \text{if } y+z<n,\\ q^x & \text{if } y+z\geqslant n\end{cases}\qquad \text{for all } x,y,z\in\{0,1,\dots,n-1\}. \tag{1.1}$$

The braidings on $\mathrm{Vect}^{\mathbb{Z}_n}$ are represented by abelian cocycles $(\phi_{\nu^n},\mathcal{R}_\nu)$ with $\nu\in k^*$ such that $\nu^{n^2}=\nu^{2n}=1$, where $\mathcal{R}(x,y)=\nu^{xy}$ for all $x,y\in\{0,\dots,n-1\}$. Note that the case $G=\mathbb{Z}_3$ has also been handled in [**2**, Propositions 6 and 7].

We continue the classification of braided monoidal structures on Vect^G. As we have explained, the cyclic case has been covered completely. In what follows, we complete the classification in the easiest remaining case, that is, the case where G is the Klein group, the product $C_2\times C_2$ of two cyclic groups of order 2. Using techniques from homological algebras, we can describe $H^3(C_2\times C_2,k^*)$ (see Proposition 2.3). However, for explicit monoidal structures we need explicit formulae for the cocycles. In § 3 we reduce this problem to the computation of so-called happy 3-cocycles and then, after some computations, we find out the explicit form of these elements (see Theorem 3.5). It turns out that there are three types of normalized happy 3-cocycles, which are denoted by ϕ_X, h_a and g_b, respectively ($X\subseteq(C_2\times C_2)\setminus\{e\}$ and $a,b\in k^*$).

The abelian cocycles are computed in § 4. Using the Eilenberg–Mac Lane theorem we can compute $H^3_{\mathrm{ab}}(C_2\times C_2,k^*)$ but, again, we want to compute the explicit formulae for the R-matrices. When k does not contain a primitive fourth root of unit, there are eight non-isomorphic braidings on $\mathrm{Vect}^{C_2\times C_2}$, all of them having the trivial cocycle as an underlying 3-cocycle (see Proposition 4.3). If k has a primitive fourth root of unity i, then we have 24 additional non-isomorphic braidings with underlying cocycles ϕ_X, with $|X|=2$ (see Proposition 4.4). In this case, $H^3_{\mathrm{ab}}(C_2\times C_2,k^*)\cong C_4\times C_4\times C_2$. In both cases, there are only four non-isomorphic symmetric monoidal structures.

The importance of having an explicit formula for the cocycles and R-matrices is illustrated in § 5, where we construct explicit examples of quasi-Hopf algebras and weak braided Hopf algebras. Quasi-Hopf algebras are obtained by explicitly determining some Harrison 3-cocycles corresponding to some 3-cocycles on C_n and $C_2\times C_2$, while the weak braided Hopf algebras are built on $k[C_n]$ (respectively, $k[C_2\times C_2]$) with the help of a coboundary abelian 3-cocycle on C_n (respectively, on $C_2\times C_2$). Note that the latter are commutative and cocommutative weak braided Hopf algebras in Vect^{C_n} (or, respectively,

$\mathrm{Vect}^{C_2\times C_2}$), which are symmetric monoidal categories with braided monoidal structures defined by a so-called 2-cochain on C_n (respectively, on $C_2\times C_2$).

2. Preliminary results

2.1. Braided monoidal categories

A monoidal category is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, called the tensor product, an object $\underline{1}\in\mathcal{C}$ called the unit object, and natural isomorphisms $a\colon \otimes\circ(\otimes\times\mathrm{Id})\to\otimes\circ(\mathrm{Id}\times\otimes)$ (the associativity constraint), $l\colon \otimes\circ(\underline{1}\times\mathrm{Id})\to\mathrm{Id}$ (the left unit constraint) and $r\colon \otimes\circ(\mathrm{Id}\times\underline{1})\to\mathrm{Id}$ (the right unit constraint). In addition, a has to satisfy the pentagon axiom, and l and r have to satisfy the triangle axiom. We refer the interested reader to [**17**, **19**] for a detailed discussion. In what follows, for any object $X\in\mathcal{C}$ we will identify $\underline{1}\otimes X\cong X\cong X\otimes\underline{1}$ using l_X and r_X.

The switch functor $\tau\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}$ is defined as $\tau(X,Y)=(Y,X)$. A braiding on a monoidal category $\mathcal{C}$ is a natural isomorphism $c\colon \otimes\to\otimes\circ\tau$, satisfying certain axioms (see, for example, [**17**, **19**]). If $c_{X,Y}=c_{Y,X}^{-1}$ for all $X,Y\in\mathcal{C}$, then we call $\mathcal{C}$ a symmetric monoidal category.

We are mainly interested in the case where $\mathcal{C}=\mathrm{Vect}^G$, the category of vector spaces graded by a group G. We will write G multiplicatively, and denote by e the unit element of G. Recall that a G-graded vector space is a vector space V together with a direct sum decomposition $V=\bigoplus_{g\in G}V_g$. An element $v\in V_g$ is called homogeneous of degree g, and we write $|v|=g\in G$. For two G-graded vector spaces V and W, a k-linear map $f\colon V\to W$ is said to preserve the grading if $f(V_g)\subseteq W_g$ for all $g\in G$. Vect^G is the category of G-graded vector spaces and grade-preserving k-linear maps.

2.2. Monoidal structures on Vect^G

For two G-graded vector spaces V and W, $V\otimes W$ is again a G-graded vector space, with $(V\otimes W)_g := \bigoplus_{\sigma\tau=g}V_\sigma\otimes W_\tau$. If the k-linear maps $f\colon V\to V'$ and $g\colon W\to W'$ are grade preserving, then $f\otimes g$ is also grade preserving, hence we have a functor $\otimes\colon \mathrm{Vect}^G\times\mathrm{Vect}^G\to\mathrm{Vect}^G$. k is a G-graded vector space, with $k_e=k$ and $k_g=0$ for all $G\ni g\neq e$. The problem is now to describe monoidal structures on Vect^G, with $\otimes$ as tensor product and k as the unit object. It is known that these correspond to 3-cocycles ϕ on G with coefficients in k^*. To solve this problem, we need to give the possible associativity and unit constraints.

To this end, let us first recall the definition of group cohomology. Let $K^n(G,k^*)$ be the set of maps from G^n to k^*. $K^n(G,k^*)$ is a group under pointwise multiplication. We have maps $\Delta_n\colon K^n(G,k^*)\to K^{n+1}(G,k^*)$. Δ_2 and Δ_3 are given by the formulae

$$\begin{aligned}\Delta_2(g)(x,y,z)&=g(y,z)g(xy,z)^{-1}g(x,yz)g(x,y)^{-1},\\ \Delta_3(f)(x,y,z,t)&=f(y,z,t)f(xy,z,t)^{-1}f(x,yz,t)f(x,y,zt)^{-1}f(x,y,z).\end{aligned}$$

It is well known that

$$B^n(G,k^*)=\operatorname{Im}\Delta_{n-1}\subseteq Z^n(G,k^*)=\operatorname{Ker}(\Delta_n).$$

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The nth cohomology group is defined as $H^n(G,k^*) = Z^n(G,k^*)/B^n(G,k^*)$, and two elements of $H^n(G,k^*)$ are called cohomologous if they lie in the same equivalence class. The elements of $Z^3(G,k^*)$ are called 3-cocycles, and the elements of $B^3(G,k^*)$ are called 3-coboundaries. $\phi \in K^3(G,k^*)$ is a 3-cocycle if and only if

$$\phi(y,z,t)\phi(x,yz,t)\phi(x,y,z) = \phi(xy,z,t)\phi(x,y,zt) \quad \text{for all } x,y,z \in G. \tag{2.1}$$

A 3-cocycle ϕ is called *normalized* if $\phi(x,e,z) = 1$ for all $x,z \in G$.

Lemma 2.1. *If ϕ is a normalized 3-cocycle, then $\phi(e,y,z) = \phi(x,y,e) = 1$ for all $x,y,z \in G$. A coboundary $\Delta_2(\psi)$ is normalized if and only if $\psi(e,x) = \psi(z,e)$ for all $x,z \in G$. Then ψ is called a normalized 2-cochain on G.*

Proof. Taking $z = e$ in (2.1), we find that $\phi(x,y,e) = 1$. Taking $y = e$, we find that $\phi(e,z,t) = 1$. The proof of the second statement is straightforward. □

Monoidal structures on Vect^G are in bijective correspondence to the elements of $Z^3(G,k^*)$. Given a 3-cocycle ϕ, the corresponding associativity constraint is given by the formula

$$a_{V,W,Z}((v \otimes w) \otimes z) = \phi(|v|,|w|,|z|)v \otimes (w \otimes z)$$

for G-graded vector spaces V, W and Z, and $v \in V$, $w \in W$ and $z \in Z$ homogeneous. The unit constraints are given by the formulae

$$r_V(v \otimes 1) = \phi(|v|,e,e)v, \qquad l_V(1 \otimes v) = \phi(e,e,|v|)^{-1}v.$$

If ϕ is normalized, then the unit constraints are trivial. Two monoidal structures on Vect^G give isomorphic monoidal categories if and only if their corresponding 3-cocycles are cohomologous. In order to solve our problem, it therefore suffices to compute the 3-cocycles that represent the elements of $H^3(G,k^*)$. Actually, we can restrict attention to normalized cocycles.

Lemma 2.2. *Every 3-cocycle ϕ is cohomologous to a normalized 3-cocycle.*

Proof. Take $y = z = e$ in (2.1). Then we find $\phi(x,e,t) = \phi(e,e,t)\phi(x,e,e)$. In particular, it follows that $\phi(e,e,e) = 1$ (take $x = t = e$). Then consider the map $g\colon G \times G \to k^*$, $g(x,y) = \phi(e,e,y)^{-1}\phi(x,e,e)$, and compute that

$$\begin{aligned}\Delta_2(g)(x,e,y) &= g(e,y)g(x,y)^{-1}g(x,y)g(x,e)^{-1}\\ &= \phi(e,e,y)^{-1}\phi(e,e,e)\phi(e,e,e)\phi(x,e,e)^{-1}\\ &= \phi(x,e,y)^{-1}.\end{aligned}$$

It then follows that $\phi\Delta_2(g)$ is normalized. □

Let $B^3_n(G,k^*)$ and $Z^3_n(G,k^*)$ be the subgroups of $B^3(G,k^*)$ and $Z^3(G,k^*)$ consisting of normalized elements. We then have a well-defined group morphism

$$Z^3_n(G,k^*)/B^3_n(G,k^*) \ni \hat{\phi} \mapsto \bar{\phi} \in Z^3(G,k^*)/B^3(G,k^*)$$

that is surjective by Lemma 2.2. It is easy to see that it is also injective, and therefore

$$H^3(G,k^*) = Z^3_n(G,k^*)/B^3_n(G,k^*).$$

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2.3. Braided monoidal structures on $\mathbf{Vect}^G$

The next problem is to describe all *braided* monoidal structures on Vect^G, with $\otimes$ as the tensor product and k as the unit object. Such a braiding can only exist in the case where G is abelian. In this case, these structures are in bijective correspondence with so-called *abelian* 3-cocycles in G [**15**, **16**]. An abelian 3-cocycle is a pair $(\phi, \mathcal{R})$, where ϕ is a normalized 3-cocycle and $\mathcal{R}\colon G \times G \to k^*$ is a map satisfying

$$\mathcal{R}(xy, z)\phi(x, z, y) = \phi(x, y, z)\mathcal{R}(x, z)\phi(z, x, y)\mathcal{R}(y, z), \tag{2.2}$$

$$\phi(x, y, z)\mathcal{R}(x, yz)\phi(y, z, x) = \mathcal{R}(x, y)\phi(y, x, z)\mathcal{R}(x, z) \tag{2.3}$$

for all $x, y, z \in G$. If we take $x = y = e$ in (2.2) and $y = z = e$ in (2.2), then we immediately obtain that

$$\mathcal{R}(e, z) = \mathcal{R}(x, e) = 1. \tag{2.4}$$

We call ϕ the underlying 3-cocycle, and $\mathcal{R}$ the R-matrix. The corresponding monoidal structure is defined by ϕ, and the braiding is described by the formula

$$c_{V,W}(v \otimes w) = \mathcal{R}(|v|, |w|)w \otimes v$$

for all $V, W \in \mathrm{Vect}^G$ and $v \in V$ and $w \in W$ homogeneous. It is easy to show that the formulae (2.2) and (2.3) express the commutativity of the hexagonal diagrams in the definition of a braiding.

Take $\psi\colon G \times G \to k^*$ satisfying $\psi(e, x) = \psi(y, e)$ for all $x, y \in G$, so that $\Delta_2(\psi)$ is normalized. Consider the map

$$\mathcal{R}_\psi\colon G \times G \to k^*, \qquad \mathcal{R}_\psi(x, y) = \psi(x, y)^{-1}\psi(y, x).$$

Then $(\Delta_2(\psi), \mathcal{R}_\psi)$ is an abelian 3-cocycle called an abelian 3-coboundary. The sets $Z^3_{\mathrm{ab}}(G, k^*)$ of abelian 3-cocycles and $B^3_{\mathrm{ab}}(G, k^*)$ of abelian 3-coboundaries are abelian groups under pointwise multiplication, and $H^3_{\mathrm{ab}}(G, k^*)$ is defined as the quotient $Z^3_{\mathrm{ab}}(G, k^*)/B^3_{\mathrm{ab}}(G, k^*)$. Two braided monoidal structures on Vect^G are isomorphic if and only if their corresponding abelian 3-cocycles are cohomologous. Finally, observe that $(\phi, \mathcal{R}) \in Z^3_{\mathrm{ab}}(G, k^*)$ defines a *symmetric* monoidal structure on Vect^G if and only if $\mathcal{R}(x, y)\mathcal{R}(y, x) = 1$ for all $x, y \in G$.

2.4. Some homological algebra

We can compute certain cohomology groups using techniques from homological algebra. For $r, s \in \mathbb{N}_0$, let (r, s) be the greatest common divisor of r and s. Denote by $\mu_r(k)$ the group of rth units in k, and by $k^{*(r)} := \{\alpha^r \mid \alpha \in k\}$. We also denote by C_s the cyclic group of order s written multiplicatively, and by $\mathbb{Z}_r$ we denote the cyclic group of order r, written additively this time.

Proposition 2.3. *Let k be a field and let $r, s \in \mathbb{N}_0$. Then*

$$\left.\begin{aligned} H^3(C_r, k^*) &= \mu_r(k), \\ H^3(C_r \times C_s) &= k^*/k^{*(r,s)} \times \mu_r(k) \times \mu_s(k) \times \mu_{(r,s)}(k). \end{aligned}\right\} \tag{2.5}$$

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Proof. We have the following consequence of the universal coefficient theorem (see, for example, [**21**, Exercise 6.1.5]):

$$H^n(G,k^*) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G,\mathbb{Z}),k^*) \times \mathrm{Hom}_{\mathbb{Z}}(H_n(G,\mathbb{Z}),k^*), \tag{2.6}$$

where $H_n(G,\mathbb{Z}) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ is the nth homology group of G with values in $\mathbb{Z}$.

(i) We apply (2.6) in the case where $n=3$ and $G = C_r \cong \mathbb{Z}_r$. From [**21**, Example 6.2.3], we recall that

$$H_0(C_r,\mathbb{Z}) = \mathbb{Z}, \qquad H_{2n-1}(C_r,\mathbb{Z}) = \mathbb{Z}_r, \qquad H_{2n}(C_r,\mathbb{Z}) = 0,$$

for $n \geqslant 1$. It follows immediately that

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_2(C_r,\mathbb{Z}),k^*) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathbb{Z}}(H_3(C_r,\mathbb{Z}),k^*) = \mathrm{Hom}_{\mathbb{Z}}(C_r,k^*) = \mu_r(k),$$

proving the first formula in (2.5).

(ii) It follows from the Künneth formula (see, for example, [**21**, Proposition 6.1.13]) that

$$\begin{aligned} H_n(G\times H,\mathbb{Z}) \cong \prod_{p+q=n} & H_p(G,\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(H,\mathbb{Z}) \\ & \times \prod_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(G,\mathbb{Z}),H_q(H,\mathbb{Z})). \end{aligned}$$

Now, for any positive integers p, q we have

$$H_p(C_r,\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(C_s,\mathbb{Z}) = \begin{cases} \mathbb{Z}_r \otimes_{\mathbb{Z}} \mathbb{Z}_s \cong \mathbb{Z}_{(r,s)} & \text{for } p,q \text{ odd,} \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_s \cong \mathbb{Z}_s & \text{for } p=0 \text{ and } q \text{ odd,} \\ \mathbb{Z}_r \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}_r & \text{for } p \text{ odd and } q=0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Tor}_1^{\mathbb{Z}}(H_p(C_r,\mathbb{Z}),H_q(C_s,\mathbb{Z})) = \begin{cases} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_r,\mathbb{Z}_s) = \mathbb{Z}_{(r,s)} & \text{for } p,q \text{ odd,} \\ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_s) = 0 & \text{for } p=0 \text{ and } q \text{ odd,} \\ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_r,\mathbb{Z}) = 0 & \text{for } p \text{ odd and } q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these formulae into the Künneth formula, we find that

$$H_2(C_r\times C_s,\mathbb{Z}) = \mathbb{Z}_{(r,s)} \quad \text{and} \quad H_3(C_r\times C_s,\mathbb{Z}) = \mathbb{Z}_r \times \mathbb{Z}_s \times \mathbb{Z}_{(r,s)}.$$

It is well known that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{(r,s)},k^*) = k^*/k^{*(r,s)}$$

(see, for example, [**20**, Theorem 7.17]) and that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_r \times \mathbb{Z}_s \times \mathbb{Z}_{(r,s)},k^*) = \mu_r(k)\times\mu_s(k)\times\mu_{(r,s)}(k).$$

The second formula in (2.5) then follows after we apply (2.6). □

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2.5. The Eilenberg–Mac Lane theorem

The Eilenberg–Mac Lane theorem gives a description of $H^3_{\mathrm{ab}}(G, k^*)$ for an arbitrary abelian group G. A function $Q\colon G \to k^*$ between abelian groups G, k^* is called a quadratic form when $Q(x^{-1}) = Q(x)$ and

$$Q(xyz)Q(x)Q(y)Q(z) = Q(xy)Q(xz)Q(yz) \tag{2.7}$$

for all $x, y, z \in G$. The set of quadratic forms on G with values in k^* is denoted by $\mathrm{QF}(G, k^*)$. It is easy to see that the pointwise product of two quadratic forms is again a quadratic form, so $\mathrm{QF}(G, k^*)$ is an abelian group.

Theorem 2.4 (Eilenberg and Mac Lane [10–13, 18]). *Let G be an abelian group and $(\phi, \mathcal{R}) \in H^3_{\mathrm{ab}}(G, k^*)$. Then $Q\colon G \to k^*$ given by $Q(x) = \mathcal{R}(x, x)$ for all $x \in G$ is a quadratic form on G with values in k^*. It is called the trace of the abelian 3-cocycle $(\phi, \mathcal{R})$. Furthermore, trace induces a group isomorphism $\mathrm{EM}\colon H^3_{\mathrm{ab}}(G, k^*) \to \mathrm{QF}(G, k^*)$.*

We refer the reader to [**15**, p. 35, Theorem 12] for a proof of the Eilenberg–Mac Lane theorem.

3. Computation of the 3-cocycles on the Klein group

The *Klein group* is the non-cyclic group of order 4, $C_2 \times C_2$. It follows from Proposition 2.3 that

$$H^3(C_2 \times C_2, k^*) \cong k^*/k^{*(2)} \times \mu_2(k) \times \mu_2(k).$$

In order to be able to describe the monoidal structures on the category of vector spaces by the Klein group, we need the cocycles explicitly. This explicit form will also be required in § 4, where we will compute the abelian 3-cocycles. In this section we compute the cocycles manually.

We will work over a field k of characteristic not equal to 2. In what follows, we write $G = C_2 \times C_2 = \{e, \sigma, \tau, \rho\}$, with $\sigma\tau = \tau\sigma = \rho$ and $\sigma^2 = \tau^2 = e$.

3.1. The normalized coboundaries

Consider $g\colon C_2 \times C_2 \to k^*$. If $\Delta_2(g)$ is a normalized coboundary, then g is determined completely by the following data (see Lemma 2.1):

$$\left.\begin{array}{lll} g(\sigma,\sigma) = a_1, & g(\tau,\tau) = a_2, & g(\rho,\rho) = a_3, \\ g(\sigma,\tau) = b_1, & g(\tau,\rho) = b_2, & g(\rho,\sigma) = b_3, \\ g(\tau,\sigma) = b_4, & g(\sigma,\rho) = b_5, & g(\rho,\tau) = b_6, \\ g(x,1) = c, & g(1,x) = c & \end{array}\right\} \tag{3.1}$$

for all $x \in G$. All other values of g are equal to 1. Normalized coboundaries thus depend on the choice of 10 parameters $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, c \in k^*$. For later use, we list

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some of the values of $\Delta_2(g)$:

$$\Delta_2(g)(\sigma,\sigma,\sigma) = \Delta_2(g)(\tau,\tau,\tau) = \Delta_2(g)(\rho,\rho,\rho) = 1, \tag{3.2}$$

$$\left.\begin{aligned}
\Delta_2(g)(\sigma,\sigma,\tau) &= b_1b_5a_1^{-1}c^{-1}, & \Delta_2(g)(\rho,\rho,\sigma) &= b_2b_6a_3^{-1}c^{-1},\\
\Delta_2(g)(\tau,\sigma,\sigma) &= b_4^{-1}b_3^{-1}a_1c, & \Delta_2(g)(\sigma,\tau,\sigma) &= b_1^{-1}b_3^{-1}b_4b_5,\\
\Delta_2(g)(\sigma,\tau,\rho) &= b_1^{-1}b_2a_1a_3^{-1}, & \Delta_2(g)(\tau,\rho,\sigma) &= b_2^{-1}b_3a_2a_1^{-1},\\
\Delta_2(g)(\rho,\sigma,\tau) &= b_3^{-1}b_1a_3a_2^{-1}, & \Delta_2(g)(\tau,\sigma,\rho) &= b_4^{-1}b_5a_2a_3^{-1},\\
\Delta_2(g)(\sigma,\rho,\tau) &= b_5^{-1}b_6a_1a_2^{-1}, & \Delta_2(g)(\rho,\tau,\sigma) &= b_6^{-1}b_4a_3a_1^{-1}.
\end{aligned}\right\} \tag{3.3}$$

3.2. The cocycle relations

Taking $x = y = z = t = \sigma$ in (2.1), we find that $\phi(\sigma,\sigma,\sigma)^2 = 1$. Thus,

$$\varepsilon_\sigma = \phi(\sigma,\sigma,\sigma) = \pm 1. \tag{3.4}$$

We have similar formulae for $\varepsilon_\tau = \phi(\tau,\tau,\tau)$ and $\varepsilon_\rho = \phi(\rho,\rho,\rho)$. Since every coboundary takes the value 1 at (σ,σ,σ), (τ,τ,τ) and (ρ,ρ,ρ), we see that ε_σ, ε_τ and ε_ρ stay invariant if we replace ϕ by a cohomologous cocycle.

Lemma 3.1. *Any normalized 3-cocycle ϕ satisfies the relations*

$$\phi(\rho,\tau,\tau) = \varepsilon_\tau\phi(\sigma,\tau,\tau), \tag{3.5}$$
$$\phi(\tau,\tau,\rho) = \varepsilon_\tau\phi(\tau,\tau,\sigma), \tag{3.6}$$
$$\phi(\tau,\rho,\tau)\phi(\tau,\sigma,\tau) = \varepsilon_\tau, \tag{3.7}$$
$$\phi(\sigma,\tau,\tau)\phi(\sigma,\sigma,\tau)\phi(\sigma,\rho,\tau) = 1, \tag{3.8}$$
$$\phi(\tau,\sigma,\tau)\phi(\sigma,\tau,\sigma)\phi(\sigma,\rho,\tau) = \phi(\rho,\sigma,\tau)\phi(\sigma,\tau,\rho), \tag{3.9}$$
$$\phi(\sigma,\tau,\tau)\phi(\tau,\tau,\sigma) = \phi(\rho,\tau,\sigma)\phi(\sigma,\tau,\rho), \tag{3.10}$$
$$\varepsilon_\sigma\phi(\tau,\rho,\sigma)\phi(\sigma,\tau,\rho) = \phi(\rho,\rho,\sigma)\phi(\sigma,\tau,\tau), \tag{3.11}$$
$$\phi(\tau,\rho,\tau)\phi(\sigma,\sigma,\tau)\phi(\sigma,\tau,\rho) = \phi(\rho,\rho,\tau)\phi(\sigma,\tau,\sigma), \tag{3.12}$$
$$\phi(\tau,\rho,\rho)\phi(\sigma,\sigma,\rho)\phi(\sigma,\tau,\rho) = \varepsilon_\rho. \tag{3.13}$$

These relations remain valid after we apply a permutation of (σ,τ,ρ).

Proof. All the formulae follow directly from the cocycle relation (2.1). We subsequently take $x = \sigma$, $y = z = t = \tau$ (3.5); $x = y = z = \tau$, $t = \sigma$ (3.6); $y = \sigma$, $x = z = t = \tau$ (3.7) (applying (3.5)); $x = y = \sigma$, $z = t = \tau$ (3.8); $x = z = \sigma$, $y = t = \tau$ (3.9); $x = t = \sigma$, $y = z = \tau$ (3.10); $x = t = \sigma$, $y = \tau$, $z = \rho$ (3.11); $x = \sigma$, $y = t = \tau$, $z = \rho$ (3.12); $x = \sigma$, $y = \tau$, $z = t = \rho$ (3.13). □

Lemma 3.2. *Let ϕ be a normalized cocycle and write*

$$p = \phi(\sigma,\tau,\rho)\phi(\tau,\rho,\sigma)\phi(\rho,\sigma,\tau), \qquad q = \phi(\rho,\tau,\sigma)\phi(\sigma,\rho,\tau)\phi(\tau,\sigma,\rho).$$

Then

$$p = q = \varepsilon_\sigma\varepsilon_\tau\varepsilon_\rho = \pm 1. \tag{3.14}$$

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Proof. We compute

$$\begin{aligned} p &= \phi(\sigma,\tau,\rho)\phi(\tau,\rho,\sigma)\phi(\rho,\sigma,\tau) \\ &\overset{(3.11)}{=} \varepsilon_\sigma\phi(\rho,\rho,\sigma)\phi(\sigma,\tau,\tau)\phi(\rho,\sigma,\tau) \\ &\overset{(3.5),(3.6)}{=} \varepsilon_\sigma\varepsilon_\rho\varepsilon_\tau\phi(\rho,\rho,\tau)\phi(\rho,\tau,\tau)\phi(\rho,\sigma,\tau) \\ &\overset{(3.8)}{=} \varepsilon_\sigma\varepsilon_\tau\varepsilon_\rho. \end{aligned}$$

In a similar way, we prove that $q = \varepsilon_\sigma\varepsilon_\tau\varepsilon_\rho$. □

3.3. Reduction to happy cocycles

A normalized 3-cocycle ϕ is called even (respectively, odd) if $p = 1$ (respectively, $p = -1$). ϕ is called happy if $\phi(x, y, z) = p$ whenever x, y and z are pairwise distinct and not equal to 1.

Proposition 3.3. *Every* 3*-cocycle* ϕ *is cohomologous to a happy normalized* 3*-cocycle.*

Proof. It follows from Lemma 2.2 that we can assume that ϕ is normalized. Let g be defined as in (3.1), with $a_1 = a_2 = a_3 = 1$, $b_1 = b_5 = p$, $b_2 = \phi(\sigma,\tau,\rho)^{-1}$, $b_3 = \phi(\rho,\sigma,\tau)$, $b_4 = \phi(\tau,\sigma,\rho)$, $b_6 = \phi(\sigma,\rho,\tau)^{-1}$ and $c = 1$. Applying (3.3), we find immediately that $\phi\Delta_2(g)$ is happy. □

3.4. Description of the happy cocycles

Assume that $\phi\colon G\times G\times G \to k^*$ is happy and normalized. This means that it satisfies the following properties:

(i) $\phi(x, y, z) = 1$ if one of the three entries is 1;

(ii) $\varepsilon_x = \phi(x, x, x) = \pm 1$ for all $x \in \{\sigma, \tau, \rho\}$;

(iii) $\phi(x, y, z) = p = \varepsilon_\sigma\varepsilon_\tau\varepsilon_\rho$ if (x, y, z) is a permutation of (σ, τ, ρ).

The cocycle relations (3.8)–(3.13) then simplify as follows:

$$\phi(\sigma,\tau,\tau)\phi(\sigma,\sigma,\tau) = p, \tag{3.15}$$
$$\phi(\tau,\sigma,\tau)\phi(\sigma,\tau,\sigma) = p, \tag{3.16}$$
$$\phi(\sigma,\tau,\tau)\phi(\tau,\tau,\sigma) = 1, \tag{3.17}$$
$$\varepsilon_\sigma = \phi(\rho,\rho,\sigma)\phi(\sigma,\tau,\tau), \tag{3.18}$$
$$p\phi(\tau,\rho,\tau)\phi(\sigma,\sigma,\tau) = \phi(\rho,\rho,\tau)\phi(\sigma,\tau,\sigma), \tag{3.19}$$
$$\phi(\tau,\rho,\rho)\phi(\sigma,\sigma,\rho) = p\varepsilon_\rho. \tag{3.20}$$

Proposition 3.4. *Let* $\phi\colon G\times G\times G \to k^*$ *be happy and normalized. Then* ϕ *is a* 3*-cocycle if and only if (3.5)–(3.7) and (3.15)–(3.17) (and their permuted versions) are satisfied. In other words, the cocycle relations (3.18)–(3.20) follow from the other cocycle relations.*

Proof. One implication is clear. Conversely, suppose now that (3.5)–(3.7) and (3.15)–(3.17) are satisfied. We show that (3.18)–(3.20) hold. Indeed, we have

$$\begin{aligned}\phi(\rho,\rho,\sigma)\phi(\sigma,\tau,\tau) &\overset{(3.5),(3.6)}{=} \varepsilon_\rho\phi(\rho,\rho,\tau)\varepsilon_\tau\phi(\rho,\tau,\tau)\\ &\overset{(3.15)}{=} \varepsilon_\rho\varepsilon_\tau p = \varepsilon_\sigma,\\ p\phi(\tau,\rho,\tau)\phi(\sigma,\sigma,\tau) &\overset{(3.7),(3.15)}{=} p\varepsilon_\tau\phi(\tau,\sigma,\tau)^{-1}p\phi(\sigma,\tau,\tau)^{-1}\\ &\overset{(3.16),(3.5)}{=} \phi(\sigma,\tau,\sigma)p\phi(\rho,\tau,\tau)^{-1}\\ &\overset{(3.15)}{=} \phi(\sigma,\tau,\sigma)\phi(\rho,\rho,\tau),\\ \phi(\tau,\rho,\rho)\phi(\sigma,\sigma,\rho) &\overset{(3.5)}{=} \varepsilon_\rho\phi(\sigma,\rho,\rho)\phi(\sigma,\sigma,\rho)\\ &\overset{(3.15)}{=} p\varepsilon_\rho,\end{aligned}$$

and this completes the proof. □

Theorem 3.5. *Let ϕ be a happy normalized 3-cocycle. ϕ is completely determined by ε_σ, ε_τ, ε_ρ, $a = \phi(\tau,\sigma,\sigma)$ and $b = \phi(\sigma,\tau,\sigma)$. More precisely, we have*

$$\left.\begin{aligned} a &= \phi(\tau,\sigma,\sigma) = p\phi(\sigma,\tau,\tau)\\ &= p\phi(\tau,\tau,\sigma)^{-1} = \phi(\sigma,\sigma,\tau)^{-1}\\ &= \varepsilon_\sigma\phi(\rho,\sigma,\sigma) = p\varepsilon_\sigma\phi(\sigma,\rho,\rho)\\ &= \varepsilon_\sigma\phi(\sigma,\sigma,\rho)^{-1} = p\varepsilon_\sigma\phi(\rho,\rho,\sigma)^{-1}\\ &= p\varepsilon_\tau\phi(\rho,\tau,\tau) = \varepsilon_\tau\phi(\tau,\rho,\rho)\\ &= p\varepsilon_\tau\phi(\tau,\tau,\rho)^{-1} = \varepsilon_\tau\phi(\rho,\rho,\tau)^{-1},\\ b &= \phi(\sigma,\tau,\sigma) = p\varepsilon_\sigma\phi(\rho,\sigma,\rho)\\ &= p\varepsilon_\tau\phi(\tau,\rho,\tau) = p\phi(\tau,\sigma,\tau)^{-1}\\ &= \varepsilon_\sigma\phi(\sigma,\rho,\sigma)^{-1} = \varepsilon_\tau\phi(\rho,\tau,\rho)^{-1}.\end{aligned}\right\}\tag{3.21}$$

Proof. Recall first that some of the cocycle conditions simplify if ϕ is happy. Using these simplified cocycle relations, we compute

$$\begin{aligned}\phi(\tau,\tau,\sigma)^{-1} &\overset{(3.15)}{=} p\phi(\tau,\sigma,\sigma) = pa,\\ \phi(\sigma,\tau,\tau) &\overset{(3.17)}{=} \phi(\tau,\tau,\sigma)^{-1}\\ &\overset{(3.15)}{=} p\phi(\tau,\sigma,\sigma) = pa,\\ \phi(\sigma,\sigma,\tau)^{-1} &\overset{(3.15)}{=} p\phi(\sigma,\tau,\tau) = a,\\ \phi(\rho,\sigma,\sigma) &\overset{(3.5)}{=} \varepsilon_\sigma\phi(\tau,\sigma,\sigma) = a\varepsilon_\sigma,\\ \phi(\rho,\tau,\tau) &\overset{(3.5)}{=} \varepsilon_\tau\phi(\sigma,\tau,\tau) = pa\varepsilon_\tau,\\ \phi(\sigma,\sigma,\rho) &\overset{(3.6)}{=} \varepsilon_\sigma\phi(\sigma,\sigma,\tau) = a^{-1}\varepsilon_\sigma,\end{aligned}$$

$$\phi(\tau,\tau,\rho) \overset{(3.6)}{=} \varepsilon_\tau\phi(\tau,\tau,\sigma) \overset{(3.15)}{=} pa^{-1}\varepsilon_\tau,$$
$$\phi(\sigma,\rho,\rho) \overset{(3.15)}{=} p\phi(\sigma,\sigma,\rho)^{-1} = pa\varepsilon_\sigma,$$
$$\phi(\tau,\rho,\rho) \overset{(3.15)}{=} p\phi(\tau,\tau,\rho)^{-1} = a\varepsilon_\tau,$$
$$\phi(\rho,\rho,\sigma) \overset{(3.15)}{=} p\phi(\rho,\sigma,\sigma)^{-1} = pa^{-1}\varepsilon_\sigma,$$
$$\phi(\rho,\rho,\tau) \overset{(3.15)}{=} p\phi(\rho,\tau,\tau)^{-1} = a^{-1}\varepsilon_\tau,$$
$$\phi(\tau,\sigma,\tau) \overset{(3.16)}{=} p\phi(\sigma,\tau,\sigma)^{-1} = pb^{-1},$$
$$\phi(\tau,\rho,\tau) \overset{(3.7)}{=} \varepsilon_\tau\phi(\tau,\sigma,\tau)^{-1} = p\varepsilon_\tau b,$$
$$\phi(\rho,\tau,\rho) \overset{(3.16)}{=} p\phi(\tau,\rho,\tau)^{-1} = \varepsilon_\tau b^{-1},$$
$$\phi(\rho,\sigma,\rho) \overset{(3.7)}{=} \varepsilon_\rho\phi(\rho,\tau,\rho)^{-1} = \varepsilon_\rho\varepsilon_\tau b = p\varepsilon_\sigma b,$$
$$\phi(\sigma,\rho,\sigma) \overset{(3.7)}{=} \varepsilon_\sigma\phi(\sigma,\tau,\sigma)^{-1} = \varepsilon_\sigma b^{-1},$$

as we claimed, so our proof is complete. □

The maps ϕ described in Theorem 3.5 are indexed by the parameters ε_σ, ε_τ, $\varepsilon_\rho \in \{-1,1\}$ and $a,b \in k^*$. It is a routine computation to verify that they all satisfy (3.5)–(3.7) and (3.15)–(3.17), hence, they are all 3-cocycles by Proposition 3.4. This tells us that there is a bijection from the subgroup $Z_h^3(C_2\times C_2,k^*)$ of $Z^3(C_2\times C_2,k^*)$ consisting of happy normalized cocycles to the set $C_2^3\times(k^*)^2$.

Let H_1 be the subset of $Z_h^3(C_2\times C_2,k^*)$ for which the corresponding parameters a and b are equal to 1. This is also the subset of $Z_h^3(C_2\times C_2,k^*)$ consisting of cocycles ϕ for which $\phi(\tau,\sigma,\sigma)=\phi(\sigma,\tau,\sigma)=1$. It is then clear that H_1 is a subgroup of $Z_h^3(C_2\times C_2,k^*)$ consisting of eight elements:

$$H_1 = \{\phi_X \mid X \subseteq \{\sigma,\tau,\rho\}\}.$$

$\phi_\emptyset$ is the trivial 3-cocycle and, for a non-empty subset X of $\{\sigma,\tau,\rho\}$, ϕ_X is the 3-cocycle defined as follows: $\varepsilon_x=-1$ if and only if $x\in X$. The multiplication on H_1 is the following:

$$\phi_X\phi_Y = \phi_{X\Delta Y},$$

where $X\Delta Y=(X\setminus Y)\cup(Y\setminus X)$ is the symmetric difference of the sets X and Y. ϕ_X is an even cocycle if and only if $|X|$ is even. It follows that $H_1\cong C_2\times C_2\times C_2$.

Since a normalized cocycle takes the value 1 if one of the three entries is equal to e, we can view them as functions $\{\sigma,\tau,\rho\}^3\to k^*$. From the description in Theorem 3.5, it follows that the eight cocycles in H_1 are invariant under permutation: $\phi_X\circ s=\phi_X$ for all $X\subset\{\sigma,\tau,\rho\}$ and $s\in S_3$.

Let $\{P_e,P_\sigma,P_\tau,P_\rho\}$ be the basis of $k[G]^*\cong k^G$ dual to the basis $\{e,\sigma,\tau,\rho\}$ of $k[G]$. Then the following elements of $k^{G\times G\times G}\cong k[G]^*\otimes k[G]^*\otimes k[G]^*$ are invariant under

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permutations:

$$X = \sum_{s\in S_3} P_{s(\sigma)} \otimes P_{s(\tau)} \otimes P_{s(\rho)},$$
$$X_x = P_x \otimes P_x \otimes P_x, \quad x \in \{\sigma,\tau,\rho\},$$
$$X_{x,y} = P_x \otimes P_y \otimes P_y + P_y \otimes P_x \otimes P_y + P_y \otimes P_y \otimes P_x, \quad x \neq y \in \{\sigma,\tau,\rho\}.$$

Viewed as maps $G \times G \times G \to k^*$, these can also be described as follows:

$$X(x,y,z) = \begin{cases} 1 & \text{if } \{x,y,z\} = \{\sigma,\tau,\rho\}, \\ 0 & \text{otherwise}, \end{cases}$$
$$X_\sigma(x,y,z) = \begin{cases} 1 & \text{if } x = y = z = \sigma, \\ 0 & \text{otherwise}, \end{cases}$$

and $X_{\sigma,\tau}(x,y,z) = 1$ if one element of (x,y,z) equals σ and the two others equal τ, and $X_{\sigma,\tau}(x,y,z) = 0$ otherwise.

Also observe that the X, X_x and $X_{x,y}$ are orthogonal. From Theorem 3.5, we now deduce the following formulae for the elements of H_1,

$$\left.\begin{aligned} \phi_{\{\sigma\}} &= 1 - 2(X_\sigma + X_{\sigma,\tau} + X_{\rho,\tau} + X_{\rho,\sigma} + X), \\ \phi_{\{\tau\}} &= 1 - 2(X_\tau + X_{\sigma,\rho} + X_{\tau,\rho} + X_{\sigma,\tau} + X), \\ \phi_{\{\rho\}} &= 1 - 2(X_\rho + X_{\sigma,\tau} + X_{\rho,\tau} + X_{\sigma,\rho} + X), \\ \phi_{\{\sigma,\tau\}} &= 1 - 2(X_\sigma + X_\tau + X_{\rho,\tau} + X_{\tau,\rho} + X_{\rho,\sigma} + X_{\sigma,\rho}), \\ \phi_{\{\sigma,\rho\}} &= 1 - 2(X_\sigma + X_\rho + X_{\rho,\sigma} + X_{\sigma,\rho}), \\ \phi_{\{\tau,\rho\}} &= 1 - 2(X_\tau + X_\rho + X_{\rho,\tau} + X_{\tau,\rho}), \\ \phi_{\{\sigma,\tau,\rho\}} &= 1 - 2(X_\sigma + X_\tau + X_\rho + X_{\rho,\sigma} + X_{\sigma,\tau} + X_{\tau,\rho} + X). \end{aligned}\right\} \tag{3.22}$$

For any $b \in k^*$, let g_b be the cocycle that we obtain taking $a = 1$, $\varepsilon_\sigma = \varepsilon_\tau = \varepsilon_\rho = 1$ in Theorem 3.5. We have

$$g_b(x,y,z) = \begin{cases} b & \text{if } (x,y,z) \in \{(\sigma,\tau,\sigma),(\rho,\sigma,\rho),(\tau,\rho,\tau)\}, \\ b^{-1} & \text{if } (x,y,z) \in \{(\tau,\sigma,\tau),(\sigma,\rho,\sigma),(\rho,\tau,\rho)\}, \\ 1 & \text{otherwise}. \end{cases} \tag{3.23}$$

For any $a \in k^*$, let h_a be the cocycle that we obtain taking $b = 1$, $\varepsilon_\sigma = \varepsilon_\tau = \varepsilon_\rho = 1$ in Theorem 3.5. Thus,

$$h_a(x,y,z) = \begin{cases} a & \text{if } e \neq x \neq y = z \neq e, \\ a^{-1} & \text{if } e \neq x = y \neq z \neq e, \\ 1 & \text{otherwise}. \end{cases}$$

It is clear that

$$H_2 = \{g_b \mid b \in k^*\} \quad \text{and} \quad H_3 = \{h_a \mid a \in k^*\}$$

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are subgroups of $Z_h^3(C_2 \times C_2, k^*)$. Therefore, $Z_h^3(C_2 \times C_2, k^*) = H_1 \times H_2 \times H_3$, and we have the following result.

Corollary 3.6. *We have an isomorphism of abelian groups*

$$Z_h^3(C_2 \times C_2, k^*) \cong C_2 \times C_2 \times C_2 \times k^* \times k^*.$$

It is easy to see that $h_a g_b$ is invariant under permutation if and only if $a = b = -1$. Observe also that

$$\begin{aligned} h_{-1}g_{-1} &= 1 - 2(X_{\rho,\sigma} + X_{\sigma,\rho} + X_{\sigma,\tau} + X_{\tau,\sigma} + X_{\tau,\rho} + X_{\rho,\tau}), \\ h_{-1}g_{-1}\phi_{\{\sigma,\tau\}} &= 1 - 2(X_\sigma + X_\tau + X_{\tau,\rho} + X_{\rho,\tau}). \end{aligned}$$

The subgroup $\tilde{H}$ of $Z_h^3(C_2 \times C_2, k^*)$ consisting of cocycles invariant under permutation is the subgroup generated by H_1 and $h_{-1}g_{-1}$, and it follows from Corollary 3.6 that $\tilde{H} \cong C_2^4$.

$H^3(C_2 \times C_2, k^*)$ is an epimorphic image of $Z_h^3(C_2 \times C_2, k^*)$. We need to figure out which happy normalized cocycles are coboundaries.

Proposition 3.7. *If $X \neq \emptyset$, then ϕ_X is not a coboundary. h_a is a coboundary for every $a \in k^*$. g_b is a coboundary if and only if b has a square root in k^*.*

Proof. The first statement follows from the fact that $\phi(x,x,x) = 1$ for every x if ϕ is a normalized coboundary (see (3.2)).

h_a can be written as a coboundary in two different ways: take $g\colon C_2 \times C_2 \to k^*$ as in (3.1), with $b_i = 1$, for $i = 1, \dots, 6$, $a_1 = a_2 = a_3 = a$ and $c = 1$, or $c = a$, and all the a_i and b_i equal to 1. It follows from (3.3) that $\Delta_2(g)$ is happy, and that $\Delta_2(g)(\tau,\sigma,\sigma) = a$ and $\Delta_2(g)(\sigma,\tau,\sigma) = 1$. Applying Theorem 3.5, we see that $\Delta_2(g) = h_a$.

Assume that $b = d^2$, and consider $g\colon C_2 \times C_2 \to k^*$ as in (3.1), now with

$$a_1 = a_2 = a_3 = b_4 = b_5 = b_6 = d, \qquad b_1 = b_2 = b_3 = 1, \qquad c = 1.$$

It follows from (3.3) that $\Delta_2(g)$ is happy, and that

$$\Delta_2(g)(\tau,\sigma,\sigma) = 1 \quad \text{and} \quad \Delta_2(g)(\sigma,\tau,\sigma) = d^2 = b.$$

Theorem 3.5 tells us that $\Delta_2(g) = g_b$, so g_b is a coboundary.

Conversely, if g_b is a coboundary, then $g_b = \Delta_2(g)$, with $g\colon C_2 \times C_2 \to k^*$ given by (3.1), for some $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, c \in k^*$. From (3.3) and the description of g_b, it follows that

$$\begin{aligned} 1 &= g_b(\sigma,\sigma,\tau) = b_1 b_5 a_1^{-1} c^{-1}, \qquad & 1 &= g_b(\tau,\sigma,\rho) = b_4^{-1} b_5 a_2 a_3^{-1}, \\ 1 &= g_b(\rho,\rho,\sigma) = b_3 b_6 a_3^{-1} c^{-1}, \qquad & 1 &= g_b(\sigma,\rho,\tau) = b_5^{-1} b_6 a_1 a_2^{-1}. \end{aligned}$$

From the first two formulae it follows that $b_1 = a_1 a_2 a_3^{-1} b_4^{-1} c$, and combining the other two formulae we obtain that $b_3 = a_1 a_2^{-1} a_3 b_5^{-1} c$. Using (3.3), we now find that

$$\begin{aligned} b = g_b(\sigma,\sigma,\tau) &= b_1^{-1} b_3^{-1} b_4 b_5 \\ &= a_1^{-1} a_2^{-1} a_3 b_4 c^{-1} a_1^{-1} a_2 a_3^{-1} b_5 c^{-1} b_4 b_5 \\ &= (a_1^{-1} b_4 b_5 c^{-1})^2 \end{aligned}$$

is a square in k^*. □

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Corollary 3.8. *We have*

$$H^3(C_2 \times C_2, k^*) = C_2 \times C_2 \times C_2 \times k^*/k^{*(2)}, \quad \text{where } k^{*(2)} = \{\alpha^2 \mid \alpha \in k^*\}.$$

Corollary 3.9. *If every element of k has a square root (for example, if k is algebraically closed), then $H^3(C_2 \times C_2, k^*) = C_2 \times C_2 \times C_2$.*

So a non-strict monoidal structure on $\mathrm{Vect}^{C_2\times C_2}$ is defined by the one of the 3-cocycles defined in (3.22), or by a 3-cocycle as in (3.23) with $b \in k^* \setminus k^{*(2)}$. All the remaining monoidal structures of $\mathrm{Vect}^{C_2\times C_2}$ are monoidal isomorphic to the strict monoidal structure of $\mathrm{Vect}^{C_2\times C_2}$.

All our computations are over fields of characteristic different from 2; they can be extended easily to the case where $\mathrm{char}(k) = 2$. Then all $\varepsilon_x = 1$, and we obtain the following result.

Proposition 3.10. *Let k be a field of characteristic 2. Then*

$$H^3(C_2 \times C_2, k^*) = k^*/k^{*(2)}.$$

If every element of k has a square root, then $H^3(C_2 \times C_2, k^*) = 1$.

Remark 3.11. Let $C_2 = \{e, \alpha\}$. If the characteristic of k is different from 2, then $Z^3(C_2, k^*)$ contains two cocycles. The non-trivial cocycle ϕ takes the value -1 at (α, α, α) and 1 elsewhere. Otherwise stated,

$$\phi = 1 - 2P_\alpha \otimes P_\alpha \otimes P_\alpha$$

if $\{P_e, P_\alpha\}$ is the basis of $k[C_2]^*$ dual to $\{e, \alpha\}$.

Now we have three Hopf algebra morphisms $t_1, t_2, t_3\colon k[C_2]^* \to k[C_2 \times C_2]^*$. These are given by the formulae

$$\begin{aligned} t_1(P_e) &= P_e + P_\sigma, & t_1(P_\alpha) &= P_\tau + P_\rho, \\ t_2(P_e) &= P_e + P_\tau, & t_2(P_\alpha) &= P_\sigma + P_\rho, \\ t_3(P_e) &= P_e + P_\rho, & t_3(P_\alpha) &= P_\tau + P_\sigma. \end{aligned}$$

The t_i induce group morphisms $t_i\colon Z^3(C_2, k^*) \to Z^3(C_2 \times C_2, k^*)$. Now we easily see that

$$\begin{aligned} t_1(\phi) &= 1 - 2(P_\tau + P_\rho) \otimes (P_\tau + P_\rho) \otimes (P_\tau + P_\rho) = \phi_{\{\tau,\rho\}}, \\ t_2(\phi) &= 1 - 2(P_\sigma + P_\rho) \otimes (P_\sigma + P_\rho) \otimes (P_\sigma + P_\rho) = \phi_{\{\sigma,\rho\}}, \\ t_3(\phi) &= 1 - 2(P_\tau + P_\sigma) \otimes (P_\tau + P_\sigma) \otimes (P_\tau + P_\sigma) = h_{-1}g_{-1}\phi_{\{\sigma,\tau\}}. \end{aligned}$$

If k contains a square root i of -1, then $h_{-1}g_{-1}$ is a coboundary, and

$$[t_1(\phi)][t_2(\phi)] = [t_3(\phi)] \quad \text{in } H^3(C_2 \times C_2, k^*).$$

4. Computation of the abelian cocycles on the Klein group

4.1. Computation of the quadratic forms

Throughout this section we assume that $\mathrm{char}(k) \neq 2$. In order to describe the braidings of $\mathrm{Vect}^{C_2 \times C_2}$, we have to compute $H^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$ (see § 2.3). To this end, we will make use of the Eilenberg–Mac Lane theorem (see § 2.5). We compute $\mathrm{QF}(C_2 \times C_2, k^*)$.

Lemma 4.1. *$Q\colon C_2 \times C_2 \to k^*$ is a quadratic form if and only if*

(i) $Q(e) = 1$,

(ii) $Q(\sigma)^4 = Q(\tau)^4 = Q(\rho)^4 = 1$,

(iii) $Q(\sigma)^2 Q(\tau)^2 Q(\rho)^2 = 1$.

Proof. Assume first that Q is a quadratic form. Part (i) follows after we take $x = y = z = e$ in (2.7). For (ii), take $x = y = z = \sigma$ in (2.7). Since $\sigma^3 = \sigma$ and $\sigma^2 = e$, it follows that $Q(\sigma)^4 = 1$. For (iii), take $x = y = \sigma$ and $\rho = \tau$ in (2.7). Then we find that $Q(\sigma)^2 Q(\tau)^2 = Q(\rho)^2$. Multiplying both sides by $Q(\rho)^2$, we find (iii).

Conversely, assume that Q satisfies conditions (i)–(iii). Then $Q(x^{-1}) = Q(x)$ is automatically satisfied, since $x = x^{-1}$ for all $x \in C_2 \times C_2$. To prove (2.7), we distinguish several cases.

Case 1 ($e \in \{x, y, z\}$, say $x = e$). Then (2.7) reduces to

$$Q(yz)Q(y)Q(z) = Q(y)Q(z)Q(yz),$$

which is satisfied.

Case 2 ($e \notin \{x, y, z\}$).

(a) $|\{x, y, z\}| = 1$: $x = y = z$. (2.7) reduces to $Q(x)^4 = Q(e)^3 = 1$.

(b) $|\{x, y, z\}| = 2$, say $x = y = \sigma$, $z = \tau$. (2.7) reduces to

$$Q(\tau)Q(\sigma)^2 Q(\tau) = Q(e)Q(\rho)Q(\rho).$$

(c) $|\{x, y, z\}| = 3$, say $x = \sigma$, $y = \tau$, $z = \rho$. (2.7) reduces to

$$Q(e)Q(\sigma)Q(\tau)Q(\rho) = Q(\rho)Q(\tau)Q(\sigma).$$

□

Assume first that k contains i, a primitive fourth root of 1. Then Q is a quadratic form if and only if $Q(e) = 1$, $Q(\sigma), Q(\tau), Q(\rho) \in \{\pm 1, \pm \mathrm{i}\}$ and $Q(\sigma)Q(\tau)Q(\rho) = \pm 1$. Then $\mathrm{QF}(C_2 \times C_2, k^*)$ has 32 elements, which are summarized in Table 1.

Thus $\mathrm{QF}(C_2 \times C_2, k^*)$ is the abelian group consisting of I, A, B, C, AB, AC, BC, ABC; E_j, AE_j, BE_j, CE_j, ABE_j, ACE_j, BCE_j, $ABCE_j$, $j = 1, 2, 3$, with relations

$$\left.\begin{aligned} A^2 = B^2 = C^2 = I, \qquad E_1^2 = BC, \qquad E_2^2 = AC, \qquad E_3^2 = AB, \\ E_1E_2 = CE_3, \qquad E_1E_3 = BE_2, \qquad E_2E_3 = AE_1, \end{aligned}\right\} \tag{4.1}$$

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Table 1. *The* 32 *elements of* $\mathrm{QF}(C_2 \times C_2, k^*)$.

	I	A	B	C	AB	AC	BC	ABC
$Q(\sigma)$	1	1	1	−1	1	−1	−1	−1
$Q(\tau)$	1	1	−1	1	−1	1	−1	−1
$Q(\rho)$	1	−1	1	1	−1	−1	1	−1

	E_1	AE_1	BE_1	CE_1	ABE_1	ACE_1	BCE_1	$ABCE_1$
$Q(\sigma)$	i	i	i	−i	i	−i	−i	−i
$Q(\tau)$	i	i	−i	i	−i	i	−i	−i
$Q(\rho)$	1	−1	1	1	−1	−1	1	−1

	E_2	AE_2	BE_2	CE_2	ABE_2	ACE_2	BCE_2	$ABCE_2$
$Q(\sigma)$	i	i	i	−i	i	−i	i	−i
$Q(\tau)$	1	1	−1	1	−1	1	−1	−1
$Q(\rho)$	i	−i	i	i	−i	−i	i	−i

	E_3	AE_3	BE_3	CE_3	ABE_3	ACE_3	BCE_3	$ABCE_3$
$Q(\sigma)$	1	1	1	−1	1	−1	−1	−1
$Q(\tau)$	i	i	−i	i	−i	i	−i	−i
$Q(\rho)$	i	−i	i	i	−i	−i	i	−i

for all $j = 1, 2, 3$. Hence $\mathrm{QF}(C_2 \times C_2, k^*) \cong C_4 \times C_4 \times C_2$, because it is an abelian group of order 32 that contains precisely seven elements of order two and all the other non-trivial elements have order four.

If k does not contain a fourth root of 1, then we clearly have

$$\mathrm{QF}(C_2 \times C_2, k^*) = \{I, A, B, C, AB, AC, BC, ABC\} \cong C_2 \times C_2 \times C_2.$$

This describes $\mathrm{QF}(C_2 \times C_2, k^*) \cong H^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$. Our aim is now to compute explicitly the abelian cocycles corresponding to the 32 quadratic forms.

4.2. Computation of the abelian cocycles

We will say that a 3-cocycle $\phi \in Z^3(C_2 \times C_2, k^*)$ is abelian if it is the underlying cocycle of an abelian cocycle $(\phi, \mathcal{R}) \in Z^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$, or, equivalently, if $\pi^{-1}(\phi) \neq \emptyset$, where $\pi\colon H^3_{\mathrm{ab}}(C_2 \times C_2, k^*) \to H^3(C_2 \times C_2, k^*)$ is induced by the projection on the first component.

Lemma 4.2. *For $b \in k^*$, g_b is an abelian 3-cocycle if and only if g_b is coboundary as a 3-cocycle.*

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Proof. Assume that g_b is an abelian 3-cocycle via a certain map $\mathcal{R}\colon G \times G \to k^*$. Taking $x = y = \sigma$ and $z = \tau$ in (2.2), we find that

$$\mathcal{R}(e,\tau)g_b(\sigma,\tau,\sigma) = g_b(\sigma,\sigma,\tau)\mathcal{R}(\sigma,\tau)^2 g_b(\tau,\sigma,\sigma).$$

By (3.23) we obtain $b = \mathcal{R}(\sigma,\tau)^2$, and so g_b is a coboundary 3-cocycle on $C_2 \times C_2$ (see Proposition 3.7). □

Our next aim is to compute $\pi^{-1}(\phi_\emptyset)$. This allows us to compute $\pi^{-1}(\phi)$ for every coboundary ϕ.

Proposition 4.3. *The subgroup* $\pi^{-1}(\phi_\emptyset)$ *of* $H^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$ *is isomorphic to* $C_2 \times C_2 \times C_2$. *Its elements are of the form* $[(1,\mathcal{R})]$, *with* $\mathcal{R}$ *given by the following data:*

$$\begin{array}{lll} \mathcal{R}(x,x) = \mu_x & \text{with } \mu_x^2 = 1 \text{ for all } x \in \{\sigma,\tau,\rho\}, & \\ \mathcal{R}(\sigma,\tau) = 1, & \mathcal{R}(\tau,\sigma) = \mu_\sigma\mu_\tau\mu_\rho, & \mathcal{R}(\sigma,\rho) = \mu_\sigma, \\ \mathcal{R}(\rho,\sigma) = \mu_\tau\mu_\rho, & \mathcal{R}(\tau,\rho) = \mu_\sigma\mu_\rho, & \mathcal{R}(\rho,\tau) = \mu_\tau. \end{array}$$

Moreover, $\mathrm{EM}(\pi^{-1}(\phi))$ *is the subgroup of* $\mathrm{QF}(C_2 \times C_2, k^*)$ *generated by* A, B *and* C.

Proof. Let $(1,\mathcal{R})$ be an abelian cocycle. The relations (2.2) and (2.3) reduce to

$$\mathcal{R}(xy,z) = \mathcal{R}(x,z)\mathcal{R}(y,z) \quad \text{and} \quad \mathcal{R}(x,yz) = \mathcal{R}(x,y)\mathcal{R}(x,z) \tag{4.2}$$

for all $x,y,z \in C_2 \times C_2$. This means that $\mathcal{R}$ is a bilinear map. Observe that

$$\mathrm{Bil}(C_2^2 \times C_2^2, k^*) \cong \mathrm{Hom}(C_2^2, (C_2^2)^*) \cong \mathrm{End}(C_2^2) = C_2^4.$$

Here $(C_2^2)^*$ is the character group of C_2^2, and the characters take values in $\{1,-1\}$. It follows that $\mathcal{R}$ takes values in $\{1,-1\}$, and there are 16 different maps for $\mathcal{R}$. These are completely determined by the values

$$\begin{array}{ll} \mathcal{R}(\sigma,\sigma) = \mu_\sigma, & \mathcal{R}(\tau,\tau) = \mu_\tau, \\ \mathcal{R}(\rho,\rho) = \mu_\rho, & \mathcal{R}(\sigma,\tau) = \alpha \in \{1,-1\}. \end{array}$$

Indeed, the other values of $\mathcal{R}$ follow from the bilinearity of $\mathcal{R}$:

$$\begin{aligned} \mathcal{R}(\sigma,\rho) &= \mathcal{R}(\sigma,\sigma)\mathcal{R}(\sigma,\tau) = \alpha\mu_\sigma, \\ \mathcal{R}(\rho,\tau) &= \mathcal{R}(\sigma,\tau)\mathcal{R}(\tau,\tau) = \alpha\mu_\tau, \\ \mathcal{R}(\rho,\sigma) &= \mathcal{R}(\rho,\tau)\mathcal{R}(\rho,\rho) = \alpha\mu_\tau\mu_\rho, \\ \mathcal{R}(\tau,\sigma) &= \mathcal{R}(\sigma,\sigma)\mathcal{R}(\rho,\sigma) = \alpha\mu_\sigma\mu_\tau\mu_\rho, \\ \mathcal{R}(\tau,\rho) &= \mathcal{R}(\sigma,\rho)\mathcal{R}(\rho,\rho) = \alpha\mu_\sigma\mu_\rho. \end{aligned}$$

Now let $\mathcal{R}$ and $\mathcal{R}_-$ be two bilinear forms that reach the same values at (σ,σ), (τ,τ) and (ρ,ρ), and assume that $\mathcal{R}(\sigma,\tau) = 1 = -\mathcal{R}_-(\sigma,\tau)$. Then $\mathcal{R}$ and $\mathcal{R}_-$ are cohomologous as abelian 3-cocycles on $C_2 \times C_2$. To see this, take $g\colon C_2 \times C_2 \to k^*$ as in (3.1) with

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Table 2. *The eight abelian* 3*-cocycles with trivial underlying* 3*-cocycle.*

$\mathrm{EM}(1,\mathcal{R})$	I	A	B	C	AB	AC	BC	ABC
$\mathcal{R}(\sigma,\sigma)$	1	1	1	−1	1	−1	−1	−1
$\mathcal{R}(\tau,\tau)$	1	1	−1	1	−1	1	−1	−1
$\mathcal{R}(\rho,\rho)$	1	−1	1	1	−1	−1	1	−1
$\mathcal{R}(\sigma,\tau)$	1	1	1	1	1	1	1	1
$\mathcal{R}(\tau,\sigma)$	1	−1	−1	−1	1	1	1	−1
$\mathcal{R}(\sigma,\rho)$	1	1	1	−1	1	−1	−1	−1
$\mathcal{R}(\rho,\sigma)$	1	−1	−1	1	1	−1	−1	1
$\mathcal{R}(\tau,\rho)$	1	−1	1	−1	−1	1	−1	1
$\mathcal{R}(\rho,\tau)$	1	1	−1	1	−1	1	−1	−1
Otherwise	1	1	1	1	1	1	1	1

$a_1 = a_2 = a_3 = b_4 = b_5 = b_6 = -1$ and $b_1 = b_2 = b_3 = c = 1$. Then one can easily verify that

$$\Delta_2(g) = 1$$

and

$$\mathcal{R}(x,y) = g(x,y)^{-1}g(y,x)\mathcal{R}_-(x,y) \quad \text{for all } x,y \in C_2 \times C_2.$$

It is easy to see that the images under EM of the eight bilinear forms that take the value 1 at (σ,τ) are I, A, B, C, AB, AC, BC and ABC. It then follows from the Eilenberg–Mac Lane theorem (Theorem 2.4) that these eight bilinear forms represent different cohomology classes in $H^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$. □

In Table 2 we present the eight abelian cocycles representing the elements of $\pi^{-1}(\phi)$. Note that the braidings associated to I, AB, AC and BC are symmetries on $\mathrm{Vect}^{C_2\times C_2}$. Indeed, by (4.2) it follows that

$$\mathcal{R}^{-1}(x,y) = \mathcal{R}(x^{-1},y) = \mathcal{R}(x,y) \quad \text{for all } x,y \in C_2 \times C_2,$$

so $\mathrm{Vect}^{C_2\times C_2}$ is symmetric monoidal if and only if $\mathcal{R}(x,y) = \mathcal{R}(y,x)$ for all $x,y \in C_2\times C_2$. Now Table 2 shows that only I, AB, AC and BC satisfy this condition.

We still have to compute the abelian 3-cocycles in $\pi^{-1}(\phi_X)$, with X a non-empty subset of $\{\sigma,\tau,\rho\}$.

Proposition 4.4. *Let $X \subseteq \{\sigma,\tau,\rho\}$ be a non-empty subset. Then ϕ_X is an abelian* 3*-cocycle if and only if it is even and k contains a primitive fourth root of* 1*.*

Proof. Assume that $(\phi_X,\mathcal{R})$ is abelian for a certain $\mathcal{R}$ and define $\mu_x := \mathcal{R}(x,x)$ for all $x \in \{\sigma,\tau,\rho\}$. Taking $x = y = z$ in (2.2) or (2.3), we get $\mu_x^2 = \varepsilon_x$ for all $x \in \{\sigma,\tau,\rho\}$. If we take $x = z \neq y$ in (2.2), we obtain

$$\mathcal{R}(xy,x) = \phi_X(x,y,x)\mu_x\mathcal{R}(y,x) \quad \text{for all } x \neq y$$

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from $\{\sigma,\tau,\rho\}$. Thus, according to (3.21) we have

$$\left.\begin{aligned}\mathcal{R}(\rho,\sigma)&=\mu_\sigma\mathcal{R}(\tau,\sigma),\\ \mathcal{R}(\tau,\rho)&=\mu_\rho\varepsilon_\rho\varepsilon_\tau\mathcal{R}(\sigma,\rho),\\ \mathcal{R}(\sigma,\tau)&=\mu_\tau\varepsilon_\sigma\varepsilon_\rho\mathcal{R}(\rho,\tau).\end{aligned}\right\}\tag{4.3}$$

We obtain the same relations if we take $y=z\neq x$ in (2.2). For $x=y\neq z$ in (2.2) we obtain $\phi_X(x,z,x)=\phi_X(x,x,z)\mathcal{R}(x,z)^2\phi_X(z,x,x)$, and therefore, by (3.21) we deduce that

$$\left.\begin{matrix}\mathcal{R}(\sigma,\tau)^2=1, & \mathcal{R}(\sigma,\rho)^2=\varepsilon_\sigma, & \mathcal{R}(\tau,\sigma)^2=p,\\ \mathcal{R}(\tau,\rho)^2=\varepsilon_\sigma\varepsilon_\rho, & \mathcal{R}(\rho,\sigma)^2=\varepsilon_\rho\varepsilon_\tau, & \mathcal{R}(\rho,\tau)^2=\varepsilon_\tau.\end{matrix}\right\}\tag{4.4}$$

Moving to (2.3), for $z=x\neq y$ we obtain $\phi_X(x,y,x)\mathcal{R}(x,xy)=\mu_x\mathcal{R}(x,y)$, so

$$\left.\begin{matrix}\mathcal{R}(\sigma,\rho)=\mu_\sigma\mathcal{R}(\sigma,\tau), & \varepsilon_\sigma\mathcal{R}(\sigma,\tau)=\mathcal{R}(\sigma,\rho)\mu_\sigma, & p\mathcal{R}(\tau,\rho)=\mathcal{R}(\tau,\sigma)\mu_\tau,\\ p\varepsilon_\tau\mathcal{R}(\tau,\sigma)=\mathcal{R}(\tau,\rho)\mu_\tau, & p\varepsilon_\sigma\mathcal{R}(\rho,\tau)=\mathcal{R}(\rho,\sigma)\mu_\rho, & \varepsilon_\tau\mathcal{R}(\rho,\sigma)=\mathcal{R}(\rho,\tau)\mu_\rho.\end{matrix}\right\}\tag{4.5}$$

The same relations are obtained if we take $x=y\neq z$ among $\{\sigma,\tau,\rho\}$ in (2.3), while for $y=z\neq x$ we obtain $\phi_X(x,y,y)\phi_X(y,y,x)=\mathcal{R}(x,y)^2\phi_X(y,x,y)$. This yields

$$\left.\begin{matrix}\mathcal{R}(\sigma,\tau)^2=p, & \mathcal{R}(\sigma,\rho)^2=\varepsilon_\tau\varepsilon_\rho, & \mathcal{R}(\tau,\sigma)^2=1,\\ \mathcal{R}(\tau,\rho)^2=\varepsilon_\tau, & \mathcal{R}(\rho,\sigma)^2=\varepsilon_\sigma, & \mathcal{R}(\rho,\tau)^2=\varepsilon_\sigma\varepsilon_\rho.\end{matrix}\right\}\tag{4.6}$$

From the first equalities in (4.4) and (4.6) it follows that $p=1$, and so ϕ_X is necessarily even. Also, there exists $x\in\{\sigma,\tau,\rho\}$ such that $\varepsilon_x=-1$, so the equation $\mu_x^2=-1$ has a solution in k, and k contains a primitive fourth root of 1.

Conversely, if ϕ_X is even and k contains a primitive fourth root of 1, then ϕ_X is completely determined by μ_x, $x\in\{\sigma,\tau,\rho\}$, and $\alpha:=\mathcal{R}(\sigma,\tau)\in\{\pm1\}$ (see the first relation in (4.6)). Actually, combining the relations in (4.3)–(4.6), we must have

$$\left.\begin{matrix} & \mathcal{R}(x,x)=\mu_x\quad\text{with }\mu_x^2=\varepsilon_x\text{ for all }x\in\{\sigma,\tau,\rho\}, & \\ \mathcal{R}(\sigma,\tau)=\alpha\in\{\pm1\}, & \mathcal{R}(\tau,\sigma)=\alpha\varepsilon_\rho\mu_\sigma\mu_\tau\mu_\rho, & \mathcal{R}(\sigma,\rho)=\alpha\mu_\sigma,\\ \mathcal{R}(\rho,\sigma)=\alpha\varepsilon_\tau\mu_\tau\mu_\rho, & \mathcal{R}(\tau,\rho)=\alpha\varepsilon_\sigma\mu_\sigma\mu_\rho, & \mathcal{R}(\rho,\tau)=\alpha\mu_\tau.\end{matrix}\right\}\tag{4.7}$$

Likewise, if $\mathcal{R}$ is defined by (4.7), then it can be easily checked that all the relations in (4.3)–(4.6) are satisfied, and so (2.2) and (2.3) are verified. □

Remark 4.5. Consider an abelian 3-cocycle $(\phi_X,\mathcal{R})$, and assume that the corresponding braided monoidal structure is symmetric. If $X\neq\emptyset$, then there exists $x\in X$ such that $\varepsilon_x=-1$ and $\mathcal{R}(x,x)=\mu_x=\pm\mathrm{i}$ by (4.7). Then $\mathcal{R}(x,x)^{-1}=-\mathcal{R}(x,x)$, contradicting the fact that the monoidal structure is symmetric. We conclude that $X=\emptyset$, and $\mathrm{Vect}^{C_2\times C_2}$ admits only four types of symmetric monoidal structures, namely the ones corresponding to the abelian 3-cocycles I, AB, AC and BC (see Table 2 for the description of these cocycles).

Table 3. *The abelian 3-cocycles with underlying 3-cocycle* $\phi_{\{\sigma,\tau\}}$.

$\mathrm{EM}(\phi_{\{\sigma,\tau\}},\mathcal{R})$	E_1	AE_1	BE_1	CE_1	ABE_1	ACE_1	BCE_1	$ABCE_1$
$\mathcal{R}(\sigma,\sigma)$	i	i	i	−i	i	−i	−i	−i
$\mathcal{R}(\tau,\tau)$	i	i	−i	i	−i	i	−i	−i
$\mathcal{R}(\rho,\rho)$	1	−1	1	1	−1	−1	1	−1
$\mathcal{R}(\sigma,\tau)$	1	1	1	1	1	1	1	1
$\mathcal{R}(\tau,\sigma)$	−1	1	1	1	−1	−1	−1	1
$\mathcal{R}(\sigma,\rho)$	i	i	i	−i	i	−i	−i	−i
$\mathcal{R}(\rho,\sigma)$	−i	i	i	−i	−i	i	i	−i
$\mathcal{R}(\tau,\rho)$	−i	i	−i	i	i	−i	i	−i
$\mathcal{R}(\rho,\tau)$	i	i	−i	i	−i	i	−i	−i
Otherwise	1	1	1	1	1	1	1	1

Table 4. *The abelian 3-cocycles with underlying 3-cocycle* $\phi_{\{\sigma,\rho\}}$.

$\mathrm{EM}(\phi_{\{\sigma,\rho\}},\mathcal{R})$	E_2	AE_2	BE_2	CE_2	ABE_2	ACE_2	BCE_2	$ABCE_2$
$\mathcal{R}(\sigma,\sigma)$	i	i	i	−i	i	−i	−i	−i
$\mathcal{R}(\tau,\tau)$	1	1	−1	1	−1	1	−1	−1
$\mathcal{R}(\rho,\rho)$	i	−i	i	i	−i	−i	i	−i
$\mathcal{R}(\sigma,\tau)$	1	1	1	1	1	1	1	1
$\mathcal{R}(\tau,\sigma)$	1	−1	−1	−1	1	1	1	−1
$\mathcal{R}(\sigma,\rho)$	i	i	i	−i	i	−i	−i	−i
$\mathcal{R}(\rho,\sigma)$	i	−i	−i	i	i	−i	−i	i
$\mathcal{R}(\tau,\rho)$	1	−1	1	−1	−1	1	−1	1
$\mathcal{R}(\rho,\tau)$	1	1	−1	1	−1	1	−1	−1
Otherwise	1	1	1	1	1	1	1	1

The abelian 3-cocycles in Proposition 4.4 with $\alpha = 1$ represent the same elements in $H^3_{\mathrm{ab}}(C_2 \times C_2, k^*)$ as the ones with $\alpha = -1$. For this, take g as in the proof of Proposition 4.3 to show that they are cohomologous as abelian 3-cocycles. The abelian 3-cocycles obtained from $\alpha = 1$ are not cohomologous because of the Eilenberg–Mac Lane theorem.

If k does not contain a primitive fourth root of unity, then $\pi^{-1}(\Phi_X) = \emptyset$ for $X \neq \emptyset$, and $H^3_{\mathrm{ab}}(C_2 \times C_2, k^*) \cong C_2 \times C_2 \times C_2$, as described in Proposition 4.3. Now assume that k contains a primitive fourth root of unity i. The inverse images under π of ϕ_X, $X = \{\sigma,\tau\}, \{\sigma,\rho\}, \{\tau,\rho\}$, each contain eight elements. Their explicit description is given in Proposition 4.4, and is summarized in Tables 3–5.

Remark 4.6. This remark is a continuation of Remark 3.11. It is easy to show that $Q\colon C_2 \times C_2 \to k^*$ is a quadratic form if and only if $Q(e) = 1$ and $Q(\alpha)^4 = 1$. Assuming that k has a primitive fourth root of 1, we have that $\mathrm{QF}(C_2, k^*) = C_4 \cong H^2_{\mathrm{ab}}(C_2, k^*)$.

Table 5. *The abelian 3-cocycles with underlying 3-cocycle* $\phi_{\{\tau,\rho\}}$.

$\mathrm{EM}(\phi_{\{\tau,\rho\}},\mathcal{R})$	E_3	AE_3	BE_3	CE_3	ABE_3	ACE_3	BCE_3	$ABCE_3$
$\mathcal{R}(\sigma,\sigma)$	1	1	1	−1	1	−1	−1	−1
$\mathcal{R}(\tau,\tau)$	i	i	−i	i	−i	i	−i	−i
$\mathcal{R}(\rho,\rho)$	i	−i	i	i	−i	−i	i	−i
$\mathcal{R}(\sigma,\tau)$	1	1	1	1	1	1	1	1
$\mathcal{R}(\tau,\sigma)$	1	−1	−1	−1	1	1	1	−1
$\mathcal{R}(\sigma,\rho)$	1	1	1	−1	1	−1	−1	−1
$\mathcal{R}(\rho,\sigma)$	1	−1	−1	1	1	−1	−1	1
$\mathcal{R}(\tau,\rho)$	i	−i	i	−i	−i	i	−i	i
$\mathcal{R}(\rho,\tau)$	i	i	−i	i	−i	i	−i	−i
Otherwise	1	1	1	1	1	1	1	1

The four cocycles in $H^2_{\mathrm{ab}}(C_2,k^*)$ are

$$(1,1),\qquad (1,\mathcal{R}_2),\qquad (\phi,\mathcal{R}_3),\qquad (\phi,\mathcal{R}_4),$$

with

$$\mathcal{R}_2=1-2P_\alpha\otimes P_\alpha,\qquad \mathcal{R}_3=1-(1-\mathrm{i})P_\alpha\otimes P_\alpha,\qquad \mathcal{R}_4=1-(1+\mathrm{i})P_\alpha\otimes P_\alpha.$$

$(1,1)$ and $(1,\mathcal{R}_2)$ give symmetries on Vect^{C_2}; the two others give braided non-symmetric monoidal structures. Now it is easy to calculate that

$$t_1(1,\mathcal{R}_2)=(1,AB),\qquad t_2(1,\mathcal{R}_2)=(1,AC),\qquad t_3(1,\mathcal{R}_2)=(1,BC),$$

and these are precisely the non-trivial symmetric abelian cocycles. In a similar way, we can compute that

$$\begin{aligned} t_1(\phi,\mathcal{R}_3)&=(\phi_{\{\tau,\rho\}},E_3), & t_1(\phi,\mathcal{R}_4)&=(\phi_{\{\tau,\rho\}},ABE_3),\\ t_2(\phi,\mathcal{R}_3)&=(\phi_{\{\sigma,\rho\}},E_2), & t_2(\phi,\mathcal{R}_4)&=(\phi_{\{\sigma,\rho\}},ACE_2). \end{aligned}$$

$t_3(\phi,\mathcal{R}_3)$ and $t_3(\phi,\mathcal{R}_4)$ are cohomologous to $(\phi_{\{\sigma,\tau\}},E_1)$ and $(\phi_{\{\sigma,\tau\}},ABE_1)$.

5. Some applications

5.1. Quasi-Hopf algebra structures

We will examine the following classification problem. Let H be a commutative, cocommutative, finite-dimensional Hopf algebra: in our case, let $H=k[C_n]$ or $k[C_2\times C_2]$, where C_n is the cyclic group of order n. Classify, up to gauge equivalence, all quasi-bialgebra structures on H. Recall briefly that a quasi-bialgebra is a unital associative algebra

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endowed with a coalgebra structure that is coassociative up to conjugation by a reassociator $\Phi \in H \otimes H \otimes H$. For a complete definition we invite the reader to consult [**17**,**19**]. Note that a quasi-bialgebra structure on a commutative Hopf algebra H is completely determined by such a reassociator, that is, a normalized Harrison 3-cocycle Φ on H. Furthermore, the quasi-bialgebra (H, Φ) is gauge equivalent to $(H, 1 \otimes 1 \otimes 1)$ if and only if Φ is a coboundary. Since every Harrison 3-cocycle is equivalent to a normalized 3-cocycle, it follows that our problem is equivalent to computing the third Harrison cohomology group $H^3_{\mathrm{Harr}}(H, k, \mathbb{G}_m)$. For the definition and generalities on Harrison cohomology, we refer the interested reader to [**8**, § 9.2].

If k contains a primitive nth root of unity (or, respectively, has characteristic not 2), then the Hopf algebras $k[C_n]$ and $k[C_2 \times C_2]$ are isomorphic to their dual Hopf algebras. This means that, for $G = C_n$ or $C_2 \times C_2$,

$$H^3_{\mathrm{Harr}}(k[G], k, \mathbb{G}_m) \cong H^3_{\mathrm{Harr}}(k[G]^*, k, \mathbb{G}_m) \cong H^3(G, k^*),$$

and we are reduced to computing group cohomology.

Proposition 5.1. *Let $C_n = \langle c \rangle$ be the cyclic group of order n written multiplicatively and let k be a field containing a primitive nth root of unit ξ. Then all the normalized Harrison 3-cocycles $\Phi \in k[C_n] \otimes k[C_n] \otimes k[C_n]$ are of the form*

$$\Phi_l = 1 - \frac{1}{n^2}(1 - c^l) \otimes \sum_{i,j=0}^{n-1} (1 - n\delta_{i,j})(\xi^j - n\delta_{j,0}) c^i \otimes c^j,$$

where $l \in \{0, 1, \dots, n-1\}$. In particular, if $n = 2$, there is a unique non-trivial 3-cocycle $\Phi_1 = 1 - 2p_- \otimes p_- \otimes p_-$, where $p_- = \frac{1}{2}(1 - c)$.

Proof. As mentioned in the introduction, we have a bijection between $H^3(C_n, k^*)$ and the nth roots of 1 in k. So we have a cocycle for every positive integer $l \in \{0, \dots, n-1\}$.

Since k contains a primitive nth root of unit ξ, we deduce that the characteristic of k does not divide n (this follows easily from $n = (1 - \xi)(1 - \xi^2) \cdots (1 - \xi^{n-1})$). Suppose $C_n = \langle c \rangle$, written multiplicatively, and let $\{P_e, P_c, \dots, P_{c^{n-1}}\}$ be the basis of $k[C_n]^* = k^{C_n}$ dual to the basis $\{e = 1, c, \dots, c^{n-1}\}$ of $k[C_n]$. Define $f \in k[C_n]^*$ as $f(c^i) = \xi^i$ for all $0 \leqslant i \leqslant n-1$. Then f is a well-defined algebra map and $f^j(c^s) = \xi^{js}$ for all $0 \leqslant s \leqslant n-1$. Furthermore, we know from [**9**, Exercise 4.3.6] that

$$\Psi \colon k[C_n] \ni c^j \mapsto f^j \in k[C_n]^*,$$

extended linearly, is a Hopf algebra isomorphism. Its inverse is defined by

$$\Psi^{-1}(P_{c^j}) = \frac{1}{n} \sum_{s=0}^{n-1} \xi^{(n-s)j} c^s \quad \text{for all } 0 \leqslant j \leqslant n-1.$$

We easily compute that

$$\begin{aligned}\sum_{j=0}^{n-1} q^j \Psi^{-1}(P_{c^j}) &= \frac{1}{n}\sum_{s=0}^{n-1}\left(\sum_{j=0}^{n-1}\xi^{lj}(\xi^{n-s})^j\right)c^s \\ &= \frac{1}{n}\sum_{s=0}^{n-1}\left(\sum_{j=0}^{n-1}(\xi^{l-s})^j\right)c^s \\ &= \frac{1}{n}\sum_{j=0}^{n-1} n\delta_{l,s}c^s \\ &= c^l.\end{aligned}$$

Thus, using the definition of ϕ_q from (1.1), we have that any normalized 3-cocycle $\Phi \in k[C_n]\otimes k[C_n]\otimes k[C_n]$ is of the form

$$\begin{aligned}\Phi_l &:= \sum_{u,v,s=0}^{n-1}\varphi_q(c^u,c^v,c^s)\Psi^{-1}(P_{c^u})\otimes\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) \\ &= \sum_{u=0}^{n-1}\Psi^{-1}(P_{c^u})\otimes\sum_{v+s<n}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) \\ &\quad+\sum_{u=1}^{n-1}q^u\Psi^{-1}(P_{c^u})\otimes\sum_{v+s\geqslant n}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) \\ &= 1\otimes\sum_{v+s<n}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) + c^l\otimes\sum_{v+s\geqslant n}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) \\ &= 1-(1-c^l)\otimes\sum_{v+s\geqslant n}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^s}) \\ &= 1-(1-c^l)\otimes\sum_{v=1}^{n-1}\sum_{t=0}^{n-2}\Psi^{-1}(P_{c^v})\otimes\Psi^{-1}(P_{c^{n+t-v}}) \\ &= 1-\frac{1}{n^2}(1-c^l)\otimes\sum_{i,j=0}^{n-1}\sum_{v=1}^{n-1}\sum_{t=0}^{n-2}\xi^{(n-i)v+(n-j)(n+t-v)}c^i\otimes c^j \\ &= 1-\frac{1}{n^2}(1-c^l)\otimes\sum_{i,j=0}^{n-1}\left(\sum_{v=1}^{n-1}(\xi^{j-i})^v\right)c^i\otimes\left(\sum_{t=0}^{n-2}(\xi^{-j})^t\right)c^j \\ &= 1-\frac{1}{n^2}(1-c^l)\otimes\sum_{i,j=0}^{n-1}(1-n\delta_{i,j})(\xi^j-n\delta_{j,0})c^i\otimes c^j,\end{aligned}$$

as stated. In the case where $n=2$, we calculate

$$\begin{aligned}\sum_{i,j=0}^{n}(1-2\delta_{i,j})((-1)^j-2\delta_{j,0})c^i\otimes c^j &= 1\otimes 1-1\otimes c-c\otimes 1+c\otimes c \\ &= (1-c)\otimes(1-c)\end{aligned}$$

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to deduce that $\Phi_1 = 1 - 2p_- \otimes p_- \otimes p_-$, as claimed. Note that it is precisely the 3-cocycle that confers to $k[C_2]$ the unique quasi-bialgebra structure that is not twist equivalent to a Hopf algebra [**14**]. $\square$

Our next aim is to describe the cocycles in

$$H^3_{\mathrm{Harr}}(k[C_2 \times C_2], k, \mathbb{G}_m) \cong H^3(C_2 \times C_2, k^*)$$

more explicitly. We use the notation of the previous sections: $C_2 \times C_2 = \{e, \sigma, \tau, \rho\}$, with $\sigma\tau = \rho$. There are three Hopf algebra maps $k[C_2] \to k[C_2 \times C_2]$, so we immediately find three Harrison 3-cocycles $\Phi_x = 1 - 2p^x_- \otimes p^x_- \otimes p^x_-$, $x = \sigma, \tau, \rho$, where $p^x_- = \frac{1}{2}(1 - x)$.

One of the isomorphisms

$$\Psi\colon k[C_2 \times C_2]^* \cong k[C_2]^* \otimes k[C_2]^* \to k[C_2] \otimes k[C_2] \cong k[C_2 \times C_2]$$

is the following:

$$\begin{aligned} \Psi(P_e) &= u_e = \tfrac{1}{4}(e + \sigma + \tau + \rho), & \Psi(P_\sigma) &= u_\sigma = \tfrac{1}{4}(e - \sigma + \tau - \rho), \\ \Psi(P_\tau) &= u_\tau = \tfrac{1}{4}(e + \sigma - \tau - \rho), & \Psi(P_\rho) &= u_\rho = \tfrac{1}{4}(e - \sigma - \tau + \rho). \end{aligned}$$

We can use this isomorphism to write down the Harrison cocycles in the basis $\{u_e, u_\sigma, u_\tau, u_\rho\}$. It is then possible to write the cocycles as sums of monomials, but this gives long formulae. Some of the cocycles can be written down elegantly. Observing that

$$X_\sigma + X_\rho + X_{\rho,\sigma} + X_{\sigma,\rho} = (P_\sigma + P_\rho) \otimes (P_\sigma + P_\rho) \otimes (P_\sigma + P_\rho) \tag{5.1}$$

and

$$\Psi(P_\sigma + P_\rho) = u_\sigma + u_\rho = \tfrac{1}{2}(e - \sigma), \tag{5.2}$$

we see that

$$\Psi(\phi_{\{\sigma,\rho\}}) = 1 - 2p^\sigma_- \otimes p^\sigma_- \otimes p^\sigma_- = \Phi_\sigma.$$

In a similar way, we can show that $\Psi(\phi_{\{\sigma,\rho\}}) = \Phi_\tau$ and $\Psi(h_{-1}g_{-1}\phi_{\{\sigma,\tau\}}) = \Phi_\rho$. If -1 has a square root in k, then it follows that Φ_ρ is cohomologous to $\Phi_\sigma\Phi_\tau$. Note that these observations are consistent with Remark 3.11.

5.2. Weak braided Hopf algebra structures

The definition of a weak Hopf algebra can be found in [**5**]. For the definition of a weak braided Hopf algebra in a symmetric monoidal category, we refer the interested reader to [**4, 6**]. We recall the following construction of weak braided Hopf algebras in Vect^G from [**6**].

Let $F\colon G \times G \to k^*$ be a normalized 2-cochain (see Lemma 2.1), with pointwise inverse F^{-1}. Consider

$$\mathcal{R}_{F^{-1}}\colon G \times G \to k^*, \qquad \mathcal{R}_{F^{-1}}(x, y) = F(x, y)F(y, x)^{-1}.$$

Then

$$(\phi_{F^{-1}} = \Delta_2(F^{-1}), \mathcal{R}_{F^{-1}})$$

is a coboundary abelian 3-cocycle on G. Let $\mathrm{Vect}^G_{F^{-1}}$ be category Vect^G equipped with the braided monoidal structure provided by $(\Delta_2(F^{-1}), \mathcal{R}_{F^{-1}})$.

Let $k_F[G]$ be the k-vector space $k[G]$ with multiplication given by the formula

$$x \bullet y = F(x, y)xy$$

for all $x, y \in G$. $k_F[G]$ is a commutative algebra in $\mathrm{Vect}^G_{F^{-1}}$ (see [**1**, Corollary 2.4]).

If $|G| \neq 0$ in k, then we can define a cocommutative coalgebra structure on $k[G]$ in $\mathrm{Vect}^G_{F^{-1}}$. $k^F[G]$, the k-vector space $k[G]$ with comultiplication and counit given by

$$\Delta_F(x) = \frac{1}{|G|}\sum_{u\in G} F(u, u^{-1}x)^{-1} u \otimes u^{-1}x \quad \text{and} \quad \varepsilon_F(x) = |G|\delta_{x,e} \quad \text{for all } x \in G,$$

is a cocommutative coalgebra in $\mathrm{Vect}^G_{F^{-1}}$ (see [**6**, Proposition 3.2]). $k^F_F[G]$, the vector space equipped with the algebra and coalgebra structure defined above, is a commutative and cocommutative weak braided Hopf algebra in $\mathrm{Vect}^G_{F^{-1}}$ (see [**6**, Proposition 4.7]). The antipode $\underline{S}$ is the identity on $k[G]$.

Examples of such braided Hopf algebras are the Cayley–Dickson and Clifford algebras [**6**, **7**]. Moreover, they are monoidal Frobenius algebras and monoidal co-Frobenius coalgebras in the appropriate braided monoidal category of graded vector spaces.

We will now apply this construction to the case where $G = C_n$ and $G = C_2 \times C_2$, in order to construct more examples of weak braided Hopf algebras. We begin with a generalization of [**2**, Corollary 10], where it is shown that the map ϕ in Lemma 5.2 is a 3-cocycle and a coboundary in the case where $n = 3$.

Lemma 5.2. *Let $C_n = \langle\sigma\rangle$ be the cyclic group of order n, let k be a field and let q be an nth root of* 1*. Then $\phi(\sigma^a, \sigma^b, \sigma^c) := q^{abc}$ is a normalized* 3*-cocycle on C_n. ϕ is a coboundary if and only if $q^{n(n-1)/2} = 1$.*

Proof. The fact that $q^n = 1$ implies that ϕ is well defined. Indeed, if $a = a' + n$, $b = b' + n$ and $c = c' + n$, then

$$abc = n^3 + (a' + b' + c')n^2 + (a'b' + a'c' + b'c')n + a'b'c',$$

and so

$$\phi(\sigma^a, \sigma^b, \sigma^c) = q^{abc} = q^{a'b'c'} = \phi(\sigma^{a'}, \sigma^{b'}, \sigma^{c'}).$$

The 3-cocycle condition reduces to $bcd + a(b + c)d + abc = ab(c + d) + (a + b)cd$ in $\mathbb{Z}$, which is clearly satisfied. It is also clear that ϕ is normalized.

If ϕ is a coboundary, then there exists $g \colon C_n \times C_n \to k^*$ such that

$$g(\sigma^b, \sigma^c)g(\sigma^{a+b}, \sigma^c)^{-1}g(\sigma^a, \sigma^{b+c})g(\sigma^a, \sigma^b)^{-1} = q^{abc} \quad \text{for all } a, b, c \in \mathbb{Z}. \tag{5.3}$$

Let $\beta := g(\sigma^a, 1) = g(1, \sigma^b)$ and $\alpha_c := g(\sigma, \sigma^c)$ for all $a, b, c \in \{1, \dots, n - 1\}$. Taking $a = 1$, $b = k$ and $c = n - 1$ in (5.3), we obtain

$$g(\sigma^{k+1}, \sigma^{n-1}) = q^k g(\sigma^k, \sigma^{n-1})g(\sigma, \sigma^{k-1})g(\sigma, \sigma^k)^{-1} \quad \text{for all } k \in \mathbb{Z}.$$

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By mathematical induction, it follows that

$$g(\sigma^k, \sigma^{n-1}) = q^{k(k-1)/2}\alpha_{n-1}\beta\alpha_{k-1}^{-1} \quad \text{for all } 2 \leqslant k \leqslant n-1.$$

We then have

$$\begin{aligned}\beta = g(1, \sigma^{n-1}) &= g(\sigma^{(n-1)+1}, \sigma^{n-1}) \\ &= q^{n-1}g(\sigma^{n-1}, \sigma^{n-1})g(\sigma, \sigma^{n-2})g(\sigma, \sigma^{n-1})^{-1} \\ &= q^{n-1}q^{(n-1)(n-2)/2}\alpha_{n-1}\beta\alpha_{n-2}^{-1}\alpha_{n-2}\alpha_{n-1}^{-1} \\ &= q^{n(n-1)/2}\beta,\end{aligned}$$

and we conclude that $q^{n(n-1)/2} = 1$.

Conversely, assume that $q^{n(n-1)/2} = 1$ and consider

$$\begin{aligned} &f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \qquad f(x, y) = -\tfrac{1}{2}((x-1)xy), \\ &g\colon C_n \times C_n \to k, \qquad g(\sigma^a, \sigma^b) = q^{f(a,b)}.\end{aligned}$$

If $a = a' + n$ and $b = b' + n$, then it can be easily checked that

$$f(a, b) - f(a', b') = -a'n^2 - (\tfrac{1}{2}a'(a'-1) + a'b')n - \tfrac{1}{2}n^2(n-1) - \tfrac{1}{2}n(n-1)b'.$$

This relation, together with $q^n = 1$ and $q^{n(n-1)/2} = 1$, implies that g is well defined. A straightforward computation now shows that

$$f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y) = xyz \quad \text{for all } x, y, z \in \mathbb{Z},$$

proving that $\Delta_2(g) = \phi$, and ϕ is a coboundary. □

Proposition 5.3. *Let k be a field and let q be an nth root of 1 such that $q^{n(n-1)/2} = 1$ (which is automatic if n is odd). If $C_n = \langle\sigma\rangle$ is the cyclic group of order n generated by σ and $F\colon C_n \times C_n \to k^*$ is given by $F(\sigma^a, \sigma^b) = q^{-(a-1)ab/2}$ for all $0 \leqslant a, b \leqslant n-1$, then $k[C_n]$ is a commutative and cocommutative weak braided Hopf algebra in $\mathrm{Vect}_{F^{-1}}^{C_n}$ via the structure*

$$\sigma^a \bullet \sigma^b = q^{-(a-1)ab/2}\sigma^{a+b}, \qquad \underline{\Delta}(\sigma^a) = \frac{1}{n}\sum_{l=0}^{n-1} q^{(l-1)l(a-l)}\sigma^l \otimes \sigma^{a-l},$$

for all $0 \leqslant a, b \leqslant n-1$. The unit is e and the counit is given by $\underline{\varepsilon}(\sigma^a) = n\delta_{a,0}$, for all $0 \leqslant a \leqslant n-1$; the antipode is the identity on $k[C_n]$.

Proof. This follows immediately from Lemma 5.2 and the general construction of $k_F^F[G]$ presented above. We leave the verification of the details to the reader. □

The 2-cochains discussed in Proposition 3.7 can also be applied to construct examples of weak braided Hopf algebras.

Proposition 5.4. *On $k[C_2 \times C_2]$ we have the following commutative and cocommutative weak braided Hopf algebra structure.*

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(i) *The multiplication table is as follows ($a \in k^*$ is a fixed scalar):*

$\bullet$	e	σ	τ	ρ
e	e	σ	τ	ρ
σ	σ	$a^{-1}e$	ρ	τ
τ	τ	ρ	$a^{-1}e$	σ
ρ	ρ	τ	σ	$a^{-1}e$

The comultiplication is given by the formulae

$$\Delta(e) = \tfrac{1}{4}(e \otimes e + a\sigma \otimes \sigma + a\tau \otimes \tau + a\rho \otimes \rho),$$
$$\Delta(\sigma) = \tfrac{1}{4}(e \otimes \sigma + \sigma \otimes e + \tau \otimes \rho + \rho \otimes \tau),$$
$$\Delta(\tau) = \tfrac{1}{4}(e \otimes \tau + \tau \otimes e + \sigma \otimes \rho + \rho \otimes \sigma),$$
$$\Delta(\rho) = \tfrac{1}{4}(e \otimes \rho + \rho \otimes e + \sigma \otimes \tau + \tau \otimes \sigma),$$

while the counit is given by $\varepsilon(e) = 4$ and $\varepsilon(x) = 0$ for $x \in \{\sigma, \tau, \rho\}$. The antipode is the identity on $k[C_2 \times C_2]$.

(ii) *For any $d \in k^*$, the multiplication table is*

$\bullet$	e	σ	τ	ρ
e	e	σ	τ	ρ
σ	σ	$d^{-1}e$	ρ	$d^{-1}\tau$
τ	τ	$d^{-1}\rho$	$d^{-1}e$	σ
ρ	ρ	τ	$d^{-1}\sigma$	$d^{-1}e$

The comultiplication is given by the formulae

$$\Delta(e) = \tfrac{1}{4}(e \otimes e + d\sigma \otimes \sigma + d\tau \otimes \tau + d\rho \otimes \rho),$$
$$\Delta(\sigma) = \tfrac{1}{4}(e \otimes \sigma + \sigma \otimes e + \tau \otimes \rho + d\rho \otimes \tau),$$
$$\Delta(\tau) = \tfrac{1}{4}(e \otimes \tau + \tau \otimes e + d\sigma \otimes \rho + \rho \otimes \sigma),$$
$$\Delta(\rho) = \tfrac{1}{4}(e \otimes \rho + \rho \otimes e + \sigma \otimes \tau + d\tau \otimes \sigma),$$

while the counit $\varepsilon(e) = 4$ and $\varepsilon(x) = 0$ for all $x \in \{\sigma, \tau, \rho\}$ makes $k[C_2 \times C_2]$ a weak braided Hopf algebra. The antipode is the identity on $k[C_2 \times C_2]$.

Proof. (i) We use the fact that h_a is a coboundary for all $a \in k^*$. Actually, if we take $g\colon C_2 \times C_2 \to k^*$ defined by $b_i = 1$ for all $1 \leqslant i \leqslant 6$, $a_1 = a_2 = a_3 = a$ and $c = 1$, then we have seen that $\Delta_2(g) = h_a$. Consequently, g is a 2-cochain on C_2. A simple inspection shows us that the claimed structure on $k[C_2 \times C_2]$ from (i) coincides with that of $k_{g^{-1}}^{g^{-1}}[C_2 \times C_2]$, so we are done. Note that we have a weak braided Hopf algebra in $\mathrm{Vect}_g^{C_2 \times C_2}$, and that $\Delta_2(g) = h_a$, while $\mathcal{R}_g(x, y) = 1$ for all $x, y \in C_2 \times C_2$.

(ii) We proceed as above, but now we take $b = d^2$ for some $d \in k^*$. Then g_d is coboundary (see Proposition 3.7). More precisely, if $g\colon C_2 \times C_2 \to k^*$ is defined as

$$a_1 = a_2 = a_3 = b_4 = b_5 = b_6 = d \quad \text{and} \quad b_1 = b_2 = b_3 = c = 1,$$

then $\Delta_2(g) = g_b = g_{d^2}$. We leave it to the reader to check that the second weak braided Hopf algebra structure from the statement is precisely the one on $k_{g^{-1}}^{g^{-1}}[C_2 \times C_2]$. We only note that, in this case, we have the braided monoidal structure on $\mathrm{Vect}^{C_2 \times C_2}$ produced by g, that is, $\Delta_2(g) = g_{d^2}$ and

$$\begin{aligned} &\mathcal{R}_g(x,x) = \mathcal{R}_g(e,y) = \mathcal{R}_g(z,e) = 1, \quad \forall x,y,z \in C_2 \times C_2,\\ &\mathcal{R}_g(\sigma,\tau) = \mathcal{R}_g(\tau,\rho) = \mathcal{R}_g(\rho,\sigma) = d,\\ &\mathcal{R}_g(\tau,\sigma) = \mathcal{R}_g(\sigma,\rho) = \mathcal{R}_g(\rho,\tau) = d^{-1}. \end{aligned}$$

□

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