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Abstract

The Dieudonné crystal of a p-divisible group over a semiperfect ring R can be endowed with a window structure. If R satisfies a boundedness condition, this construction gives an equivalence of categories. As an application we obtain a classification of p-divisible groups and commutative finite locally free p-group schemes over perfectoid rings by Breuil–Kisin–Fargues modules if $p \geqslant 3$.

Contents

1. Introduction

Let p be a prime. A semiperfect ring is an $\mathbb{F}_p$-algebra R such that the Frobenius endomorphism $\phi_R : R \to R$ is surjective. In the first part of this article we study the classification of p-divisible groups over semiperfect rings by Dieudonné crystals and related objects. This was initiated in [SW13]. In the second part we draw conclusions for perfectoid rings.

1.1 Crystalline Dieudonné windows

Every semiperfect ring R has a universal p-adic divided power extension $A_{\mathrm{cris}}(R)$. By a lemma of [SW13], this ring carries a natural structure of a frame $\underline{A}_{\mathrm{cris}}(R)$, which means that the Frobenius of $A_{\mathrm{cris}}(R)$ is divided by p on the kernel of $A_{\mathrm{cris}}(R) \to R$. This is not clear *a priori* because in general $A_{\mathrm{cris}}(R)$ has p-torsion.

The following result has been suggested in [SW13].

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Theorem 1.1. *Let R be a semiperfect ring. There is a natural functor*

$$\Phi_R^{\mathrm{cris}} : \mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R))$$

from p-divisible groups over R to windows over $\underline{A}_{\mathrm{cris}}(R)$, such that the underlying module of $\Phi_R^{\mathrm{cris}}(G)$ is given by the Dieudonné crystal of G.

See Theorem 6.3. The functor Φ_R^{cris} is a variant of the functor Φ_R of [Lau13] from p-divisible groups to displays for an arbitrary p-adic ring R, and of the functor Φ_R of [Lau14] from p-divisible groups to Dieudonné displays for a local Artin ring R with perfect residue field.

Our main result on the functor Φ_R^{cris} depends on the following boundedness condition. We call R balanced if $\operatorname{Ker}(\phi_R)^p = 0$, and we call R iso-balanced if there is a nilpotent ideal $\mathfrak{a} \subseteq R$ such that $R/\mathfrak{a}$ is balanced. Every f-semiperfect ring in the sense of [SW13] is iso-balanced.

Theorem 1.2. *If R is iso-balanced, the functor Φ_R^{cris} is an equivalence.*

See Theorem 7.10. Theorem 1.2 implies that for iso-balanced semiperfect rings the crystalline Dieudonné functor

$$\mathbb{D}_R : \mathrm{BT}(\operatorname{Spec} R) \to (\text{Dieudonné crystals over } \operatorname{Spec} R)$$

is fully faithful up to isogeny. When R is f-semiperfect, this is proved in [SW13] using perfectoid spaces.

Assume that R is a complete intersection in the sense that R is the quotient of a perfect ring by a regular sequence. Then $A_{\mathrm{cris}}(R)$ is p-torsion free, and windows over $\underline{A}_{\mathrm{cris}}(R)$ are equivalent to Dieudonné crystals over $\operatorname{Spec} R$ with an admissible filtration in the sense of [Gro74]; this filtration is unique if it exists. Thus for complete intersections, Theorem 1.2 means that the functor $\mathbb{D}_R$ is fully faithful and that its essential image consists of those Dieudonné crystals which admit an admissible filtration. Full faithfulness is already proved in [SW13] as an easy consequence of full faithfulness up to isogeny.

For a general semiperfect ring, $A_{\mathrm{cris}}(R)$ can have p-torsion, and the functor $\mathbb{D}_R$ cannot be expected to be fully faithful. The phenomenon that passing from Dieudonné modules to windows can compensate for this failure is familiar from the classification of formal p-divisible groups over arbitrary p-adic rings by nilpotent displays, and from the classification of arbitrary p-divisible groups over local Artin rings by Dieudonné displays.

1.2 Dieudonné modules via lifts

The proof of Theorem 1.2 relies on another construction of Dieudonné modules, which is independent of the functors Φ_R^{cris}. A lift of an $\mathbb{F}_p$-algebra R is a p-adically complete and p-torsion free ring A with $A/pA = R$ and with a Frobenius lift $\sigma : A \to A$. Then there is an evident frame structure $\underline{A}$ and a functor

$$\Phi_A : \mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\underline{A}).$$

Here $\underline{A}$-windows are equivalent to locally free Dieudonné modules over A in the usual sense. The functor Φ_A also induces a functor Φ_A^{tor} from commutative finite locally free p-group schemes over R to p-torsion Dieudonné modules over A which are of projective dimension less than or equal to 1 as A-modules. In general the properties of Φ_A depend on the lift.

Theorem 1.3. *If R is a complete intersection or balanced semiperfect ring, there is a lift A of R such that the functors Φ_A and Φ_A^{tor} are equivalences.*

See Theorem 5.7 and Corollary 10.14. When R is perfect, then $A = W(R)$ is the unique lift of R, and Theorem 1.3 holds by a result of Gabber. The general case is reduced to the perfect case by a specialization argument along $R^\flat \to R$, where $R^\flat$ is the limit perfection of R.

We note that for an arbitrary $\mathbb{F}_p$-algebra R with a lift (A, σ) the functor Φ_A gives an equivalence between formal p-divisible groups and nilpotent windows by [Zin01] and the extensions of [Zin02] provided by [Lau08, Lau13]. So the new aspect of Theorem 1.3 is that it applies to all p-divisible groups.

The functors Φ_R^{cris} and Φ_A are related as follows. For every lift A of a semiperfect ring R there is a natural homomorphism of frames

$$\varkappa : \underline{A}_{\mathrm{cris}}(R) \to \underline{A},$$

and the base change under $\varkappa$ of $\Phi_R^{\mathrm{cris}}(G)$ coincides with $\Phi_A(G)$.

Lemma 1.4. *If R is a complete intersection or balanced semiperfect ring, there is a lift A of R as in Theorem 1.3 such that $\varkappa$ induces an equivalence of the window categories.*

See Proposition 5.10. Theorem 1.3 and Lemma 1.4 give Theorem 1.2 when R is balanced or a complete intersection, and the general case follows by a deformation argument, using a weak version of lifts for iso-balanced rings, for which an analogue of Lemma 1.4 holds; see Proposition 7.8.

1.3 Breuil–Kisin–Fargues modules

Now let R be a perfectoid ring in the sense of [BMS16]. This class of rings includes all perfect rings and all bounded open integrally closed subrings of perfectoid Tate rings in the sense of [Fon13]. Let $R^\flat$ be the tilt of R, which is a perfect ring, and $A_{\mathrm{inf}} = W(R^\flat)$.

The kernel of the natural homomorphism $\theta : A_{\mathrm{inf}} \to R$ is generated by a non-zero divisor ξ. In the following, a Breuil–Kisin–Fargues module for R is a finite projective A_{inf}-module $\mathfrak{M}$ with a linear map $\varphi : \mathfrak{M}^\sigma \to \mathfrak{M}$ whose cokernel is annihilated by ξ.[1] As an application of Theorem 1.2 we obtain the following result.[2]

Theorem 1.5. *If $p \geqslant 3$, for each perfectoid ring R the category $\mathrm{BT}(\operatorname{Spec} R)$ is equivalent to the category of Breuil–Kisin–Fargues modules for R.*

See Theorem 9.8. When $R = \mathcal{O}_C$ for an algebraically closed perfectoid field C, the result is due to Fargues [Far15, Far13]. Theorem 1.5 is a variant of the classical equivalence between p-divisible groups over a mixed characteristic complete discrete valuation ring with perfect residue field and Breuil–Kisin modules.

To prove Theorem 1.5 we consider the ring R/p, which is semiperfect and balanced. The universal p-adic divided power extension $A_{\mathrm{cris}}(R)$ coincides with $A_{\mathrm{cris}}(R/p)$ as a ring and carries a natural frame structure. The equivalence of Theorem 1.2 for R/p (which is covered by Theorem 1.3 and Lemma 1.4 in this case) extends for $p \geqslant 3$ to an equivalence

$$\mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)).$$

[1] In general these modules should be called *minuscule* Breuil–Kisin–Fargues modules, but since other Breuil–Kisin–Fargues modules do not appear in this text, for simplicity we omit 'minuscule'.

[2] A different proof of Theorem 1.5, which also holds for $p = 2$, was given recently in [SW17, Theorem 17.5.2].

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Moreover there is a base change functor

$$(\text{Breuil–Kisin–Fargues modules for } R) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)),$$

which is an equivalence for $p \geqslant 3$ by a descent from A_{cris} to A_{inf} that generalizes the 'descent from S to $\mathfrak{S}$' used in the classical case. Theorem 1.5 follows. One can expect that Theorem 1.5 also holds for $p = 2$, but the present proof does not extend to that case directly.

As in the classical case, Theorem 1.5 induces a similar result for finite group schemes. Namely, a torsion Breuil–Kisin–Fargues module for R is a triple $(\mathfrak{M}, \varphi, \psi)$ where $\mathfrak{M}$ is a p-torsion finitely presented A_{inf}-module of projective dimension less than or equal to 1 with linear maps

$$\xi A_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathfrak{M} \xrightarrow{\psi} \mathfrak{M}^{\sigma} \xrightarrow{\varphi} \mathfrak{M}$$

such that $\varphi \circ \psi$ and $\psi \circ (1 \otimes \varphi)$ are the multiplication maps. If R is p-torsion free then ξ is $\mathfrak{M}$-regular and ψ is determined by φ.

COROLLARY 1.6 (Theorem 10.12). *If $p \geqslant 3$, for each perfectoid ring R the category of commutative finite locally free p-group schemes over R is equivalent to the category of torsion Breuil–Kisin–Fargues modules for R.*

2. Notation

We fix a prime p.

An abelian group A is called p-adically complete if $A \cong \varprojlim_n A/p^n A$.

A PD extension is a surjective ring homomorphism whose kernel is equipped with divided powers. A p-adic PD extension is a PD extension of p-adically complete rings such that the divided powers are compatible with the divided powers on $p\mathbb{Z}_p$. Divided powers γ are also denoted by $\gamma_n(x) = x^{[n]}$.

Following [Lau10], a frame $\underline{S} = (S, \mathrm{Fil}\, S, R, \sigma, \sigma_1)$ consists of rings S and $R = S/\mathrm{Fil}\, S$ such that $pS + \mathrm{Fil}\, S \subseteq \mathrm{Rad}\, S$, together with a Frobenius lift $\sigma : S \to S$ and a σ-linear map $\sigma_1 : \mathrm{Fil}\, S \to S$ whose image generates the unit ideal.[3] A window over the frame $\underline{S}$ is a collection $\underline{M} = (M, \mathrm{Fil}\, M, F, F_1)$ where M is a finite projective S-module, $\mathrm{Fil}\, M \subseteq M$ is a submodule which takes the form $\mathrm{Fil}\, M = L \oplus (\mathrm{Fil}\, S)T$ for some decomposition $M = L \oplus T$, and $F : M \to M$ and $F_1 : \mathrm{Fil}\, M \to M$ are σ-linear maps such that the image of F_1 generates M, and $F_1(ax) = \sigma_1(a)F(x)$ for $a \in \mathrm{Fil}\, S$ and $x \in M$. We denote by $\mathrm{Win}(\underline{S})$ the category of windows over $\underline{S}$. A frame homomorphism $\alpha : \underline{S} \to \underline{S}'$ is a ring homomorphism $S \to S'$ with $\mathrm{Fil}\, S \to \mathrm{Fil}\, S'$ such that $\sigma'\alpha = \alpha\sigma$ and $\sigma'_1\alpha = u \cdot \alpha\sigma_1$ for a unit $u \in S'$. There is a base change functor $\alpha^* : \mathrm{Win}(\underline{S}) \to \mathrm{Win}(\underline{S}')$. If this functor is an equivalence, α is called crystalline.

A frame $\underline{S}$ is called a p-frame if $p\sigma_1 = \sigma$ on $\mathrm{Fil}\, S$, i.e. in the notation of [Lau10, Lemma 2.2] we have $\theta = p$. A PD frame is a p-frame $\underline{S}$ where $S \to R$ is a p-adic PD extension such that σ preserves the resulting divided powers on the ideal $\mathrm{Fil}\, S + pS$. If in addition S is p-torsion free, then (S, σ) is a frame for R in the sense of [Zin01].

[3] Form a systematic perspective, it would be better to drop this condition; see for example [CL14, § 2.1]. The condition is satisfied for all frames considered in this article.

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3. Dieudonné crystals and modules

In this section we fix notation and recall some standard results.

For a scheme X on which p is nilpotent, or more generally a p-adic formal scheme, let $\mathrm{BT}(X)$ be the category of p-divisible groups over X, let $\mathrm{D}(X)$ be the category of locally free Dieudonné crystals over X, and let $\mathrm{DF}(X)$ be the category of locally free Dieudonné crystals $\mathcal{M}$ over X equipped with an admissible filtration $\operatorname{Fil}\mathcal{M}_X \subseteq \mathcal{M}_X$ as in [Gro74]; see [CL14, Definition 2.4.1]. Let

$$\mathbb{D}_X : \mathrm{BT}(X) \to \mathrm{D}(X) \tag{3.1}$$

be the contravariant crystalline Dieudonné functor defined in [MM74] and in [BBM82], and let

$$\mathbb{D}\mathrm{F}_X : \mathrm{BT}(X) \to \mathrm{DF}(X) \tag{3.2}$$

be its extension defined by the Hodge filtration; see [CL14, Proposition 2.4.3]. If $\underline{S} = (S, \operatorname{Fil} S, R, \sigma, \sigma_1)$ is a p-torsion free PD frame as in § 2, the evaluation of the filtered Dieudonné crystal at $\underline{S}$ gives a contravariant functor

$$\Phi_{\underline{S}} : \mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\underline{S}), \quad G \mapsto (M, \operatorname{Fil} M, F, F_1), \tag{3.3}$$

where $M = \mathbb{D}(G)_S$, the submodule $\operatorname{Fil} M \subseteq M$ is the inverse image of the Hodge filtration $\operatorname{Lie}(G)^* \subseteq \mathbb{D}(G)_R$ of G, F is induced by the Frobenius of G, and $F_1 = p^{-1}F$ on $\operatorname{Fil} M$; see [Lau14, Proposition 3.17] or [CL14, Proposition 2.5.2].

3.1 Explicit Dieudonné modules

Let R be an $\mathbb{F}_p$-algebra. A lift of R is a pair (A, σ) where A is a p-adically complete and p-torsion free ring with $R = A/pA$, and $\sigma : A \to A$ is a Frobenius lift.

In the following let (A, σ) be a lift of R. A (locally free) Dieudonné module over A is a triple $\underline{M} = (M, \varphi, \psi)$ where M a finite projective A-module and $\varphi : M^\sigma \to M$ and $\psi : M \to M^\sigma$ are linear maps with $\varphi\psi = p$ and $\psi\varphi = p$, where $M^\sigma = M \otimes_{A,\sigma} A$. We write $\mathrm{DM}(A)$ for the category of Dieudonné modules over A.

Lemma 3.1. *For $(M, \varphi, \psi) \in \mathrm{DM}(A)$ the R-module $\operatorname{Coker}(\varphi)$ is projective.*

Proof. Let $\bar{M} = M \otimes_A R$. There is an exact sequence of finite projective R-modules

$$\bar{M}^\sigma \xrightarrow{\bar{\varphi}} \bar{M} \xrightarrow{\bar{\psi}} \bar{M}^\sigma \xrightarrow{\bar{\varphi}} \bar{M}, \tag{3.4}$$

and we have to show that $\operatorname{Im}(\bar{\psi})$ is a direct summand of $\bar{M}^\sigma$. This holds if and only if for each maximal ideal $\mathfrak{m} \subset R$ the base change of (3.4) to $k = R/\mathfrak{m}$ is exact, or equivalently if the base change to k^{per} is exact.

Let $\Delta : A \to W(A)$ be the homomorphism with $w_n \circ \Delta = \sigma^n$, where w_n is the nth Witt polynomial; see [Bou83, IX, § 1.2, Proposition 2]. The composition of Δ with the homomorphism $W(A) \to W(R) \to W(k^{\mathrm{per}})$ is a homomorphism $A \to W(k^{\mathrm{per}})$ that commutes with σ. Then $M \otimes_A W(k^{\mathrm{per}})$ is a Dieudonné module whose reduction mod p is (3.4) $\otimes_R k^{\mathrm{per}}$, which is therefore exact as required. □

We have a frame $\underline{A} = (A, pA, R, \sigma, \sigma_1)$ with $\sigma_1(pa) = \sigma(a)$, and $\underline{A}$ is a p-torsion free PD frame as defined in § 2. Using Lemma 3.1 one verifies that there is an equivalence of categories

$$\mathrm{Win}(\underline{A}) \to \mathrm{DM}(A), \quad (M, \operatorname{Fil} M, F, F_1) \mapsto (N, \varphi, \psi) \tag{3.5}$$

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defined by $N = \mathrm{Fil}\, M$ and $\varphi(x \otimes 1) = F(x)$ for $x \in \mathrm{Fil}\, M$; see [CL14, Lemma 2.1.15] with $E = p$. Thus the functor $\Phi_{\underline{S}}$ of (3.3) for $\underline{S} = \underline{A}$ can be viewed as a contravariant functor

$$\Phi_A : \mathrm{BT}(\mathrm{Spec}\, R) \to \mathrm{DM}(A). \tag{3.6}$$

In certain cases one can hope that Φ_A is an equivalence of categories; see [deJ93] for the case of complete regular local rings.

Remark 3.2. The functor Φ_A always induces an equivalence between formal p-divisible groups and φ-nilpotent Dieudonné modules, which correspond to F-nilpotent $\underline{A}$-windows. This follows from [Zin01] together with the extension of [Zin02, Theorem 9] to general base rings in [Lau08, Lau13].

4. Semiperfect rings

Let p be a prime. Following [SW13], an $\mathbb{F}_p$-algebra R is called semiperfect if the Frobenius endomorphism $\phi : R \to R$ is surjective. An isogeny of semiperfect rings is a surjective ring homomorphism whose kernel is annihilated by a power of ϕ. Let R be semiperfect. There is a universal homomorphism

$$R^\flat \to R$$

from a perfect ring to R, and there is a universal p-adic PD extension

$$A_{\mathrm{cris}}(R) \to R.$$

Explicitly, we have $R^\flat = \varprojlim(R, \phi)$, and $A_{\mathrm{cris}}(R)$ is the p-adic completion of the PD envelope of the natural map $W(R^\flat) \to R$. We will often write $J = \mathrm{Ker}(R^\flat \to R)$. Two classes of semiperfect rings will play a special role: complete intersections and balanced rings.

4.1 Complete intersection semiperfect rings

Definition 4.1. A semiperfect ring R is called a complete intersection if $R \cong R_0/J_0$ where R_0 is a perfect ring and where the ideal J_0 is generated by a regular sequence.

Lemma 4.2. *Let $R = R_0/J_0$ as in Definition 4.1 where J_0 is generated by the regular sequence $\underline{u} = (u_1, \ldots, u_r)$. The natural homomorphism $R_0 \to R^\flat$ maps $\underline{u}$ to a regular sequence that generates the kernel of $R^\flat \to R$.*

Proof. Since the ideal J_0 is finitely generated, the J_0-adic topology of R_0 coincides with the linear topology defined by the ideals $\phi^n(J_0)$ for $n \geqslant 0$. Thus $R^\flat$ is the J_0-adic completion of R_0. Then the assertion is clear. □

Remark 4.3. Lemma 4.2 implies that in Definition 4.1 one can take $R_0 = R^\flat$. It follows that for a complete intersection semiperfect ring R the ring $A_{\mathrm{cris}}(R)$ is p-torsion free; see for example [CL14, Lemma 2.6.1].

4.2 Balanced semiperfect rings

Definition 4.4. A semiperfect ring R is called balanced if the ideal $\bar{J} = \mathrm{Ker}(\phi : R \to R)$ satisfies $\bar{J}^p = 0$, and R is called iso-balanced if R is isogenous to a balanced semiperfect ring.

Lemma 4.5. *For a homomorphism of semiperfect rings $\alpha : R' \to R$ where R is balanced we have* $\mathrm{Ker}(\alpha)^p = \phi(\mathrm{Ker}(\alpha))$.

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Proof. Clearly $\phi(\mathrm{Ker}(\alpha)) \subseteq \mathrm{Ker}(\alpha)^p$. To prove the opposite inclusion, let $x_1, \ldots, x_p \in \mathrm{Ker}(\alpha)$ be given, and choose $y_i \in R'$ with $\phi(y_i) = x_i$. Then $\alpha(y_i) \in \bar{J} = \mathrm{Ker}(\phi : R \to R)$. Since R is balanced we have $\alpha(\prod y_i) = 0$, thus $\prod x_i = \phi(\prod y_i) \in \phi(\mathrm{Ker}(\alpha))$ as required. □

Lemma 4.6. *A semiperfect ring R is balanced if and only if the ideal $J = \mathrm{Ker}(R^\flat \to R)$ satisfies $J^p = \phi(J)$.*

Proof. If R is balanced then $J^p = \phi(J)$ by Lemma 4.5. The rest is clear. □

Remark 4.7. For every semiperfect ring R there is a universal homomorphism to a balanced semiperfect ring $R \to R^{\mathrm{bal}}$, namely $R^{\mathrm{bal}} = R^\flat / J^{\mathrm{bal}}$ where J^{bal} is the ascending union of the ideals $\phi^{-n}(J)^{p^n}$ for $n \geqslant 0$. The ring R is iso-balanced if and only if $R \to R^{\mathrm{bal}}$ is an isogeny.

Lemma 4.8. *Let $\pi : R' \to R$ be an isogeny of iso-balanced semiperfect rings. Then the ideal $\mathrm{Ker}(\pi)$ is nilpotent.*

Proof. The composition $\alpha : R' \xrightarrow{\pi} R \to R^{\mathrm{bal}}$ is an isogeny since R is iso-balanced. Lemma 4.5 implies that $\mathrm{Ker}(\alpha)^{p^n} = \phi^n(\mathrm{Ker}(\alpha))$, which is zero for large n. □

Remark 4.9. A semiperfect ring R is called f-semiperfect [SW13, Definition 4.1.2] if it is isogenous to the quotient of a perfect ring by a finitely generated ideal. Each f-semiperfect ring R is iso-balanced. Indeed, assume that $R = R_0/J_0$ where R_0 is perfect and $J_0 = (a_1, \ldots, a_r)$ is finitely generated. Let J_1 be the union of $\phi^{-n}(J_0)^{p^n}$ for $n \geqslant 0$. Then R_0/J_1 is balanced. Explicitly, J_1 is generated by all monomials $\prod a_i^{m_i}$ with $m_i \in \mathbb{Z}[1/p]$ and $m_i \geqslant 0$ such that $\sum m_i = 1$, which implies that $m_i \geqslant 1/r$ for at least one i. Choose s such that $p^s \geqslant r$. Then $\phi^s(J_1) \subseteq J_0$, hence $R \to R_0/J_1$ is an isogeny.

4.3 Lifts of semiperfect rings

Let R be a semiperfect ring, and let $J = \mathrm{Ker}(R^\flat \to R)$.

Definition 4.10. A lift of R is a p-adically complete and p-torsion free ring A with $A/pA = R$ which carries a ring endomorphism $\sigma : A \to A$ that induces ϕ on R.

Remark 4.11. The endomorphism $\sigma : A \to A$ is unique if it exists. Indeed, the universal property of the ring of Witt vectors [Gro74, ch. IV, Proposition 4.3] gives a unique homomorphism $\psi : W(R^\flat) \to A$ that induces the projection $R^\flat \to R$ modulo p, and we have $\psi \circ \sigma = \sigma \circ \psi$ by the universal property. Moreover ψ is surjective, and the uniqueness of σ follows. This reasoning shows that lifts A of R correspond to closed ideals $J' \subseteq W(R^\flat)$ such that $\sigma(J') \subseteq J'$ and $J' \cap pW(R^\flat) = pJ'$ and $J'/pJ' = J$.

Definition 4.12. A lift A of R is called straight if $A = W(R^\flat)/J'$ such that the set of all $a \in J$ with $[a] \in J'$ generates J.

Lemma 4.13. *Let R be a semiperfect ring which is a complete intersection or balanced, see Definitions 4.1 and 4.4. Then a straight lift of R exists.*

Proof. If R is a complete intersection, J is generated by a regular sequence $(u_1, \ldots, u_r)$; see Lemma 4.2. Let $J' = ([u_1], \ldots, [u_r])$ in $W(R^\flat)$. The ring $A = W(R^\flat)/J'$ is p-adically complete

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and p-torsion free with $A/pA = R$. The ideal J' is stable under σ since $\sigma([u_i]) = [u_i]^p$. Thus A is a straight lift of R.

Assume that R is balanced. Let $J' \subseteq W(R^\flat)$ be the set of all Witt vectors $a = (a_0, a_1, \dots)$ with $a_i \in \phi^i(J)$. We claim that J' is an ideal. Indeed, the ring structure of $W(R^\flat)$ is given by $(x_0, x_1, \dots) * (y_0, y_1, \dots) = (g_0^*(x, y), g_1^*(x, y), \dots)$ where $*$ is $+$ or $\times$, with certain polynomials g_n^*. If the variables x_i, y_i have degree p^i, then p_n^+ is homogeneous of degree p^n, and $p_n^\times$ is bihomogeneous of bidegree (p^n, p^n). Since R is balanced we have $\phi(J) = J^p$; see Lemma 4.6. It follows that J' is an ideal. We have $J' \cap pW(R^\flat) = pJ'$, and J' is the closure of the ideal generated by the elements $[a]$ for all $a \in J$. Clearly J' is stable under σ. Thus $A = W(R^\flat)/J'$ is a straight lift of R. □

Lemma 4.14. *If A is a lift of the semiperfect ring R, then $\sigma : A \to A$ is surjective, and*

$$\varprojlim(A, \sigma) = W(R^\flat).$$

Proof. The first assertion holds because the natural σ-equivariant homomorphism $W(R^\flat) \to A$ is surjective, and σ is bijective on $W(R^\flat)$; see Remark 4.11. Let $B = \varprojlim(A, \sigma)$. Since A is p-torsion free the same holds for B. We take the limit over σ of the exact sequence $0 \to A \to A \to A_n \to 0$, where the first map is p^n. It follows that $B/p^nB = \varprojlim(A_n, \sigma)$, which implies that $\varprojlim_n(B/p^nB) = B$; moreover $B/pB = R^\flat$. Therefore $B = W(R^\flat)$. □

5. Dieudonné modules via lifts

Let R be a semiperfect ring and let A be lift of R; see Definition 4.10.

5.1 Frames associated to a lift

To the lift A of R we associate two frames. First, there is the p-torsion free PD frame

$$\underline{A} = (A, pA, R, \sigma, \sigma_1)$$

with $\sigma_1 = p^{-1}\sigma$; see § 3. Second, let

$$\tilde{\operatorname{Fil}} A = \operatorname{Ker}(A \to R \xrightarrow{\phi} R).$$

Lemma 5.1. *We have $\sigma(\tilde{\operatorname{Fil}} A) \subseteq pA$, and $\tilde{\operatorname{Fil}} A$ is a PD ideal of A.*

Proof. Since σ is a lift of ϕ, for $a \in A$ we have $a \in \tilde{\operatorname{Fil}} A$ if and only if $\sigma(a) \in pA$ if and only if $a^p \in pA$. For $a \in \tilde{\operatorname{Fil}} A$ let $b = a^p/p \in A$. We have to show that $b \in \tilde{\operatorname{Fil}} A$, or equivalently that $\sigma(b) \in pA$. But $\sigma(b) = \sigma(a)^p/p = p^{p-1}(\sigma(a)/p)^p$. □

Since R is semiperfect, σ induces an isomorphism $A/\tilde{\operatorname{Fil}} A \xrightarrow{\sim} R$. By Lemma 5.1 we can define a p-torsion free PD frame

$$\underline{A}/\phi = (A, \tilde{\operatorname{Fil}} A, R, \sigma, \sigma_1)$$

with $\sigma_1 = p^{-1}\sigma$. The endomorphism σ of A is a frame endomorphism $\sigma : \underline{A} \to \underline{A}$ over $\phi : R \to R$, which factors into frame homomorphisms

$$\underline{A} \xrightarrow{\iota} \underline{A}/\phi \xrightarrow{\pi} \underline{A}, \tag{5.1}$$

where ι is given by the identity on A and by ϕ on R, while π is given by σ on A and by the identity on R.

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Lemma 5.2. *The frame homomorphism $\pi : \underline{A}/\phi \to \underline{A}$ is crystalline, i.e. it induces an equivalence of the window categories.*

Proof. Let I be the kernel of the surjective homomorphism $\sigma : A \to A$. If we write $A = W(R^\flat)/J'$ (see Remark 4.11) then $I = J'/\sigma(J')$. Thus $\sigma = p\sigma_1$ is zero on I. Since A is p-torsion free it follows that $\sigma_1 : I \to I$ is zero, and the lemma follows from the general deformation lemma [Lau10, Theorem 3.2]. □

Remark 5.3. The divided powers on $\tilde{\mathrm{Fil}}\, A$, which exist by Lemma 5.1, induce divided powers on the ideal $(\tilde{\mathrm{Fil}}\, A)/pA = \mathrm{Ker}(\phi : R \to R)$ of R. Thus the given lift A of R determines divided powers on $\mathrm{Ker}(\phi)$.

Lemma 5.4. *If A is a straight lift of R in the sense of Definition 4.12, then the associated divided powers on $\mathrm{Ker}(\phi)$ are pointwise nilpotent.*

Proof. Let $J = \mathrm{Ker}(R^\flat \to R)$ and $A = W(R^\flat)/J'$. Since A is straight, there are generators a_i of J with $[a_i] \in J'$. The elements $b_i = \phi^{-1}(a_i) + J$ of R generate the ideal $\mathrm{Ker}(\phi)$. We claim that $b_i^{[p]} = 0$, which proves the lemma. The element $c_i = [\phi^{-1}(a_i)] + J'$ of A is an inverse image of b_i. We have $c_i^p = [a_i] + J' = 0$ in A, thus $c_i^{[p]} = 0$ in A, and thus $b_i^{[p]} = 0$ in R. □

5.2 Evaluation of crystals

We consider the functor

$$\Phi_A : \mathrm{BT}(\mathrm{Spec}\, R) \to \mathrm{Win}(\underline{A}) \tag{5.2}$$

given by (3.3) for $\underline{S} = \underline{A}$. Here $\mathrm{Win}(\underline{A})$ is equivalent to the category $\mathrm{DM}(A)$ of Dieudonné modules over A by (3.5).

Proposition 5.5. *If the divided powers on $\mathrm{Ker}(\phi)$ given by Remark 5.3 are pointwise nilpotent, then the commutative diagram of categories*

$$\begin{array}{ccc} \mathrm{BT}(\mathrm{Spec}\, R) & \xrightarrow{\phi^*} & \mathrm{BT}(\mathrm{Spec}\, R) \\ \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_A} \\ \mathrm{Win}(\underline{A}) & \xrightarrow{\sigma^*} & \mathrm{Win}(\underline{A}) \end{array} \tag{5.3}$$

is cartesian.

Proof. The diagram (5.3) commutes by the functoriality of Φ_A with respect to the frame endomorphism $\sigma : \underline{A} \to \underline{A}$. The factorization (5.1) of σ induces the following extension of (5.3).

$$\begin{array}{ccccc} \mathrm{BT}(\mathrm{Spec}\, R) & \xrightarrow{\phi^*} & \mathrm{BT}(\mathrm{Spec}\, R) & \xrightarrow{\mathrm{id}} & \mathrm{BT}(\mathrm{Spec}\, R) \\ \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_{\underline{A}/\phi}} & & \downarrow{\scriptstyle \Phi_A} \\ \mathrm{Win}(\underline{A}) & \xrightarrow{\iota^*} & \mathrm{Win}(\underline{A}/\phi) & \xrightarrow[\sim]{\pi^*} & \mathrm{Win}(\underline{A}) \end{array} \tag{5.4}$$

Here π^* is an equivalence by Lemma 5.2. Thus (5.3) is equivalent to the left-hand square of (5.4). For a p-divisible group G over R let $\underline{M} = \Phi_{\underline{A}/\phi}(G)$ in $\mathrm{Win}(\underline{A}/\phi)$. Then $M \otimes_A R$ is the value of $\mathbb{D}(G)$ at the PD extension $\phi : R \to R$. Thus lifts of the Hodge filtration of G under ϕ correspond to lifts of the Hodge filtration of $\underline{M}$ under ι^*. The latter correspond to lifts of $\underline{M}$ under ι^* by [Lau10, Lemma 4.2], and the former correspond to lifts of G under ϕ by the Grothendieck–Messing Theorem [Mes72] since the divided powers on $\mathrm{Ker}(\phi)$ are pointwise nilpotent. □

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COROLLARY 5.6. *If the divided powers on* $\mathrm{Ker}(\phi)$ *given by Remark 5.3 are pointwise nilpotent, then the commutative diagram of categories*

$$\begin{array}{ccc} \mathrm{BT}(\operatorname{Spec} R^\flat) & \longrightarrow & \mathrm{BT}(\operatorname{Spec} R) \\ {\scriptstyle \Phi_{W(R^\flat)}}\downarrow & & \downarrow{\scriptstyle \Phi_A} \\ \mathrm{Win}(\underline{W}(R^\flat)) & \longrightarrow & \mathrm{Win}(\underline{A}) \end{array}$$

is cartesian. □

Proof. Proposition 5.5 gives the following cartesian diagram.

$$\begin{array}{ccc} \varprojlim(\mathrm{BT}(\operatorname{Spec} R), \phi^*) & \longrightarrow & \mathrm{BT}(\operatorname{Spec} R) \\ {\scriptstyle \varprojlim \Phi_A}\downarrow & & \downarrow{\scriptstyle \Phi_A} \\ \varprojlim(\mathrm{Win}(\underline{A}), \sigma^*) & \longrightarrow & \mathrm{Win}(\underline{A}) \end{array}$$

The upper limit category is equivalent to $\mathrm{BT}(\operatorname{Spec} R^\flat)$ by the obvious analogue of [Mes72, ch. II, Lemma 4.16]; see also [deJ95, Lemma 2.4.4]. Since we have $\varprojlim(\underline{A}, \sigma) = \underline{W}(R^\flat)$ by Lemma 4.14, the lower limit category is equivalent to $\mathrm{Win}(\underline{W}(R^\flat))$ by [Lau10, Lemma 2.12]. □

THEOREM 5.7. *Let* R *be a semiperfect ring with a lift* A *such that the associated divided powers on* $\mathrm{Ker}(\phi)$ *given by Remark 5.3 are pointwise nilpotent. Then the functor* Φ_A *is an equivalence.*

Remark 5.8. If R is a complete intersection or balanced, there is a straight lift by Lemma 4.13, and the associated divided powers on $\mathrm{Ker}(\phi)$ are pointwise nilpotent by Lemma 5.4. Thus Theorem 5.7 applies in these cases.

Proof of Theorem 5.7. Since $R^\flat$ is a perfect ring, the functor $\Phi_{W(R^\flat)}$ is an equivalence by a theorem of Gabber; see [Lau13, Theorem 6.4]. Every window over $\underline{A}$ can be lifted to a window over $\underline{W}(R^\flat)$. Indeed, the projections $A \to R$ and $W(R^\flat) \to R^\flat \to R$ induce bijective maps of the sets of isomorphisms classes of finite projective modules, and thus the same holds for $W(R^\flat) \to A$. Hence a normal representation of an $\underline{A}$-window in the sense of [Lau10, Lemma 2.6] can be lifted to $\underline{W}(R^\flat)$. Now Lemma 5.9 below applied to the diagram of Corollary 5.6 gives the result. □

LEMMA 5.9. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\psi} & \mathcal{C} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{B} & \xrightarrow{\pi} & \mathcal{D} \end{array}$$

be a cartesian diagram of additive categories or of groupoids. If f *is an equivalence and* π *is essentially surjective, then* g *is an equivalence.*

Proof. The case of additive categories is reduced to the case of groupoids using that a homomorphism $u : X \to Y$ can be encoded by the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ of $X \oplus Y$. Consider the groupoid case. We may assume that $\mathcal{A}$ is equal to the fibered product of $\mathcal{C}$ and $\mathcal{B}$ over $\mathcal{D}$, which is the category of triples (C, δ, B) with $C \in \mathcal{C}$, $B \in \mathcal{B}$, and $\delta : g(C) \cong \pi(B)$.

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(1) The functor g is surjective on isomorphism classes: This holds for π and f.
(2) The functor ψ is surjective on isomorphism classes: Let $C \in \mathcal{C}$. Find $B \in \mathcal{B}$ and $\delta : g(C) \cong \pi(B)$. Then $A = (C, \delta, B)$ satisfies $\psi(A) = C$.
(3) The functor g is faithful: We have to show that if $C \in \mathcal{C}$ and $\gamma \in \mathrm{Aut}(C)$ with $g(\gamma) = \mathrm{id}$ then $\gamma = \mathrm{id}$. Extend C to $A = (C, \delta, B) \in \mathcal{A}$. Then $\alpha = (\gamma, \mathrm{id}_B)$ lies in $\mathrm{Aut}(A)$ with $f(\alpha) = \mathrm{id}$. Thus $\alpha = \mathrm{id}$ and $\gamma = \mathrm{id}$.
(4) The functor g is full: Let $C, C' \in \mathcal{C}$ and $\delta : g(C) \cong g(C')$. Extend C' to $A' = (C', \delta', B') \in \mathcal{A}$. Let $A = (C, \delta'\delta, B') \in \mathcal{A}$. Then $f(A) = B' = f(A')$, and $\mathrm{id}_{B'}$ lifts to a unique $\alpha : A \cong A'$, which consists of $(\gamma, \mathrm{id}_{B'})$ with $\gamma : C \cong C'$ such that $\delta' \circ g(\gamma) = \delta'\delta$, thus $g(\gamma) = \delta$. □

5.3 The passage to A_{cris}

For a moment let R be an arbitrary semiperfect ring. By the universal property of $A_{\mathrm{cris}}(R)$ there is a unique lift of $\phi : R \to R$ to a PD endomorphism σ of $A_{\mathrm{cris}}(R)$, and one verifies that σ is a Frobenius lift. Let $\mathrm{Fil}\, A_{\mathrm{cris}}(R)$ be the kernel of $A_{\mathrm{cris}}(R) \to R$. By [SW13, Lemma 4.1.8] there is a unique functorial σ-linear map $\sigma_1 : \mathrm{Fil}\, A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R)$ such that $p\sigma_1 = \sigma$, which means that

$$\underline{A}_{\mathrm{cris}}(R) = (A_{\mathrm{cris}}(R), \mathrm{Fil}\, A_{\mathrm{cris}}(R), R, \sigma, \sigma_1) \tag{5.5}$$

is a p-frame, and even a PD frame; see § 2. A homomorphism of semiperfect rings $R \to R'$ induces a strict frame homomorphism $\underline{A}_{\mathrm{cris}}(R) \to \underline{A}_{\mathrm{cris}}(R')$.

Assume now that A is a lift of R as earlier. The universal property of $A_{\mathrm{cris}}(R)$ gives a homomorphism $\varkappa : A_{\mathrm{cris}}(R) \to A$ of extensions of R, and $\varkappa$ commutes with σ. Since A is p-torsion free, $\varkappa$ is a frame homomorphism

$$\varkappa : \underline{A}_{\mathrm{cris}}(R) \to \underline{A}. \tag{5.6}$$

Proposition 5.10. *If A is a straight lift of R in the sense of Definition 4.12, the frame homomorphism $\varkappa$ is crystalline.*

See also Proposition 7.8 below.

Proof. Let $N \subseteq A_{\mathrm{cris}}(R)$ be the kernel of $\varkappa$. Since A is p-torsion free we have $N \cap p^n A_{\mathrm{cris}}(R) = p^n N$. Since $\varkappa$ is continuous for the p-adic topology and A is p-adically complete, N is closed in the p-adic topology of $A_{\mathrm{cris}}(R)$, and it follows that N is p-adically complete. We have an exact sequence $0 \to N/p \to A_{\mathrm{cris}}(R)/p \to R \to 0$, and this is the PD envelope over $\mathbb{F}_p$ of the ideal $J = \mathrm{Ker}(R^\flat \to R)$.

Clearly N is stable under σ_1. We claim that σ_1 is nilpotent on N/p; cf. [SW13, Lemma 4.2.4]. Let $A = W(R^\flat)/J'$. The hypothesis means that there are generators a_i of J such that $[a_i] \in J'$. The ideal N/pN of $A_{\mathrm{cris}}(R)/p$ is generated by the elements $a_i^{[n]}$ for $n \geqslant 1$. The elements $[a_i]^{[n]} \in N$ satisfy

$$\sigma_1([a_i]^{[n]}) = \frac{(pn)!}{p \cdot n!}[a_i]^{[pn]}; \tag{5.7}$$

see [SW13, Lemma 4.1.8]. Since the integer $(pn)!/(p \cdot n!)$ is divisible by p when $n \geqslant p$ it follows that $\sigma_1 \circ \sigma_1 = 0$ on N/pN.

We consider the frames $\underline{B}_n = (A_{\mathrm{cris}}(R)/p^n N, \mathrm{Fil}\, A_{\mathrm{cris}}(R)/p^n N, R, \sigma, \sigma_1)$ for $n \geqslant 0$. Since σ_1 is nilpotent on $N/p^n N$, the projection $\underline{B}_n \to \underline{B}_0 = \underline{A}$ is crystalline by the general deformation lemma [Lau10, Theorem 3.2]. We have $\varprojlim \underline{B}_n = \underline{A}_{\mathrm{cris}}(R)$, and the proposition follows; see [Lau10, Lemma 2.12]. □

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Corollary 5.11. *If the semiperfect ring R admits a straight lift A, there is an equivalence of categories*

$$\mathrm{BT}(\operatorname{Spec} R) \cong \operatorname{Win}(\underline{A}_{\mathrm{cris}}(R)). \tag{5.8}$$

Proof. By Theorem 5.7 and Proposition 5.10 we have equivalences

$$\mathrm{BT}(\operatorname{Spec} R) \xrightarrow{\Phi_A} \operatorname{Win}(\underline{A}) \xleftarrow{\varkappa^*} \operatorname{Win}(\underline{A}_{\mathrm{cris}}(R)). \qquad \square$$

Remark 5.12. When $A_{\mathrm{cris}}(R)$ is p-torsion free, the equivalence (5.8) is given by the functor $\Phi_{\underline{S}}$ of (3.3) for $\underline{S} = \underline{A}_{\mathrm{cris}}$. A variant of this holds in general; see Corollary 6.5, which shows in particular that the equivalence (5.8) does not depend on the choice of the lift A.

Corollary 5.13. *If R is a complete intersection semiperfect ring, the functor $\mathbb{D}\mathrm{F}_{\operatorname{Spec} R}$ of (3.2) is an equivalence.*

Proof. If R is a complete intersection, the ring $A_{\mathrm{cris}}(R)$ is p-torsion free; see Remark 4.3. Therefore we have a sequence of functors

$$\mathrm{BT}(\operatorname{Spec} R) \xrightarrow{\mathbb{D}\mathrm{F}_R} \mathrm{DF}(\operatorname{Spec} R) \xrightarrow{e} \operatorname{Win}(A_{\mathrm{cris}}(R)) \xrightarrow{\varkappa^*} \operatorname{Win}(\underline{A}),$$

where e is the evaluation functor, and the composition is Φ_A. The functor e is an equivalence; see [CL14, Proposition 2.6.4]. Here no connection appears because $R^\flat$ is perfect, and thus $\Omega_{R^\flat} = 0$. The functors $\varkappa^*$ and Φ_A are equivalences by Theorem 5.7 and Proposition 5.10. Thus $\mathbb{D}\mathrm{F}_R$ is an equivalence as well. $\square$

Lemma 5.14. *If the semiperfect ring R has a lift A, then the forgetful functor $\mathrm{D}(\operatorname{Spec} R) \to \mathrm{DF}(\operatorname{Spec} R)$ is fully faithful.*

Proof. For a PD extension $S \xrightarrow{\pi} R$ of $\mathbb{F}_p$-algebras the Frobenius ϕ_S factors through a homomorphism $\phi_{S/R} : R \to S$, i.e. $\phi_{S/R} \circ \pi = \phi_S$. An object of $\mathrm{DF}(\operatorname{Spec} R)$ is a triple $(\mathcal{M}, F, V) \in \mathrm{D}(\operatorname{Spec} R)$ together with a direct summand of $\mathcal{M}_R$ whose base change under each $\phi_{S/R}$ is determined by $(\mathcal{M}, F)$; see [CL14, Definition 2.4.1]. The lift A of R makes $\phi : R \to R$ into a PD extension, which we write as $S \to R$; see Remark 5.3. The corresponding $\phi_{S/R}$ is the identity of R, and the lemma follows. $\square$

Corollary 5.13 together with Lemmas 4.13 and 5.14 gives the following.

Corollary 5.15 [SW13, Corollary 4.1.12]. *If R is a complete intersection semiperfect ring, the crystalline Dieudonné functor $\mathbb{D}_{\operatorname{Spec} R}$ is fully faithful.*

6. Crystalline Dieudonné windows

In this section we associate to a p-divisible group over an arbitrary semiperfect ring R a window over the frame $\underline{A}_{\mathrm{cris}}(R)$ of (5.5).

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6.1 Relative deformation rings

We need a relative version of the universal deformation of a p-divisible group. Let $\Lambda \to R$ be a homomorphism of $\mathbb{F}_p$-algebras. (More generally one could take p-adic rings.)

Let $\mathrm{Aug}_{\Lambda/R}$ be the category of Λ-algebras A equipped with a Λ-linear homomorphism $A \to R$, and let $\mathrm{Nil}_{\Lambda/R} \subseteq \mathrm{Aug}_{\Lambda/R}$ be the full subcategory of all A such that $A \to R$ is surjective and $J_A = \mathrm{Ker}(A \to R)$ is a nilpotent ideal. For a p-divisible group G over R we consider the deformation functor

$$\mathrm{Def}_G : \mathrm{Nil}_{\Lambda/R} \to \mathrm{Set},$$

where $\mathrm{Def}_G(A)$ is the set of isomorphism classes of deformations of G to A. If $\Lambda = R$, then Def_G is pro-represented by the twisted power series ring $B = \Lambda[[Q]] \in \mathrm{Aug}_{\Lambda/R}$, where Q is the projective Λ-module $\mathrm{Lie}(G^\vee)^* \otimes_\Lambda \mathrm{Lie}(G)^*$; see [Lau14, Proposition 3.11].

Lemma 6.1. *Assume that G' is a p-divisible group over Λ with an isomorphism $G' \otimes_\Lambda R \cong G$. If $B = \Lambda[[Q]]$ represents $\mathrm{Def}_{G'} : \mathrm{Nil}_{\Lambda/\Lambda} \to \mathrm{Set}$, then B also represents $\mathrm{Def}_G : \mathrm{Nil}_{\Lambda/R} \to \mathrm{Set}$.*

Proof. For $A \in \mathrm{Nil}_{\Lambda/R}$ the fiber product $A' = A \times_R \Lambda$ lies in $\mathrm{Nil}_{\Lambda/\Lambda}$. Let $\mathrm{LF}(A)$ denote the category of finite projective A-modules. Then the obvious functor $\mathrm{LF}(A') \to \mathrm{LF}(A) \times_{\mathrm{LF}(R)} \mathrm{LF}(\Lambda)$ is an equivalence. It follows that the natural map $\mathrm{Def}_{G'}(A') \to \mathrm{Def}_G(A)$ is bijective, which proves the lemma. □

Let $\tilde{\mathrm{Nil}}_{\Lambda/R}$ be the category of all $A \in \mathrm{Aug}_{\Lambda/R}$ such that $A \to R$ is surjective and the ideal J_A is bounded nilpotent, i.e. there is an $n \geqslant 1$ with $x^n = 0$ for all $x \in J_A$. We define $\mathrm{Def}_G : \tilde{\mathrm{Nil}}_{\Lambda/R} \to \mathrm{Set}$ as before.

Lemma 6.2. *In the situation of Lemma 6.1 the functor Def_G on $\tilde{\mathrm{Nil}}_{\Lambda/R}$ is also represented by B.*

Proof. Let $A \in \tilde{\mathrm{Nil}}_{\Lambda/R}$. We have to show that the natural map $\mathrm{Hom}(B, A) \to \mathrm{Def}_G(A)$ is bijective. For each pair of homomorphisms $f_1, f_2 : B \to A$ in $\mathrm{Aug}_{\Lambda/R}$ there is a finitely generated ideal $\mathfrak{b} \subseteq J_A$ such that the projection $A \to \bar{A} = A/\mathfrak{b}$ equalizes f and g. For each pair of deformations G_1, G_2 of G over A the reduction map $\mathrm{Hom}_A(G_1, G_2) \to \mathrm{End}_R(G)$ is injective with cokernel annihilated by p^r for some r; see [Lau14, Lemma 3.4]. Thus there is a unique isogeny $\psi : G_1 \to G_2$ which lifts $p^r \mathrm{id}_G$. Its kernel is finitely presented; see [Lau14, Lemma 3.6]. Thus there is a finitely generated ideal $\mathfrak{b} \subseteq A$ such that $\mathrm{Ker}(\psi)$ and $G_1[p^r]$ coincide over $A/\mathfrak{b}$, which means that G_1 and G_2 map to the same element of $\mathrm{Def}_G(A/\mathfrak{b})$. Moreover G_1 and G_2 are equal as deformations of G if and only if they are equal as deformations of $G_1 \otimes_A A/\mathfrak{b}$. In view of these remarks it suffices to show that $B(A) \to \mathrm{Def}_G(A)$ is bijective when R is replaced by $R' = A/\mathfrak{b}$ for varying finitely generated ideals $\mathfrak{b}$. Then A lies in $\mathrm{Nil}_{\Lambda/R'}$, and the lemma follows from Lemma 6.1. □

6.2 Construction of the crystalline window functor

Theorem 6.3. *For semiperfect rings R there are unique functors*

$$\Phi_R^{\mathrm{cris}} : \mathrm{BT}(\mathrm{Spec}\, R) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)), \quad G \mapsto \underline{M} = (M, \mathrm{Fil}\, M, F, F_1)$$

which are functorial in R, such that the triple $(M, \mathrm{Fil}\, M, F)$ is given by the filtered Dieudonné crystal $\mathbb{D}\mathrm{F}(G)$ of (3.2) as usual, i.e. $M = \mathbb{D}(G)_{A_{\mathrm{cris}}(R)}$, the submodule $\mathrm{Fil}\, M \subseteq M$ is the inverse image of the Hodge filtration $\mathrm{Lie}(G)^* \subseteq \mathbb{D}(G)_R$, and $F : M \to M$ is induced by $F : \phi^*\mathbb{D}(G) \to \mathbb{D}(G)$.

The existence of such a functor has been suggested in [SW13, Remark 4.1.9]. We call $\underline{M}$ the crystalline Dieudonné window of G.

Proof. This is similar to [Lau14, Theorem 3.19].

Let $G \mapsto (M(G), \operatorname{Fil} M(G), F)$ be as defined in the theorem. We have to find a functorial map $F_1 : \operatorname{Fil} M(G) \to M(G)$ which gives a window $\underline{M}(G)$, and verify that F_1 is unique. If $A_{\mathrm{cris}}(R)$ is p-torsion free then F_1 and thus $\underline{M}(G)$ are well defined; see [Lau14, Proposition 3.17]. This applies in particular when R is perfect since then $A_{\mathrm{cris}}(R) = W(R)$.

In general let $\pi : R^\flat \to R$ be the projection. We write π^* for the base change functor of modules or windows from $W(R^\flat)$ to $A_{\mathrm{cris}}(R)$. Note that p-divisible groups can be lifted under $\phi : R \to R$ by [Ill85, Theorem 4.4], and thus p-divisible groups can be lifted under π. Let $G \in \mathrm{BT}(\operatorname{Spec} R)$ be given. We choose a lift $G_1 \in \mathrm{BT}(\operatorname{Spec} R^\flat)$ of G. Then $M(G) = \pi^* M(G_1)$ as modules with Fil and F, and necessarily we have to define $\underline{M}(G) = \pi^* \underline{M}(G_1)$ as windows. We have to show that this construction of F_1 does not depend on the choice of G_1, i.e. if $G_2 \in \mathrm{BT}(\operatorname{Spec} R^\flat)$ is another lift of G, then the composite isomorphism of modules

$$\pi^* M(G_1) \cong M(G) \cong \pi^* M(G_2)$$

preserves the homomorphisms F_1 defined on the outer terms by the windows $\underline{M}(G_i)$.

We want to lift the situation to perfect rings. More precisely, we claim that one can find a commutative diagram of rings

$$\begin{array}{ccc} S' & \xrightarrow{u} & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ R^\flat & \xrightarrow{\pi} & R \end{array}$$

where S and S' are perfect, and p-divisible groups $H_1, H_2 \in \mathrm{BT}(\operatorname{Spec} S')$ together with an isomorphism $\alpha : u^* H_1 \cong u^* H_2$ over S and isomorphisms $f^* H_i \cong G_i$ over $R^\flat$ for $i = 1, 2$ such that α induces the given isomorphism $\pi^* G_1 \cong \pi^* G_2$ over R, i.e. the composition

$$\pi^* G_1 \cong \pi^* f^* H_1 \cong g^* u^* H_1 \xrightarrow{g^*\alpha} g^* u^* H_2 \cong \pi^* f^* H_2 \cong \pi^* G_2$$

is the given isomorphism. Then the homomorphisms F_1 of H_1 and of H_2 coincide over S since S is perfect, and by base change under g it follows that the homomorphisms F_1 of G_1 and of G_2 coincide over R as required.

Let us prove the claim. Let G' be a lift of G to $R^\flat$, for example $G' = G_1$. Let $B = R^\flat[[Q]]$ be the universal deformation ring of G' as in § 6.1 and let $\mathcal{G}$ over B be the universal deformation. By Lemma 6.2, B represents the deformation functor Def_G on the category $\widetilde{\mathrm{Nil}}_{R^\flat/R}$ of augmented algebras $R^\flat \to A \to R$ such that the kernel of $A \to R$ is bounded nilpotent. The system $(\phi^n : R \to R)_n$ is a pro-object of $\widetilde{\mathrm{Nil}}_{R^\flat/R}$ with limit $R^\flat \to R$ in $\mathrm{Aug}_{R^\flat/R}$. Thus there are homomorphisms $\beta_i : B \to R^\flat$ in $\mathrm{Aug}_{R^\flat/R}$ with $\beta_i^* \mathcal{G} \cong G_i$ as deformations of G over $R^\flat$.

We put $S = R^\flat$ with $g = \pi$ and $S' = B^{\mathrm{per}} = \varinjlim(B, \phi)$ with $u = \beta_1^{\mathrm{per}}$ and $f = \beta_2^{\mathrm{per}}$. Let H_1 be the base change of G_1 under $R^\flat \to B \to S'$ and let H_2 be the base change of $\mathcal{G}$ under $B \to S'$. Then $u^* H_1 \cong G_1 \cong u^* H_2$ and $f^* H_1 \cong G_1$ and $f^* H_2 \cong G_2$ as deformations of G. This proves the claim; the required equality of isomorphisms $\pi^* G_1 \cong \pi^* G_2$ is automatic because the reduction map $\operatorname{Hom}(G_1, G_2) \to \operatorname{End}(G)$ is injective. □

The functors Φ_R^{cris} are related with the functors Φ_A of (5.2) as follows.

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Lemma 6.4. *If A is a lift of the semiperfect ring R, there is a natural isomorphism of $\underline{A}$-windows $\varkappa^* \circ \Phi_R^{\mathrm{cris}}(G) \cong \Phi_A(G)$, where $\varkappa$ is defined in (5.6).*

Proof. The functor Φ_R^{cris} without F_1 is given by the Dieudonné crystal evaluated at $A_{\mathrm{cris}}(R)$. Thus the functor $\varkappa^* \circ \Phi_R^{\mathrm{cris}}$ without F_1 is given by the Dieudonné crystal evaluated at A. Since A is p-torsion free, for the frame $\underline{A}$ the functor of forgetting F_1 is fully faithful, and the lemma follows. □

Corollary 6.5. *If the semiperfect ring R admits a straight lift A, the functor Φ_R^{cris} is an equivalence and coincides with the equivalence of Corollary 5.11.*

Proof. If A is a straight lift of R, the functor Φ_A is an equivalence by Theorem 5.7 together with Lemma 5.4, and the functor $\varkappa^*$ is an equivalence by Proposition 5.10. By Lemma 6.4 it follows that Φ_R^{cris} is an equivalence. The final assertion is clear. □

Remark 6.6. Corollary 6.5 is a special case of Theorem 7.10 below. Corollary 6.5 applies in particular when R is a complete intersection or balanced; see Lemma 4.13. For complete intersections, Corollary 6.5 is essentially a restatement of Corollary 5.13, but the balanced case contains new information.

Corollary 6.7. *If R is an iso-balanced semiperfect ring, then the crystalline Dieudonné functor $\mathbb{D}_R : \mathrm{BT}(\operatorname{Spec} R) \to \mathrm{D}(\operatorname{Spec} R)$ is fully faithful up to isogeny.*

For f-semiperfect rings, this is [SW13, Theorem 4.1.4]; see Remark 4.9.

Proof. To prove the assertion we may replace R by an isogenous ring; see [SW13, Proposition 4.1.5]. Thus we can assume that R is balanced, so R has a straight lift. For $G \in \mathrm{BT}(\operatorname{Spec} R)$ and $\Phi_R^{\mathrm{cris}}(G) = \underline{M} = (M, \operatorname{Fil} M, F, F_1)$, the Dieudenné crystal $\mathbb{D}(G)$ is given by the pair (M, F). For $G, G' \in \mathrm{BT}(\operatorname{Spec} R)$ we have to show that the composition

$$\operatorname{Hom}(G, G') \otimes \mathbb{Q} \to \operatorname{Hom}(\underline{M}, \underline{M}') \otimes \mathbb{Q} \to \operatorname{Hom}((M, F), (M', F)) \otimes \mathbb{Q}$$

is bijective. The first map is bijective without $\otimes \mathbb{Q}$ by Corollary 6.5, the second map is bijective because the $A_{\mathrm{cris}}(R)$-module M is of finite type. □

7. The crystalline equivalence

In this section we extend Corollary 6.5 to arbitrary iso-balanced semiperfect rings. Let R be a semiperfect ring, and let $J = \operatorname{Ker}(R^\flat \to R)$.

7.1 Weak lifts

We use a weak version of lifts which may have p-torsion.

Definition 7.1. A weak lift of R is a p-adically complete ring A with $A/pA = R$ which carries a ring endomorphism $\sigma : A \to A$ that induces ϕ on R, and a σ-linear map $\sigma_1 : pA \to A$ with $\sigma_1(p) = 1$.

Remark 7.2. The maps σ and σ_1 are unique if they exist. This is analogous to Remark 4.11. There is a unique homomorphism $\psi : W(R^\flat) \to A$ of extensions of R, and ψ commutes with σ. Since ψ is surjective, σ is unique. Then $\sigma_1(px) = \sigma(x)$ is unique as well.

By definition, a weak lift A of R gives a PD frame $\underline{A} = (A, pA, R, \sigma, \sigma_1)$.

DEFINITION 7.3. A weak lift A of R is called straight if $A = W(R^\flat)/J'$ such that J is generated by elements a with $[a] \in J'$.

LEMMA 7.4. *For each straight weak lift A of R there is a unique homomorphism of PD frames*

$$\varkappa : \underline{A}_{\mathrm{cris}}(R) \to \underline{A}$$

over the identity of R.

Proof. The universal property of $A_{\mathrm{cris}}(R)$ gives a PD homomorphism $\varkappa : A_{\mathrm{cris}} \to A$ over the identity of R, and $\varkappa$ commutes with σ. To show that $\varkappa$ is a frame homomorphism it suffices to verify that $\varkappa(\sigma_1(y)) = \sigma_1(\varkappa(y))$ for generators y of the ideal $\mathrm{Fil}\, A_{\mathrm{cris}}(R)$. A set of generators of this ideal is formed by p and the elements $[x]^{[n]}$ for generators $x \in J$ and $n \geqslant 1$. We have $\varkappa(\sigma_1(p)) = 1 = \sigma_1(\varkappa(p))$, moreover (5.7) gives

$$\varkappa(\sigma_1([x]^{[n]})) = \frac{(np)!}{p \cdot n!}\varkappa([x]^{[np]}) = \frac{(np)!}{p \cdot n!}\varkappa([x])^{[np]} \tag{7.1}$$

and

$$\sigma_1(\varkappa([x]^{[n]})) = \sigma_1(\varkappa([x])^{[n]}). \tag{7.2}$$

Since the weak lift A is straight, the generators x of J can be chosen such that $[x] \in J'$. Then $\varkappa([x]) = 0$, and (7.1) and (7.2) are both zero. □

Next we observe that every semiperfect ring has many straight weak lifts.

DEFINITION 7.5. A descending sequence of ideals $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ of $R^\flat$ is called admissible if $J_0 = J$ and $J_i^p \subseteq J_{i+1}$. In this case let $W(J_*) \subseteq W(R^\flat)$ be the set of all Witt vectors $a = (a_0, a_1, a_2, \ldots)$ with $a_i \in J_i$, which is an ideal by Lemma 7.6 below, and let $A(J_*) = W(R^\flat)/W(J_*)$.

LEMMA 7.6. *Let J_* be an admissible sequence of ideals of $R^\flat$. Then $W(J_*)$ is an ideal of $W(R^\flat)$, and the ring $A(J_*)$ is a straight weak lift of R.*

Let $\underline{A}(J_*) = (A(J_*), pA(J_*), R, \sigma, \sigma_1)$ be the corresponding PD frame.

Proof. As in the proof of Lemma 4.13 we see that $W(J_*)$ is an ideal. This ideal is closed in $W(R^\flat)$, and thus $A = A(J_*)$ is p-adically complete. Since $J_0 = J$ we have $A/pA = R$. Clearly $W(J_*)$ is stable under the endomorphism σ of $W(R^\flat)$, so σ induces $\sigma : A \to A$. We have $pA = pW(R^\flat)/(W(J_*) \cap pW(R^\flat))$, and an element $a \in W(J_*)$ lies in $pW(R^\flat)$ if and only if $a_0 = 0$. Since the sequence J_* is descending we have $\sigma_1(W(J_*) \cap pW(R^\flat)) \subseteq W(J_*)$, so σ_1 induces $\sigma_1 : pA \to A$. It follows that A is a weak lift, which is straight because for every $a \in J$ we have $[a] \in W(J_*)$. □

Windows over $\underline{A}(J_*)$ are insensitive to bounded variations of J_* in the following sense.

LEMMA 7.7. *Let J_* and J'_* be two admissible sequences of ideals of $R^\flat$ such that there is an $n \geqslant 0$ with $J'_{i+n} \subseteq J_i \subseteq J'_i$ for all $i \geqslant 0$. Then there is a natural frame homomorphism $\pi : \underline{A}(J_*) \to \underline{A}(J'_*)$, which is crystalline.*

Proof. The homomorphism π exists because $J_i \subseteq J_i'$. Let $\mathfrak{a} = \mathrm{Ker}(\pi) = W(J_*')/W(J_*)$. Then σ_1 induces an endomorphism of $\mathfrak{a}$, and $(\sigma_1)^n$ is zero on $\mathfrak{a}$ because $J_{i+n}' \subseteq J_i$. The result follows from the deformation lemma [Lau10, Theorem 3.2] if we find a sequence of ideals $\mathfrak{a} = \mathfrak{a}_0 \supseteq \cdots \supseteq \mathfrak{a}_n = 0$ which are stable under σ_1 such that $\sigma(\mathfrak{a}_m) \subseteq \mathfrak{a}_{m+1}$ for $m < n$.

This sequence can be constructed as follows. We have $\phi^n(J_i') \subseteq J_i'^{p^n} \subseteq J_{i+n}' \subseteq J_i$ and thus $J_i \subseteq J_i' \subseteq \phi^{-n}(J_i)$. For each m with $0 \leqslant m \leqslant n$ let $K_{m,i} = J_i' \cap \phi^{m-n}(J_i)$. Then $K_{m,*}$ is an admissible sequence, moreover $J_i' = K_{0,i} \supseteq K_{1,i} \supseteq \cdots \supseteq K_{n,i} = J_i$ and thus $W(J_*') = W(K_{0,*}) \supseteq \cdots \supseteq W(K_{n,*}) = W(J_*)$. Let $\mathfrak{a}_m = W(K_{m,*})/W(J_*)$. Then $\mathfrak{a}_m$ is stable under σ_1 because $K_{m,*}$ is a decreasing sequence; see the proof of Lemma 7.6. We have $\sigma(\mathfrak{a}_m) \subseteq \mathfrak{a}_{m+1}$ because $\phi(K_{m,i}) \subseteq K_{m+1,i}$. □

7.2 The passage to A_{cris}

Proposition 7.8. *Assume that R is iso-balanced, and let $J_i = J^{p^i}$ for all i. Then the frame homomorphism $\varkappa : \underline{A}_{\mathrm{cris}}(R) \to \underline{A}(J_*)$ is crystalline.*

The homomorphism $\varkappa$ is given by Lemma 7.4. See also Proposition 5.10.

Proof. Let $R \to R'$ be an isogeny with balanced R' whose kernel is annihilated by ϕ^n. Then $R' = R^\flat/J'$ with $\phi(J') = J'^p$ and $\phi^n(J') \subseteq J \subseteq J'$. Let $K_i = J \cap \phi^i(J')$ for $i \geqslant 0$. The sequence K_* is admissible. For $i \geqslant 0$ we have $K_{n+i} = \phi^{n+i}(J') \subseteq J_i \subseteq K_i$. Thus the natural frame homomorphism $\pi : \underline{A}(J_*) \to \underline{A}(K_*)$ is crystalline by Lemma 7.7, and it suffices to show that the composition

$$\varkappa' = \pi \circ \varkappa : \underline{A}_{\mathrm{cris}}(R) \to \underline{A}(K_*)$$

is crystalline. Let $N = \mathrm{Ker}(\varkappa')$.

Lemma 7.9. (i) *The p-power torsion of $A(K_*)$ is annihilated by p^n.*

(ii) *For $i \geqslant 0$ we have $N \cap p^{n+i} A_{\mathrm{cris}}(R) \subseteq p^i N$, in particular the p-adic topology of N is induced by the p-adic topology of $A_{\mathrm{cris}}(R)$.*

(iii) *The endomorphism $\sigma_1 : N/pN \to N/pN$ is nilpotent.*

Proof. Let $J_i' = \phi^i(J')$. Then J_*' is an admissible sequence of ideals of $W(R^\flat)$ with respect to $R' = R^\flat/J'$; note that $R^\flat = R'^\flat$. We have $K_i \subseteq J_i'$ with equality for $i \geqslant n$, so there is a projection $A(K_*) \to A(J_*')$ whose kernel is annihilated by p^n. The ring $A(J_*')$ is the straight lift of R' constructed in Lemma 4.13, which is p-torsion free. This proves (i), and (ii) follows.

Let us prove (iii). The ring $A_{\mathrm{cris}}(R)$ is the p-adic completion of a $W(R^\flat)$-algebra generated by the elements $[x]^{[i]}$ for $x \in J$ and $i \geqslant 1$, and these elements map to zero in $A(K_*)$. Thus for each $m \geqslant 1$ the image of N in $A_{\mathrm{cris}}(R)/p^m$ is generated as an ideal by $W(K_*)$ and the elements $[x]^{[i]}$. By (ii) it follows that N/pN is generated as an $A_{\mathrm{cris}}(R)$-module by $W(K_*)$ and the elements $[x]^{[i]}$. We check these elements separately.

First, the explicit formula (5.7) for σ_1 implies that for $x \in J$ the element $(\sigma_1)^2([x]^{[i]})$ lies in pN; see the proof of Proposition 5.10. Second, since $\phi^i(K_n) = K_{n+i}$ for $i \geqslant 0$, each element of $W(K_*)/pW(K_*)$ is represented by an element $a = (a_0, a_1, a_2, \ldots) \in W(K_*)$ with $a_i = 0$ for $i > n$. Then

$$(\sigma_1)^{n+2}(a) = (\sigma_1)^{n+2}([a_0]) + \cdots + (\sigma_1)^2([a_n])$$

lies in pN, using that $a_i \in K_i \subseteq J$. Thus $(\sigma_1)^{n+2}$ is zero on N/pN, and Lemma 7.9 is proved. □

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We continue the proof of Proposition 7.8. Since $A(K_*)$ is p-adically complete, the ideal N is closed in $A_{\mathrm{cris}}(R)$, and thus N is p-adically complete by Lemma 7.9(ii). Since $\sigma_1 : N \to N$ stabilizes $p^m N$, the ring $A_{\mathrm{cris}}(R)/p^m N$ carries a natural frame structure, denoted by $\underline{A}_{\mathrm{cris}}(R)/p^m N$. We have $\underline{A}_{\mathrm{cris}}(R)/N = \underline{A}(K_*)$ and $\underline{A}_{\mathrm{cris}}(R) = \varprojlim \underline{A}_{\mathrm{cris}}(R)/p^m N$. This limit preserves the window categories by [Lau10, Lemma 2.12]. Thus it suffices to show that the frame homomorphism $\underline{A}_{\mathrm{cris}}(R)/p^m N \to \underline{A}(K_*)$ is crystalline for each m. Since $\sigma_1 : N/p^m N \to N/p^m N$ is nilpotent by Lemma 7.9(iii), this follows from [Lau10, Theorem 3.2]. □

Theorem 7.10. *If R is an iso-balanced semiperfect ring, the functor Φ_R^{cris} of Theorem 6.3 is an equivalence of categories.*

Proof. By Corollary 6.5 the theorem holds for balanced rings. An isogeny from R to a balanced ring has nilpotent kernel by Lemma 4.5. Therefore it suffices to show the following. Let $\pi : R' \to R$ be an isogeny of iso-balanced rings such that $\mathrm{Ker}(\pi)^p = 0$. If Φ_R^{cris} is an equivalence then so is $\Phi_{R'}^{\mathrm{cris}}$.

To prove this we use some auxiliary frames. Let $J = \mathrm{Ker}(R^\flat \to R)$ and $J' = \mathrm{Ker}(R^\flat \to R')$, thus $J^p \subseteq J' \subseteq J$. We define $J_i = J^{p^i}$ and $J'_i = J'^{p^i}$ for $i \geqslant 0$, and we define $K_0 = J'$ and $K_i = J_i$ for $i \geqslant 1$. Then J_* is an admissible sequence with respect to R, while K_* and J'_* are admissible sequences with respect to R'. There are obvious frame homomorphisms

$$\underline{A}(J'_*) \xrightarrow{a} \underline{A}(K_*) \xrightarrow{q} \underline{A}(J_*),$$

where a lies over $\mathrm{id}_{R'}$ and q lies over π. Here a is crystalline by Lemma 7.7, using that $K_{i+1} \subseteq J'_i \subseteq K_i$. We want to factor q over another frame $\mathcal{F} = (A, I, R, \sigma, \sigma_1)$ with $A = A(K_*)$, thus I is the kernel of $A(K_*) \to A(J_*) \to R$. We only have to define $\sigma_1 : I \to A$. It is easy to see that the natural map $\mathrm{Ker}(q) \to \mathrm{Ker}(\pi)$ is bijective and that

$$I = \mathrm{Ker}(q) \oplus pA$$

as a direct sum of ideals. We have $\mathrm{Ker}(q)^p = 0$, and $\sigma(x) = 0$ for $x \in \mathrm{Ker}(q)$. We extend the homomorphism σ_1 and the divided powers defined on pA to I by $\sigma_1(x) = 0$ and $x^{[p]} = 0$ for $x \in \mathrm{Ker}(q)$. This defines a PD frame $\mathcal{F}$ as above. Together we have homomorphisms of PD frames

$$\underline{A}(J'_*) \xrightarrow{a} \underline{A}(K_*) \xrightarrow{b} \mathcal{F} \xrightarrow{c} \underline{A}(J_*)$$

over $R' \xrightarrow{\mathrm{id}} R' \xrightarrow{\pi} R \xrightarrow{\mathrm{id}} R$, where c is given by q and b is given by id_A. Since σ_1 is zero on $\mathrm{Ker}(c) = \mathrm{Ker}(q)$, c is crystalline by [Lau10, Theorem 3.2].

Since $A \to R$ is a p-adic PD extension, the universal property of $A_{\mathrm{cris}}(R)$ gives a unique homomorphism $\tilde{\varkappa} : A_{\mathrm{cris}}(R) \to A$ of PD extensions of R, and $\tilde{\varkappa}$ commutes with σ. We claim that $\tilde{\varkappa}$ is a frame homomorphism $\underline{A}_{\mathrm{cris}}(R) \to \mathcal{F}$, i.e. that $\tilde{\varkappa}$ commutes with σ_1. As in the proof of Lemma 7.4 it suffices to show that $\tilde{\varkappa}(\sigma_1(y)) = \sigma_1(\tilde{\varkappa}(y))$ when $y = [x]^{[n]}$ with $x \in J$ and $n \geqslant 1$. Let $z = \tilde{\varkappa}([x])$ in A. Then $z \in \mathrm{Ker}(q)$ and thus $z^{[np]} = 0$, moreover $z^{[n]} \in \mathrm{Ker}(q)$ as well and thus $\sigma_1(z^{[n]}) = 0$. Therefore (7.1) and (7.2) with $\tilde{\varkappa}$ in place of $\varkappa$ show that $\tilde{\varkappa}(\sigma_1(y)) = 0$ and $\sigma_1(\tilde{\varkappa}(y)) = 0$. Thus $\tilde{\varkappa}$ is a frame homomorphism. Since a, b, c are PD homomorphisms, we obtain a commutative diagram of frames

$$\begin{array}{ccccccc}
\underline{A}_{\mathrm{cris}}(R') & & \xrightarrow{\quad A_{\mathrm{cris}}(\pi) \quad} & & & & \underline{A}_{\mathrm{cris}}(R) \\
\downarrow \varkappa & \searrow \varkappa' & & & & \swarrow \tilde{\varkappa} & \downarrow \varkappa \\
\underline{A}(J'_*) & \xrightarrow{a} & \underline{A}(K_*) & \xrightarrow{b} & \mathcal{F} & \xrightarrow{c} & \underline{A}(J_*)
\end{array}$$

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where the homomorphisms $\varkappa$ are given by Lemma 7.4, and $\varkappa' = a \circ \varkappa$. The two homomorphisms $\varkappa$ are crystalline by Proposition 7.8. Since a and c are crystalline, the same holds for $\varkappa'$ and $\tilde{\varkappa}$. We have the following commutative diagram of categories.

$$\begin{array}{ccccc} \mathrm{BT}(\operatorname{Spec} R') & \xrightarrow{\Phi^{\mathrm{cris}}_{R'}} & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R')) & \xrightarrow[\sim]{\varkappa'} & \mathrm{Win}(\underline{A}(K_*)) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow & & \downarrow{\scriptstyle b} \\ \mathrm{BT}(\operatorname{Spec} R) & \xrightarrow{\Phi^{\mathrm{cris}}_{R}} & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)) & \xrightarrow[\sim]{\tilde{\varkappa}} & \mathrm{Win}(\mathcal{F}) \end{array}$$

The functors $\mathrm{BT}(\operatorname{Spec} R') \to \mathrm{Win}(\underline{A}(K_*))$ and $\mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\mathcal{F})$ are given by the Dieudonné crystal with an additional F_1. Since $\mathrm{Ker}(\pi)^p = 0$, the ideal $\mathrm{Ker}(\pi)$ can be equipped with the trivial divided powers. Then the projection $A(K_*) = A \to R'$ is a homomorphism of PD extensions of R. It follows that for $G \in \mathrm{BT}(\operatorname{Spec} R)$ with associated $\underline{M} \in \mathrm{Win}(\mathcal{F})$ there is a natural isomorphism $M \otimes_A R' \cong \mathbb{D}(G)_{R'}$. By the Grothendieck–Messing theorem [Mes72] and its trivial counterpart for the frame homomorphism b in [Lau10, Lemma 4.2] it follows that the lifts of G under π and the lifts of $\underline{M}$ under b coincide; cf. the proof of Proposition 5.5. Therefore if Φ^{cris}_R is an equivalence, the same holds for $\Phi^{\mathrm{cris}}_{R'}$. This finishes the proof of Theorem 7.10. □

8. Perfectoid rings

We use the definition of perfectoid rings of [BMS16] in a slightly different formulation. We begin with an easy remark on perfect rings.

Lemma 8.1. *Let S be a perfect ring and $a \in S$. Let $J = (a^{p^{-\infty}})$ and $I = \mathrm{Ann}(a)$. Then $J = \phi(J)$ and $I = \phi(I) = \mathrm{Ann}(J)$ and $I \cap J = 0$, thus we have an exact sequence*

$$0 \to S \to S/I \oplus S/J \to S/(I+J) \to 0, \tag{8.1}$$

where the first map is the diagonal map and the second map is the difference. The element $a \in S/I$ is a non-zero divisor. If S is a-adically complete, the same holds for S/I.

Proof. Clearly $J = \phi(J)$, moreover $I = \mathrm{Ann}(a) \subseteq \mathrm{Ann}(a^p) = \phi(I) \subseteq I$, and thus $I = \phi(I) = \mathrm{Ann}(J)$. Since S is reduced we have $I \cap J = IJ = 0$, and (8.1) is exact. Since $I = \mathrm{Ann}(a) = \mathrm{Ann}(a^2)$ the element $a \in S/I$ is a non-zero divisor. The last assertion follows from (8.1) because S/J and $S/(I+J)$ are annihilated by a. □

The exact sequence (8.1) can also be expressed by the following cartesian and cocartesian diagram of perfect rings.

$$\begin{array}{ccc} S & \longrightarrow & S/J \\ \downarrow & & \downarrow \\ S/I & \longrightarrow & S/(I+J) \end{array} \tag{8.2}$$

The following is contained in [GR17, Proposition 9.3.45].

Lemma 8.2. *Let S be a perfect ring and let $\xi = (\xi_0, \xi_1, \ldots) \in W(S)$ such that $\xi_0, \ldots, \xi_r$ generate the unit ideal of S. Then ξ is a non-zero divisor, and for $n \geqslant 0$ we have*

$$\xi W(S) \cap p^{n+r} W(S) = p^n(\xi W(S) \cap p^r W(S)). \tag{8.3}$$

In particular, $\xi W(S)$ is p-adically closed in $W(S)$ and p-adically complete.

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Proof. Using Lemma 8.1 with $a = \xi_0$ one reduces to the case where $\xi_0 = 0$ or where ξ_0 is a non-zero divisor. In the second case ξ is a non-zero divisor, and $\xi W(S) \cap p^n W(S) = p^n \xi W(S)$. If $\xi_0 = 0$ then $\xi = p\xi'$, and the proof of (8.3) is finished by induction on r. The last assertion follows easily. □

Definition 8.3. For a perfect ring S, an element $\xi = (\xi_0, \xi_1, \ldots) \in W(S)$ is called distinguished if $\xi_1 \in S$ is a unit and S is ξ_0-adically complete.

Remark 8.4. If a ring R is complete with respect to some linear topology and $x \in R$ is topologically nilpotent, then R is also x-adically complete; see the proof of [SPA16, Tag 090T].

Definition 8.5. A ring R is called perfectoid if there is an isomorphism $R \cong W(S)/\xi$ where S is perfect and $\xi \in W(S)$ is distinguished.

Remark 8.6. Definition 8.5 is equivalent to [BMS16, Definition 3.5]; moreover for $R = W(S)/\xi$ as in Definition 8.5 we have

$$S = R^\flat := \varprojlim(R/p, \phi) \tag{8.4}$$

canonically. Indeed, if $R = W(S)/\xi$ then $R/p = S/\xi_0$, the projective system $R/p \leftarrow R/p \leftarrow \cdots$ with arrows ϕ is identified with $S/\xi_0 \leftarrow S/\xi_0^p \leftarrow \cdots$ where the arrows are the projection maps, and (8.4) follows since S is ξ_0-adically complete. Moreover R is p-adically complete because this holds for $W(S)$ and because $\xi W(S)$ is p-adically closed by Lemma 8.2. If $\pi \in R$ is the image of $[\xi_0^{1/p}] \in W(S)$ then $\pi^p R = pR$. Thus R satisfies [BMS16, Definition 3.5]. Conversely, if the latter holds, then $R = W(R^\flat)/\xi$ where $R^\flat$ is perfect and ξ is distinguished. See also [GR17, 16.2.19].

Remark 8.7. If $R = W(S)/\xi$ is perfectoid then the ring $R/p = S/\xi_0$ is semiperfect and balanced (Definition 4.4). This is straightforward.

Remark 8.8. The perfectoid ring $R = W(S)/\xi$ is p-torsion free if and only if $\xi_0 \in S$ is a non-zero divisor. Indeed, since $p, \xi \in W(S)$ are regular elements, the kernels of $p : R \to R$ and of $\xi_0 : S \to S$ are isomorphic.

Remark 8.9. If $R = W(S)/\xi$ is perfectoid, the decomposition (8.2) of S with respect to $a = \xi_0$ gives a similar decomposition of R. More precisely, let $S_1 = S/\mathrm{Ann}(\xi_0)$ and $S_2 = S/(\xi_0^{p^{-\infty}})$ and $S_{12} = S_1 \otimes_S S_2$. Then $\xi \in W(S_i)$ is distinguished, and $R_i = W(S_i)/\xi$ is perfectoid. The sequence (8.1) gives an exact sequence

$$0 \to W(S) \to W(S_1) \oplus W(S_2) \to W(S_{12}) \to 0. \tag{8.5}$$

Since ξ is a non-zero divisor in $W(S_{12})$, we obtain an exact sequence

$$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0. \tag{8.6}$$

Here $R_2 = S_2$ and $R_{12} = S_{12}$ are perfect, while R_1 is p-torsion free perfectoid.

As an easy consequence we observe the following.

Lemma 8.10. *Every perfectoid ring R is reduced.*

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Proof. By (8.6) we can assume that R is either perfect (thus reduced) or p-torsion free. For $\pi \in R$ as in Remark 8.6 we have $\pi^p R = pR$, and $\phi : R/\pi \to R/p$ is bijective. Hence, if $a \in R$ satisfies $a^p = 0$ then $a = \pi b$. If R is p-torsion free it follows that $b^p = 0$, thus $a \in \pi^n R$ for all n, whence $a = 0$. □

We need the following form of tilting.

LEMMA 8.11. *Let R be a perfectoid ring and $B = R/p$. The functor $R' \mapsto R'/p$ from perfectoid R-algebras to B-algebras has a left adjoint $B' \mapsto B'^\sharp$. If B' is an étale B-algebra then $R' = B'^\sharp$ is the unique p-adically complete R-algebra such that $R'/p = B'$ and $R/p^n \to R'/p^n$ is étale for all n.*

Proof. Let $R = W(S)/\xi$ where ξ is distinguished, thus $S = B^\flat = \varprojlim(B, \phi)$, see Remark 8.6. For a B-algebra B' let $B'^\sharp = W(B'^\flat)/\xi$. This defines the left adjoint functor. Assume that $B \to B'$ is étale and let $R' = B'^\sharp$. We have to show that $R'/p = B'$ and that $R/p^n \to R'/p^n$ is flat. Let $x_n \in R/p$ be the image of ξ_0^{1/p^n}, so $x_n(R/p)$ is the kernel of $\phi^n : R/p \to R/p$. Since $B \to B'$ is étale, the diagram of rings

$$\begin{array}{ccc} B & \longrightarrow & B' \\ {\scriptstyle \phi^n}\big\downarrow & & \big\downarrow{\scriptstyle \phi^n} \\ B & \longrightarrow & B' \end{array}$$

is cocartesian, in particular $x_n B$ is the kernel of $\phi^n : B' \to B'$. It follows that $B' = B'^\flat/\xi_0$ (see [GD60, ch. 0, Proposition 7.2.7]) and thus $R'/p = B'^\flat/\xi_0 = B'$. We have $R/p^n = W(B^\flat)/([\xi_0^n], p^n, \xi)$ and similarly for R'. For fixed n let

$$C = W(B^\flat)/([\xi_0^n], p^n), \quad C' = W(B'^\flat)/([\xi_0^n], p^n).$$

In order to verify that $R/p^n \to R'/p^n$ is flat it suffices to show that $C \to C'$ is flat, or equivalently that $C/p \to C'/p$ is flat and that the associated graded rings satisfy $\mathrm{gr}_p(C') = \mathrm{gr}_p(C) \otimes_{C/p} C'/p$ (local flatness criterion). But $C/p = B^\flat/\xi_0^n \cong B/x_r^n$ when $p^r \geqslant n$, and $gr_p(C) \cong (C/p)[T]/T^n$; and similarly for C'. The assertion follows. □

8.1 The ring A_{cris} for perfectoid rings

Let $R = W(S)/\xi$ be a perfectoid ring where S is perfect and ξ is distinguished. Let $A_{\mathrm{inf}}(R) = W(S)$ and let $A_{\mathrm{cris}}(R) \to R$ be the universal p-adic PD extension. We have $A_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R/p)$ as rings. If R is perfect then $A_{\mathrm{cris}}(R) = A_{\mathrm{inf}}(R) = W(R)$. If R is p-torsion free, which means that the semiperfect ring R/p is a complete intersection in the sense of Definition 4.1 (see Remark 8.8), then $A_{\mathrm{cris}}(R)$ is p-torsion free. Let us verify that this also holds in general.

PROPOSITION 8.12. *Let $R = W(S)/\xi$ be a perfectoid ring as above and $R_i = W(S_i)/\xi$ as in Remark 8.9, for $i = 1, 2, 12$. We have an exact sequence*

$$0 \to A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R_1) \oplus W(R_2) \to W(R_{12}) \to 0.$$

In particular, the ring $A_{\mathrm{cris}}(R)$ is p-torsion free.

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Proof. Recall that

$$S_1 = S/\mathrm{Ann}(\xi_0), \quad S_2 = S/(\xi_0^{p^{-\infty}}) = R_2, \quad S_{12} = S_1 \otimes_S S_2 = R_{12}.$$

To simplify the notation, in the following we consider the empty index $\emptyset$ so that $S_\emptyset = S$ and $S_{\emptyset 2} = S_2$. For $i = \emptyset$ or $i = 1$ let A_i be the PD envelope of $[\xi_0]W(S_i) \subseteq W(S_i)$ relative to $p\mathbb{Z}_p \subset \mathbb{Z}_p$. Then $A_{\mathrm{cris}}(R_i)$ is the p-adic completion of A_i. The projection $W(S_i) \to W(S_{i2})$ extends to a PD homomorphism $g_i : A_i \to W(S_{i2})$, and we have the following commutative diagram of rings, where f_i is the canonical map.

$$\begin{array}{ccccc} W(S) & \xrightarrow{f} & A & \xrightarrow{g} & W(S_2) \\ \downarrow & & \downarrow & & \downarrow \\ W(S_1) & \xrightarrow{f_1} & A_1 & \xrightarrow{g_1} & W(S_{12}) \end{array}$$

We claim that $\mathrm{Coker}(f) \to \mathrm{Coker}(f_1)$ is bijective, f_1 is injective, and A_1 is p-torsion free. Assume this holds. The diagonal map $W(S) \to W(S_1) \times W(S_2)$ is injective, thus $W(S) \to W(S_1) \times A$ is injective. Since f_1 is injective, it follows that f is injective. Consider the homomorphisms of complexes

$$[W(S) \to W(S_1)] \xrightarrow{f_*} [A \to A_1] \xrightarrow{g_*} [W(S_2) \to W(S_{12})].$$

Here f_* and $g_* \circ f_*$ are quasi-isomorphism, thus g_* is a quasi-isomorphism. This remains true after p-adic completion, and the lemma follows.

To prove the claim we need a closer look on the construction of A and A_1. Let $\Lambda_0 = \mathbb{Z}_p[T]$ and let $\Lambda = \mathbb{Z}_p\langle T\rangle$ be the PD polynomial algebra, i.e. the $\mathbb{Z}_p$-subalgebra of $\mathbb{Q}_p[T]$ generated by $T^n/n!$ for $n \geqslant 1$. Define $\Lambda_0 \to W(S)$ by $T \mapsto [\xi_0]$. This extends to a PD homomorphism $\Lambda \to A$, and the resulting homomorphisms

$$h : W(S) \otimes_{\Lambda_0} \Lambda \to A, \quad h_1 : W(S_1) \otimes_{\Lambda_0} \Lambda \to A_1$$

are surjective. Since ξ_0 is a non-zero divisor in S_1, the homomorphism h_1 is bijective, and A_1 is torsion free.[4]

We consider the following ascending filtration of Λ and the associated filtrations of A and of A_1. For $m \geqslant 0$ let $F^m\Lambda = \Lambda \cap p^{-m}\mathbb{Z}_p[T]$. Then $\Lambda = \bigcup F^m\Lambda$, and $\mathrm{gr}^m\Lambda = F^m\Lambda/F^{m-1}\Lambda$ is a free Λ_0/p-module of rank 1 generated by $p^{-m}T^{d_m}$ where d_m is minimal such that p^m divides $d_m!$. For $i = \emptyset$ or 1 let $F^mA_i \subseteq A_i$ be the image of $W(S_i) \otimes_{\Lambda_0} F^m\Lambda$ and let $\mathrm{gr}^mA_i = F^mA_i/F^{m-1}A_i$. The homomorphism h_i induces surjective maps

$$F^m h_i : W(S_i) \otimes_{\Lambda_0} F^m\Lambda \to F^mA_i$$

and surjective maps

$$\mathrm{gr}^m h_i : S_i \cong W(S_i) \otimes_{\Lambda_0} \mathrm{gr}^m\Lambda \to \mathrm{gr}^mA_i$$

which map $1 \in S_i$ to $(p^{-m}d_m!)\gamma_{d_m}([\xi_0])$. The transition homomorphisms

$$W(S_i) \otimes_{\Lambda_0} F^{m-1}\Lambda \to W(S_i) \otimes_{\Lambda_0} F^m\Lambda$$

[4] In more detail, since $[\xi_0]$ is not a zero divisor in $W(S_1)$, the ring $A_1' = W(S_1) \otimes_{\Lambda_0} \Lambda$ is the absolute PD envelope of $[\xi_0]W(S_1) \subseteq W(S_1)$; see [Ber74, p. 64, (3.4.8)]. Since ξ_0 is not a zero divisor in S_1 we have $\mathrm{Tor}_1^{\Lambda_0}(W(S_1), \mathbb{F}_p) = 0$. Then $\mathrm{Tor}_1^{\Lambda_0}(W(S_1), \Lambda/p) = 0$ because the Λ_0-module Λ/p is isomorphic to the direct sum of infinitely many copies of $\mathbb{F}_p[T]/T^p$. Since Λ is p-torsion free it follows that A_1' is p-torsion free, and therefore $A_1' = A_1$.

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are injective because

$$\operatorname{Tor}_1^{\Lambda_0}(W(S_i), \operatorname{gr}^m \Lambda) \cong \operatorname{Tor}_1^{\mathbb{Z}_p}(W(S_i), \mathbb{F}_p) = 0.$$

Since h_1 is bijective it follows that $F^m h_1$ is bijective for $m \geqslant 0$, which for $m = 0$ means that $f_1 : W(S_1) \to A_1$ is injective, moreover $\operatorname{gr}^m h_1$ is bijective for $m \geqslant 1$. We consider the following commutative diagram of surjective maps.

$$\begin{array}{ccc} S & \longrightarrow & S_1 \\ {\scriptstyle \operatorname{gr}^m h}\downarrow & & \cong \downarrow {\scriptstyle \operatorname{gr}^m h_1} \\ \operatorname{gr}^m A & \longrightarrow & \operatorname{gr}^m A_1 \end{array}$$

We claim that $\operatorname{Ker}(S \to S_1) = \operatorname{Ann}(\xi_0)$ maps to zero in $\operatorname{gr}^m A$. Indeed, choose r such that $p^r \geqslant d_m$. For $a \in \operatorname{Ann}(\xi_0)$ we have $b = a^{p^{-r}} \in \operatorname{Ann}(\xi_0)$ and therefore $[a]\gamma_{d_m}([\xi_0]) = [b^{p^r - d_m}]\gamma_{d_m}([b\xi_0]) = 0$. It follows that $\operatorname{gr}^m A \to \operatorname{gr}^m A_1$ is bijective for $m \geqslant 1$, and thus $A/F^0 A \to A_1/F^0 A_1$ is bijective, which means that $\operatorname{Coker}(f) \to \operatorname{Coker}(f_1)$ is bijective as required. □

9. Windows and modules for perfectoid rings

As earlier, let $R = W(S)/\xi$ be a perfectoid ring where S is perfect and ξ is distinguished. The rings $A_{\mathrm{inf}}(R) = W(S)$ and $A_{\mathrm{cris}}(R)$ carry natural frame structures:

$$\underline{A}_{\mathrm{inf}}(R) = (A_{\mathrm{inf}}(R), \operatorname{Fil} A_{\mathrm{inf}}(R), R, \sigma, \sigma_1^{\mathrm{inf}}),$$

where $\operatorname{Fil} A_{\mathrm{inf}}(R) = \xi A_{\mathrm{inf}}(R)$ and $\sigma_1^{\mathrm{inf}}(\xi a) = \sigma(a)$, and

$$\underline{A}_{\mathrm{cris}}(R) = (A_{\mathrm{cris}}(R), \operatorname{Fil} A_{\mathrm{cris}}(R), R, \sigma, \sigma_1),$$

where $\operatorname{Fil} A_{\mathrm{cris}}(R)$ is the kernel of $A_{\mathrm{cris}}(R) \to R$, and $\sigma_1(a) = p^{-1}\sigma(a)$; this is well defined since $A_{\mathrm{cris}}(R)$ is p-torsion free by Proposition 8.12. The natural map $A_{\mathrm{inf}}(R) \to A_{\mathrm{cris}}(R)$ is a frame homomorphism

$$\lambda : \underline{A}_{\mathrm{inf}}(R) \to \underline{A}_{\mathrm{cris}}(R). \tag{9.1}$$

Indeed, let $c = \sigma_1(\xi)$ in $A_{\mathrm{cris}}(R)$. Then $c \equiv [\xi_0]^p/p + [\xi_1]^p \bmod pA_{\mathrm{cris}}(R)$ and thus $c \equiv [\xi_1]^p \bmod pA_{\mathrm{cris}}(R) + \operatorname{Fil} A_{\mathrm{cris}}(R)$, so c is a unit since ξ_1 is a unit. We have $\sigma_1 \circ \lambda = c \cdot \lambda \circ \sigma_1$ on $\xi A_{\mathrm{inf}}(R)$, so λ is a c-homomorphism of frames in the sense of [Lau10]. If R is perfect, λ is the identity and $c = 1$.

9.1 Descent of windows under λ

We need the following standard lemma. For a ring A let $\operatorname{LF}(A)$ be the category of finite projective A-modules.

Lemma 9.1. *Let $A_1 \to A_3 \leftarrow A_2$ be rings with surjective homomorphisms and $A = A_1 \times_{A_3} A_2$. Then the corresponding diagram of categories*

$$\begin{array}{ccc} \operatorname{LF}(A) & \longrightarrow & \operatorname{LF}(A_2) \\ \downarrow & & \downarrow \\ \operatorname{LF}(A_1) & \longrightarrow & \operatorname{LF}(A_3) \end{array}$$

is 2-cartesian.

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Proof. For a flat A-module M and $M_i = M \otimes_A A_i$ the natural map $M \to M_1 \times_{M_3} M_2$ is bijective. Thus the functor $\mathrm{LF}(A) \to \mathrm{LF}(A_1) \times_{\mathrm{LF}(A_3)} \mathrm{LF}(A_2)$ is fully faithful. For given $M_i \in \mathrm{LF}(A_i)$ and isomorphisms $M_1 \otimes_{A_1} A_3 \cong M_3 \cong M_2 \otimes_{A_2} A_3$ let $M = M_1 \times_{M_3} M_2$. We have to show that $M \in \mathrm{LF}(A)$ and that $M \otimes_A A_i \to M_i$ is bijective. One can choose a finite free A-module F and compatible surjective maps $g_i : F_i \to M_i$ where $F_i = F \otimes_A A_i$. Indeed, clearly one can arrange that g_1 or g_3 is surjective, and then take the direct sum. Next one can find compatible maps $s_i : M_i \to F_i$ with $g_i s_i = \mathrm{id}$. Indeed, choose s_1, which induces s_3, and use that $F_2 \to F_3 \times_{M_3} M_2$ is surjective to get s_2. This gives compatible isomorphisms $F_i \cong M_i \oplus \mathrm{Ker}(g_i)$, so M is a direct summand of F, and the assertion follows. □

Lemma 9.2. *Let R be a perfectoid ring. For R_1, R_2, R_{12} as in Remark 8.9 the natural diagrams of window categories*

$$\begin{array}{ccc} \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R)) & \longrightarrow & \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R_2)) \\ \downarrow & & \downarrow \\ \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R_1)) & \longrightarrow & \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R_{12})) \end{array} \tag{9.2}$$

and

$$\begin{array}{ccc} \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)) & \longrightarrow & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R_2)) \\ \downarrow & & \downarrow \\ \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R_1)) & \longrightarrow & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R_{12})) \end{array} \tag{9.3}$$

are 2-cartesian.

Proof. The rings R, R_1, R_2, R_{12} form a cartesian diagram with surjective maps, and the same holds for the associated rings A_{inf} and A_{cris}, the latter by Proposition 8.12. Thus the diagrams of frames that arise from (9.2) and (9.3) by deleting 'Win' are cartesian with surjective maps in all components. Using Lemma 9.1 the assertion follows easily. □

Proposition 9.3. *If $p \geqslant 3$, for every perfectoid ring R the functor*

$$\lambda^* : \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R)) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)) \tag{9.4}$$

associated to (9.1) is an equivalence of categories.

Proof. By Lemma 9.2 we can assume that R is either perfect or p-torsion free. In the perfect case λ is bijective. Let $R = W(S)/\xi$ where S is perfect and ξ is distinguished. If R is p-torsion free, (p, ξ) is a regular sequence in $W(S)$, and λ^* is an equivalence by [CL14, Proposition 2.3.1] (which requires $p \geqslant 3$). □

9.2 Breuil–Kisin–Fargues modules

Let $R = W(S)/\xi$ be perfectoid as before. In the following we write $A_{\mathrm{inf}} = A_{\mathrm{inf}}(R) = W(S)$.

Definition 9.4. A (locally free) Breuil–Kisin–Fargues module for R is a pair $(\mathfrak{M}, \varphi)$ where $\mathfrak{M}$ is a finite projective A_{inf}-module and where $\varphi : \mathfrak{M}^\sigma \to \mathfrak{M}$ is a linear map whose cokernel is annihilated by ξ. We denote by $\mathrm{BK}(R)$ the category of Breuil–Kisin–Fargues modules for R.

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In the case $R = \mathcal{O}_K$ for a perfectoid field K, free φ-modules over A_{inf} are studied by Fargues [Far15] in analogy with the classical theory of Breuil–Kisin modules [Kis06], and are called Breuil–Kisin–Fargues modules in [BMS16]. Here we only consider minuscule φ-modules, which correspond to p-divisible groups. When R is a perfect ring, then $A = W(R)$, and $\mathrm{BK}(R)$ is the category of Dieudonné modules over R in the usual sense.

Lemma 9.5. *For $(\mathfrak{M}, \varphi) \in \mathrm{BK}(R)$ the R-module $\mathrm{Coker}(\varphi)$ is projective.*

Proof. Cf. Lemma 3.1. Let $\mathfrak{N} = \mathfrak{M}^\sigma$ and $\bar{\mathfrak{M}} = \mathfrak{M} \otimes_A R$ and $\bar{\mathfrak{N}} = \mathfrak{N} \otimes_A R$. There is a unique linear map $\psi : \mathfrak{M} \to \mathfrak{M}^\sigma$ such that $\varphi \circ \psi = \xi$, and we obtain an exact sequence of finite projective R-modules

$$\bar{\mathfrak{N}} \xrightarrow{\bar{\varphi}} \bar{\mathfrak{M}} \xrightarrow{\bar{\psi}} \bar{\mathfrak{N}} \xrightarrow{\bar{\varphi}} \bar{\mathfrak{M}}. \tag{9.5}$$

We have to show that $\mathrm{Im}(\bar{\psi})$ is a direct summand of $\bar{\mathfrak{N}}$. This holds if and only if for each maximal ideal $\mathfrak{m} \subset R$ the base change of (9.5) to $k = R/\mathfrak{m}$ is exact. We have $p \in \mathfrak{m}$, so k is a perfect field of characteristic p. The natural homomorphism $A \to A_{\mathrm{inf}}(k) = W(k)$ maps ξ to p. Thus $\mathfrak{M} \otimes_A W(k)$ is a Dieudonné module over k, and it follows that the base change of (9.5) under $R \to k$ is exact as required. □

Lemma 9.5 implies that there is an equivalence of categories

$$\mathrm{Win}(\underline{A}_{\mathrm{inf}}(R)) \to \mathrm{BK}(R), \tag{9.6}$$

given by $(M, \mathrm{Fil}\, M, F, F_1) \mapsto (\mathfrak{M}, \varphi)$ with $\mathfrak{M} = \mathrm{Fil}\, M$ and $\varphi(1 \otimes x) = \xi F_1(x)$, see [CL14, Lemma 2.1.15]. The inverse functor is determined by $M = \mathfrak{M}^\sigma$ and $\mathrm{Fil}\, M = \{x \in M \mid \varphi(x) \in \xi \mathfrak{M}\}$ and $F(x) = 1 \otimes \varphi(x)$ for $x \in M$.

Remark 9.6. The frame $\underline{A}_{\mathrm{inf}}(R)$ depends on the choice of ξ, but the functor

$$\mathrm{BK}(R) \to \mathrm{Win}(\underline{A}_{\mathrm{inf}}(R)) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R))$$

defined as the composition of (9.4) and the inverse of (9.6) is independent of ξ as is easily verified.

9.3 *p*-divisible groups over perfectoid rings

Let R be a perfectoid ring. The functor $\Phi_{\underline{S}}$ of (3.3) for $\underline{S} = \underline{A}_{\mathrm{cris}}(R)$ defined by evaluation of the crystalline Dieudonné module is a functor

$$\Phi_R^{\mathrm{cris}} : \mathrm{BT}(\mathrm{Spec}\, R) \to \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)).$$

Proposition 9.7. *If $p \geqslant 3$ then the functor Φ_R^{cris} is an equivalence.*

Proof. Since the ring $A_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R/p)$ is torsion free by Proposition 8.12, there is another frame

$$\underline{A}_{\mathrm{cris}}(R/p) = (A_{\mathrm{cris}}(R/p), \mathrm{Fil}\, A_{\mathrm{cris}}(R/p), R/p, \sigma, \sigma_1)$$

defined by $\mathrm{Fil}\, A_{\mathrm{cris}}(R/p) = \mathrm{Fil}\, A_{\mathrm{cris}}(R) + pA_{\mathrm{cris}}(R)$ and $\sigma_1(x) = p^{-1}\sigma(x)$. The identity is a strict frame homomorphism $j : \underline{A}_{\mathrm{cris}}(R) \to \underline{A}_{\mathrm{cris}}(R/p)$ over the projection $\pi : R \to R/p$, and we obtain a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{BT}(\mathrm{Spec}\, R) & \xrightarrow{\Phi_R^{\mathrm{cris}}} & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle j} \\ \mathrm{BT}(\mathrm{Spec}\, R/p) & \xrightarrow{\Phi_{R/p}^{\mathrm{cris}}} & \mathrm{Win}(\underline{A}_{\mathrm{cris}}(R/p)) \end{array}$$

where $\Phi^{\text{cris}}_{R/p}$ is the functor $\Phi_{\underline{S}}$ for $\underline{S} = \underline{A}_{\text{cris}}(R/p)$. Here $\Phi^{\text{cris}}_{R/p}$ coincides with the functor of Theorem 6.3, but this is not needed. Since R/p is a balanced semiperfect ring (see Remark 8.7), the functor $\Phi^{\text{cris}}_{R/p}$ is an equivalence by Corollary 5.11; see also Remark 5.12.

For $G \in \mathrm{BT}(\operatorname{Spec} R/p)$ and $\underline{M} = \Phi^{\text{cris}}_{R/p}(G)$ there is a natural isomorphism of R-modules $M \otimes_{A_{\text{cris}}(R/p)} R \cong \mathbb{D}(G)_R$. Since $p \geqslant 3$, the divided powers on the ideal pR are topologically nilpotent. By the Grothendieck–Messing theorem [Mes72] and by [Lau10, Lemma 4.2] it follows that lifts of G under π and lifts of $\underline{M}$ under j correspond to lifts of the Hodge filtration in the same way. Therefore the functor Φ^{cris}_R is an equivalence. □

Theorem 9.8. *If $p \geqslant 3$, for every perfectoid ring R there is an equivalence*

$$\mathrm{BT}(\operatorname{Spec} R) \cong \mathrm{BK}(R)$$

between p-divisible groups and Breuil–Kisin–Fargues modules.

Proof. We have a chain of functors

$$\mathrm{BT}(\operatorname{Spec} R) \to \mathrm{Win}(\underline{A}_{\text{cris}}(R)) \leftarrow \mathrm{Win}(\underline{A}_{\text{inf}}(R)) \cong \mathrm{BK}(R), \tag{9.7}$$

where the last equivalence is (9.6). For $p \geqslant 3$ the two arrows are equivalences by Propositions 9.7 and 9.3. □

The equivalence of Theorem 9.8 is independent of the choice of the generator ξ of the kernel of $A_{\text{inf}} \to R$; see Remark 9.6.

10. Classification of finite group schemes

The equivalence between p-divisible groups and Breuil–Kisin–Fargues modules over perfectoid rings induces a similar equivalence for finite group schemes. For a scheme X let $p\mathrm{Gr}(X)$ be the category of commutative finite locally free p-group schemes over X.

10.1 A category of torsion modules

If A is a p-adically complete and p-torsion free ring, let $\mathrm{T}(A)$ be the category of finitely presented A-modules of projective dimension less than or equal to 1 which are annihilated by a power of p.

Lemma 10.1. *For a homomorphism of p-adically complete and p-torsion free rings $A \to A'$ and $M \in \mathrm{T}(A)$ we have $M \otimes_A A' \in \mathrm{T}(A')$.*

Proof. Let $0 \to Q \xrightarrow{u} P \to M \to 0$ be exact where P and Q are finite projective A-modules. Let $p^r M = 0$. There is a homomorphism $w : P \to Q$ such that $uw = p^r$ and $wu = p^r$. Let $Q' = Q \otimes_A A'$ etc. Since Q' is p-torsion free it follows that $0 \to Q' \to P' \to M' \to 0$ is exact, thus $M' \in \mathrm{T}(A')$. □

The category $\mathrm{T}(A)$ can be described in terms of the rings A/p^n as follows.

Lemma 10.2. *Let A be a p-adically complete p-torsion free ring, $A_n = A/p^n$. Let M be a finite A-module annihilated by p^r. We have $M \in \mathrm{T}(A)$ if and only if for every exact sequence $0 \to Q_n \to P_n \to M \to 0$ where P_n is a finite projective A_n-module with $n \geqslant r$, the A_{n-r}-module $Q_n/p^{n-r}Q_n$ is finite projective.*

Proof. Assume that $M \in \mathrm{T}(A)$ and let $0 \to Q_n \to P_n \to M \to 0$ be as in the lemma. Choose a finite projective A-module P with $P/p^n = P_n$ and let Q be the kernel of $P \to M$. Then Q is finite projective over A, and $Q_n = Q/p^n P$. We have $p^n P \subseteq p^{n-r} Q$, and thus $Q_n/p^{n-r} = Q/p^{n-r}$ is finite projective over A_{n-r}. Conversely, assume that the condition on M holds and let $0 \to Q \to P \to M \to 0$ be exact where P is finite projective over A. For $n \geqslant r$ let $P_n = P/p^n$ and $Q_n = Q/p^n P$. Then the A_n-module $\tilde{Q}_n = Q_{n+r}/p^n Q_{n+r}$ is finite projective, and we have $\tilde{Q}_{n+1}/p^n = \tilde{Q}_n$. It follows that $Q = \varprojlim Q_n = \varprojlim \tilde{Q}_n$ is finite projective over A. □

The category $\mathrm{T}(A)$ satisfies fpqc descent in the following sense.

Lemma 10.3. *Let $A \to A'$ be a homomorphism of p-adically complete p-torsion free rings such that $A/p \to A'/p$ is faithfully flat. Let A'' and A''' be the p-adic completions of $A' \otimes_A A'$ and $A' \otimes_A A' \otimes_A A'$. Then $\mathrm{T}(A)$ is equivalent to the category of pairs (M', α) where $M' \in \mathrm{T}(A')$ and $\alpha : M' \otimes_A A' \cong A' \otimes_A M'$ is an isomorphism that satisfies the usual cocycle condition over A'''.*

Proof. Lemma 10.1 gives a functor $M \mapsto (M', \alpha)$. By the local flatness criterion $A/p^n \to A'/p^n$ is faithfully flat for each n. It follows that the functor $M \mapsto (M', \alpha)$ is fully faithful, moreover each (M', α) with $M' \in \mathrm{T}(A)$ comes from an A-module M annihilated by a power of p, and it remains to show that $M \in \mathrm{T}(A)$. This is an easy consequence of Lemma 10.2. □

The category $\mathrm{T}(A)$ preserves projective limits of nilpotent immersions as follows.

Lemma 10.4. *Let $A = \varprojlim_n A^n$ for a surjective system $A^1 \leftarrow A^2 \leftarrow \cdots$ of p-adically complete p-torsion free rings such that $\operatorname{Ker}(A^{n+1} \to A^n)$ is nilpotent for each n. Then the obvious functor $\rho : \mathrm{T}(A) \to \varprojlim_n \mathrm{T}(A^n)$ is an equivalence.*

Proof. For $M \in \mathrm{T}(A)$ let $0 \to Q \to P \to M \to 0$ be exact where P and Q are finite projective over A. By the proof of Lemma 10.1 the base change under $A \to A^n$ gives an exact sequence $0 \to Q^n \to P^n \to M^n \to 0$. Since $P = \varprojlim_n P^n$ and $Q = \varprojlim_n Q^n$ it follows that $M = \varprojlim_n M^n$. In particular the functor ρ is fully faithful.

Conversely, let $M^n \in \mathrm{T}(A^n)$ with isomorphisms $M^{n+1} \otimes_{A^{n+1}} A^n \cong M^n$ be given. Let $M = \varprojlim_n M^n$ and choose a homomorphism $P \to M$ where P is finite projective over A such that $P^1 \to M^1$ is surjective, $P^n = P \otimes_A A^n$. Then $P^n \to M^n$ is surjective by Nakayama's lemma. The module $Q^n = \operatorname{Ker}(P^n \to M^n)$ is finite projective over A^n, and $Q^n = Q^{n+1} \otimes_{A^{n+1}} A^n$ by the proof of Lemma 10.1. It follows that $Q = \varprojlim_n Q^n$ is finite projective over A, and $0 \to Q \to P \to M \to 0$ is exact, thus $M \in \mathrm{T}(A)$. The base change under $A \to A^n$ of the last sequence remains exact, so $M^n = M \otimes_A A^n$. □

10.2 Torsion Breuil–Kisin–Fargues modules

Let $R = W(S)/\xi$ be a perfectoid ring where S is perfect and ξ is distinguished. We write again $A_{\mathrm{inf}} = A_{\mathrm{inf}}(R) = W(S)$, and $\operatorname{Fil} A_{\mathrm{inf}} = \operatorname{Ker}(A_{\mathrm{inf}} \to R) = \xi A_{\mathrm{inf}}$.

Definition 10.5. A torsion Breuil–Kisin–Fargues module for R is a triple $(\mathfrak{M}, \varphi, \psi)$ where $\mathfrak{M} \in \mathrm{T}(A_{\mathrm{inf}})$ and where

$$\operatorname{Fil} A_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathfrak{M} \xrightarrow{\psi} \mathfrak{M}^{\sigma} \xrightarrow{\varphi} \mathfrak{M} \tag{10.1}$$

are linear maps such that $\varphi \circ \psi$ and $\psi \circ (1 \otimes \varphi)$ are the multiplication maps. We denote by $\mathrm{BK}_{\mathrm{tor}}(R)$ the category of torsion Breuil–Kisin–Fargues modules over R.

Remark 10.6. For a homomorphism of perfectoid rings $R \to R'$ there is an obvious base change functor $\mathrm{BK}_{\mathrm{tor}}(R) \to \mathrm{BK}_{\mathrm{tor}}(R')$; see Lemma 10.1.

Remark 10.7. If R is p-torsion free, (p, ξ) is a regular sequence in A_{inf}, thus ξ is $\mathfrak{M}$-regular for each $\mathfrak{M} \in \mathrm{T}(A_{\mathrm{inf}})$, and torsion Breuil–Kisin–Fargues are equivalent to pairs $(\mathfrak{M}, \varphi)$ where the cokernel of φ is annihilated by ξ.

Remark 10.8. For a locally free Breuil–Kisin–Fargues module $\underline{\mathfrak{M}} = (\mathfrak{M}, \varphi)$ as in Definition 9.4 there is a unique ψ as in (10.1). In the following we will view $\underline{\mathfrak{M}}$ as a triple $(\mathfrak{M}, \varphi, \psi)$. A homomorphism $u : \underline{\mathfrak{M}} \to \underline{\mathfrak{M}}'$ in $\mathrm{BK}(R)$ is called an isogeny if it becomes bijective over $A[1/p]$. Then u is injective, and its cokernel lies in $\mathrm{BK}_{\mathrm{tor}}(R)$.

Étale descent. By an abuse of notation, let $(\operatorname{Spec} R/p)_{\text{ét}}$ denote the site of all affine étale R/p-schemes, with surjective families as coverings. For an étale R/p-algebra B' there is a unique homomorphism of perfectoid rings $R \to R'$ with $R'/p = B'$; see Lemma 8.11. We define presheaves of rings $\mathcal{A}_{\mathrm{inf}}$ and $\mathcal{R}$ on $(\operatorname{Spec} R/p)_{\text{ét}}$ by

$$\mathcal{R}(\operatorname{Spec} B') = R', \quad \mathcal{A}_{\mathrm{inf}}(\operatorname{Spec} B') = A_{\mathrm{inf}}(R').$$

For varying étale R/p-algebras B', the categories $\mathrm{LF}(R')$ of locally free R'-modules form a fibered category $\mathrm{LF}(\mathcal{R})$ over $(\operatorname{Spec} R/p)_{\text{ét}}$. Similarly we have fibered categories $\mathrm{LF}(\mathcal{A}_{\mathrm{inf}})$, $\mathrm{T}(\mathcal{A}_{\mathrm{inf}})$, $\mathrm{BK}(\mathcal{R})$, $\mathrm{BK}_{\mathrm{tor}}(\mathcal{R})$, $\mathrm{BT}(\operatorname{Spec} \mathcal{R})$, and $p\mathrm{Gr}(\operatorname{Spec} \mathcal{R})$ over $(\operatorname{Spec} R/p)_{\text{ét}}$; see Lemma 10.1 for $\mathrm{T}(\mathcal{A}_{\mathrm{inf}})$.

LEMMA 10.9. *The presheaves of rings $\mathcal{A}_{\mathrm{inf}}$ and $\mathcal{R}$ on $(\operatorname{Spec} R/p)_{\text{ét}}$ are sheaves. The fibered categories $\mathrm{LF}(\mathcal{R})$, $\mathrm{LF}(\mathcal{A}_{\mathrm{inf}})$, $\mathrm{T}(\mathcal{A}_{\mathrm{inf}})$, $\mathrm{BK}(\mathcal{R})$, $\mathrm{BK}_{\mathrm{tor}}(\mathcal{R})$, $\mathrm{BT}(\operatorname{Spec} \mathcal{R})$, and $p\mathrm{Gr}(\operatorname{Spec} \mathcal{R})$ over $(\operatorname{Spec} R/p)_{\text{ét}}$ are stacks.*

Proof. Let $x = [\xi_0] \in A := A_{\mathrm{inf}}$ and let $I = (x, p)$ as an ideal of A. Then A is I-adically complete. Let $B = R/p$. We fix a faithfully flat étale homomorphism $B \to B'$ and write $A' = A_{\mathrm{inf}}(B')$ and $A'' = A_{\mathrm{inf}}(B' \otimes_B B')$ and $A''' = A_{\mathrm{inf}}(B' \otimes_B B' \otimes_B B')$. The reduction modulo I^n of $A \to A'$ is étale, and the reductions modulo I^n of $A' \otimes_A A' \to A''$ and of $A' \otimes_A A' \otimes_A A' \to A'''$ are isomorphisms. Since the category $\mathrm{LF}(A)$ is equivalent to $\varprojlim_n \mathrm{LF}(A/I^n)$, étale descent of locally free modules shows that $\mathcal{A}_{\mathrm{inf}}$ is a sheaf and $\mathrm{LF}(\mathcal{A}_{\mathrm{inf}})$ and $\mathrm{BK}(\mathcal{R})$ are stacks. A similar argument shows that $\mathcal{R}$ is a sheaf and that $\mathrm{LF}(\mathcal{R})$, $p\mathrm{Gr}(\operatorname{Spec} \mathcal{R})$, and $\mathrm{BT}(\operatorname{Spec} \mathcal{R})$ are stacks.

We claim that A is x-adically complete and that the quotients A/x^n are p-adically complete and p-torsion free. Indeed, this is clear when R is perfect and thus $x = 0$, or when R is torsion free; in that case (x, p) is a regular sequence in A. In general, we use the exact sequence (8.5) where $A = W(S)$. Let $A_i = W(S_i)$. Since x is zero in A_2 and in A_{12} we get an exact sequence $0 \to A/x^n \to A_1/x^n \oplus A_2 \to A_{12} \to 0$. Here all rings except possibly A/x^n are p-adically complete and p-torsion free, thus the same holds for A/x^n. The limit over n shows that A is x-adically complete. The claim is proved.

Lemma 10.4 implies that $\mathrm{T}(A)$ is equivalent to $\varprojlim_n \mathrm{T}(A/x^n)$, and similarly for A' and A'' and A'''. The homomorphism $A/(x^n, p) \to A'/(x^n, p)$ is faithfully flat étale, hence Lemma 10.3 implies that $\mathrm{T}(A/x^n)$ is equivalent to the category of modules in $\mathrm{T}(A'/x^n)$ with a descent datum in $\mathrm{T}(A''/x^n)$. This proves that $\mathrm{T}(\mathcal{A})$ and $\mathrm{BK}_{\mathrm{tor}}(\mathcal{R})$ are stacks. □

Let us now continue the discussion of Remark 10.8.

LEMMA 10.10. *For $\underline{\mathfrak{M}} \in \mathrm{BK}_{\mathrm{tor}}(R)$, Zariski locally in $\operatorname{Spec}(R/p)$ there is an isogeny of locally free Breuil–Kisin–Fargues modules with cokernel $\underline{\mathfrak{M}}$.*

Proof. This is similar to [Kis06, Lemma (2.3.4)]. We have to find locally in $\mathrm{Spec}(R/p)$ a surjective map $\underline{\mathfrak{N}} \to \underline{\mathfrak{M}}$ where $\underline{\mathfrak{N}}$ is a locally free Breuil–Kisin–Fargues module. One can choose finite free A-modules Q and $\mathfrak{N}$ of equal rank and a commutative diagram with surjective vertical maps

$$\begin{array}{ccccc} \mathrm{Fil}\,A_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathfrak{N} & \xrightarrow{g} & Q & \xrightarrow{f} & \mathfrak{N} \\ {\scriptstyle 1\otimes\pi}\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi} \\ \mathrm{Fil}\,A_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathfrak{M} & \xrightarrow{\psi} & \mathfrak{M}^{\sigma} & \xrightarrow{\varphi} & \mathfrak{M} \end{array}$$

such that $f \circ g$ and $g \circ (1 \otimes f)$ are the multiplication maps. Assume that $u : \mathfrak{N}^{\sigma} \to Q$ is an isomorphism with $\rho u = \sigma^{*}(\pi)$. Then $\underline{\mathfrak{N}} = (\mathfrak{N}, fu, u^{-1}g)$ solves the problem. It is easy to see that u exists locally in $\mathrm{Spec}\,A$ and therefore also locally in $\mathrm{Spec}(R/p)$. □

Lemma 10.11. *For $H \in p\mathrm{Gr}(\mathrm{Spec}\,R)$, Zariski locally in $\mathrm{Spec}(R/p)$ there is an isogeny of p-divisible groups with kernel H.*

Proof. We have to find locally in $\mathrm{Spec}(R/p)$ an embedding of H into a p-divisible group. By [BBM82, Theorem 3.1.1] such an embedding exists Zariski locally in $\mathrm{Spec}(R)$, and therefore also Zariski locally in $\mathrm{Spec}(R/p)$. □

Theorem 10.12. *If $p \geqslant 3$, for every perfectoid ring R there is an equivalence*

$$p\mathrm{Gr}(\mathrm{Spec}\,R) \cong \mathrm{BK}_{\mathrm{tor}}(R).$$

Proof. This follows from Theorem 9.8 as in [Kis06, Theorem (2.3.5)]. More precisely, let $p\mathrm{Gr}(\mathrm{Spec}\,R)^{\circ}$ be the category of all $H \in p\mathrm{Gr}(\mathrm{Spec}\,R)$ with are the kernel of an isogeny in $\mathrm{BT}(\mathrm{Spec}\,R)$, and let $\mathrm{BK}_{\mathrm{tor}}(R)^{\circ}$ be the category of all $\underline{\mathfrak{M}} \in \mathrm{BK}_{\mathrm{tor}}(R)$ which are the cokernel of an isogeny in $\mathrm{BK}(R)$. The corresponding fibered categories $p\mathrm{Gr}(\mathrm{Spec}\,\mathcal{R})^{\circ}$ and $\mathrm{BK}_{\mathrm{tor}}(\mathcal{R})^{\circ}$ over $(\mathrm{Spec}\,R/p)_{\text{ét}}$ have associated stacks $p\mathrm{Gr}(\mathrm{Spec}\,\mathcal{R})$ and $\mathrm{BK}_{\mathrm{tor}}(\mathcal{R})$ by Lemmas 10.10 and 10.11. Moreover $p\mathrm{Gr}(\mathrm{Spec}\,R)^{\circ}$ (respectively $\mathrm{BK}_{\mathrm{tor}}(R)^{\circ}$) is equivalent to the full subcategory of the derived category of the exact category $\mathrm{BT}(\mathrm{Spec}\,R)$ (respectively $\mathrm{BK}(R)$) whose objects are isogenies $G^{0} \to G^{1}$ (respectively isogenies $\underline{\mathfrak{M}}_{1} \to \underline{\mathfrak{M}}_{0}$). Thus the equivalence of fibered categories $\mathrm{BT}(\mathrm{Spec}\,\mathcal{R}) \cong \mathrm{BK}(\mathcal{R})$ given by Theorem 9.8 induces an equivalence $p\mathrm{Gr}(\mathrm{Spec}\,\mathcal{R}) \cong \mathrm{BK}_{\mathrm{tor}}(\mathcal{R})$. □

10.3 Torsion Dieudonné modules

For completeness we record a similar classification of finite group schemes in the context of §§ 3 and 5.

Let R be an $\mathbb{F}_p$-algebra and let (A, σ) be a lift of R as in § 3. A torsion Dieudonné module over A is a triple $\underline{M} = (M, \varphi, \psi)$ where $M \in \mathrm{T}(A)$ and $\varphi : M^{\sigma} \to M$ and $\psi : M \to M^{\sigma}$ are linear maps with $\varphi\psi = p$ and $\psi\varphi = p$. We write $\mathrm{DM}_{\mathrm{tor}}(A)$ for the category of torsion Dieudonné modules over A.

An étale ring homomorphism $R \to R'$ extends to a unique homomorphism of lifts $(A, \sigma) \to (A', \sigma)$; each $A/p^{r} \to A'/p^{r}$ is the unique étale homomorphism that lifts $R \to R'$.

Lemma 10.13. *The functors $\Phi_{A'} : \mathrm{BT}(\mathrm{Spec}\,R') \to \mathrm{DM}(A')$ of (3.6) for all étale R-algebras R' induce functors*

$$\Phi_{A'}^{\mathrm{tor}} : p\mathrm{Gr}(\mathrm{Spec}\,R') \to \mathrm{DM}_{\mathrm{tor}}(A'). \tag{10.2}$$

If all functors $\Phi_{A'}$ are equivalences, then so are the functors $\Phi_{A'}^{\mathrm{tor}}$.

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Proof. The functor Φ_A induces a functor from the category of all $H \in p\mathrm{Gr}(\operatorname{Spec} R)$ which are the kernel of an isogeny of p-divisible groups to the category of all $\mathfrak{M} \in \mathrm{DM}_{\mathrm{tor}}(A)$ which are the cokernel of an isogeny of locally free Dieudonné modules. For given H or $\mathfrak{M}$, such isogenies exist locally in $\operatorname{Spec} R$. The lemma follows by descent; see Lemma 10.3. □

Corollary 10.14. *For a semiperfect ring R with a lift (A, σ) as in Theorem 5.7, the functor $\Phi_A^{\mathrm{tor}} : p\mathrm{Gr}(\operatorname{Spec} R) \to \mathrm{DM}_{\mathrm{tor}}(\underline{A})$ is an equivalence.*

Proof. For each étale R-algebra R' with associated lift A' the resulting divided powers on the kernel of $\phi : R' \to R'$ are induced from the divided powers on the kernel of $\phi : R \to R$ and are thus pointwise nilpotent. Hence $\Phi_{A'}$ is an equivalence by Theorem 5.7, and Lemma 10.13 applies. □

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