J. Inst. Math. Jussieu (2023), **22**(6), 2857–2877 2857
doi:10.1017/S1474748022000251

LINEAR SERIES ON GENERAL CURVES WITH PRESCRIBED INCIDENCE CONDITIONS

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(Received 24 May 2021; revised 27 February 2022; accepted 17 March 2022; first published online 6 June 2022)

Abstract Using degeneration and Schubert calculus, we consider the problem of computing the number of linear series of given degree d and dimension r on a general curve of genus g satisfying prescribed incidence conditions at n points. We determine these numbers completely for linear series of arbitrary dimension when d is sufficiently large, and for all d when either $r=1$ or $n=r+2$. Our formulas generalise and give new proofs of recent results of Tevelev and of Cela, Pandharipande and Schmitt.

2020 Mathematics Subject Classification: 14H10

1. Introduction

Having fixed positive integers r and s and setting $g=rs+s$ and $d=rs+r$, in a celebrated paper [3], Castelnuovo computed the number of linear series of type g^r_d on a general curve C of genus g. By degeneration to a g-nodal rational curve, he argued that this number equals the degree of the Grassmannian $\mathrm{Gr}(r+1,d+1)$ in its Plücker embedding, that is,

$$g!\cdot\frac{1!\cdot 2!\cdot\cdots\cdot r!}{s!\cdot(s+1)!\cdot\cdots\cdot(s+r)!}.$$

A rigorous modern presentation of Castelnuovo's argument[1] was first carried out by Griffiths and Harris [11]. More generally, the theory of limit linear series developed by Eisenbud and Harris [6, 7] allows one to compute the number of linear series on a general curve with ramification conditions imposed at fixed marked points, see also [18] for a more recent treatment.

Motivated by two recent papers of Tevelev [19] and Cela, Pandharipande and Schmitt [5], we consider a variant of this problem, where we impose incidence conditions on the

[1]The fact that Castelnuovo provided a plausibility argument rather than a complete proof has been immediately recognised. We quote from the Zentralblatt MATH review [14] of [3]: *Das Resultat, welches Herr Castelnuovo bekommen hat, gibt mit grosser Wahrscheinlichkeit den wahren Wert, weil sein Forderungssatz ...sehr leicht angenommen werden kann; doch können wir unseren Wunsch nicht unterdrücken, die obige Aufgabe auf einspruchsfreie Weise aufgelöst zu sehen.*

corresponding maps to projective spaces. Let $[C,x_1,\ldots,x_n]\in\mathcal{M}_{g,n}$ be a general n-pointed complex curve of genus g. We denote by $G^r_d(C)$ the variety of linear systems $\ell=(L,V)$ of type g^r_d on C. A general $\ell\in G^r_d(C)$ corresponds to a regular map $\phi_\ell\colon C\to\mathbb{P}^r$. Evaluation at the points $x_1,\ldots,x_n$ induces a rational map:

$$\mathrm{ev}_{(x_1,\ldots,x_n)}\colon G^r_d(C)\dashrightarrow\left(\mathbb{P}^r\right)^n//PGL(r+1)=:P^n_r, \tag{1}$$

to the moduli spaces of n points in $\mathbb{P}^r$.[2] We study the degree $L_{g,r,d}$ of the map $\mathrm{ev}_{(x_1,\ldots,x_n)}$ in the case when this map is generically finite and both spaces have nonnegative dimension. Since $G^r_d(C)$ is a smooth variety of dimension $\rho(g,r,d)=g-(r+1)(g-d+r)$, whereas $\dim(P^n_r)=rn-r^2-2r$ as long as $n\geq r+2$, one expects $\mathrm{ev}_{(x_1,\ldots,x_n)}$ to be generically finite precisely when:

$$n=\frac{dr+d+r-rg}{r}. \tag{2}$$

Equivalently, $L_{g,r,d}$ may be understood as the degree of the morphism:

$$\tau:\mathcal{M}_{g,n}(\mathbb{P}^r,d)\to\mathcal{M}_{g,n}\times(\mathbb{P}^r)^n,$$

where $\mathcal{M}_{g,n}(\mathbb{P}^r,d)$ is the moduli space of degree d maps $f:C\to\mathbb{P}^r$, with smooth domain and distinct marked points $x_1,\ldots,x_n\in C$, and the map τ remembers the pointed domain and the images of the x_i under f. Again, the map τ is expected to be generically finite exactly when (2) holds, and Brill-Noether theory guarantees that this is indeed the case as long as $d>0$.

If $y_1,\ldots,y_n\in\mathbb{P}^r$ are general points, $L_{g,r,d}$ counts the number of morphisms $f\colon C\to\mathbb{P}^r$ of degree d satisfying $f(x_i)=y_i$ for $i=1,\ldots,n$. When the points y_i are considered up to projective equivalence, these incidence conditions are intrinsic to ℓ. For large d, it turns out there is a very simple formula for this degree:

Theorem 1.1. *Suppose $d\geq rg+r$, or equivalently, $n\geq d+2$. Then:*

$$L_{g,r,d}=(r+1)^g.$$

We remark that the hypothesis $n\geq d+2$ is automatically satisfied whenever $g\leq 1$. Indeed, if instead $n\leq d+1$ and $g\leq 1$, then $d+1\geq n=d+1+\frac{d}{r}-g\geq d+\frac{d}{r}$, hence, $n\leq d+1\leq r+1$, a contradiction. On the other hand, we will also see that the inequality $d\geq rg+r$ is sharp in the sense that $L_{g,r,d}=(r+1)^g-(d+1)$ when $d=rg$, see Remark 3.4.

When $r=1$, the special case $d=g+1$ was studied under the guise of *scattering amplitudes* by Tevelev [19], who found the strikingly simple formula $L_{g,1,g+1}=2^g$. This raised the possibility, confirmed by Theorem 1.1, that in the range when d is relatively large, the degree $L_{g,r,d}$ has a simple expression. Using Hurwitz space techniques, Cela, Pandharipande and Schmitt [5] obtained general formulas for $L_{g,1,d}$, which they called *Tevelev degrees*; in particular, when $d\geq g+1$, they found again, $L_{g,1,d}=2^g$.

[2]The geometric invariant theory (GIT) quotient $\left(\mathbb{P}^r\right)^n//PGL(r+1)$ depends on a choice of linearisation, but our main point of study, the degree of $\mathrm{ev}_{(x_1,\ldots,x_n)}$, is independent of this choice.

The result of Theorem 1.1 can be compared to a certain virtual count of Bertram, Daskalopoulous and Wentworth [1, Theorem 2.9] in the range $d > 2g-2$, predating the theory of virtual fundamental classes on moduli spaces of stable maps.

When either $r = 1$ or $n = r+2$, or under the hypotheses of Theorem 1.1, we obtain a more general formula for $L_{g,r,d}$ in terms of Schubert calculus. For a positive integer a, we recall the notation σ_a for the class of the special Schubert cycle of codimension a consisting of those $(r+1)$-planes $V \in \mathrm{Gr}(r+1, d+1)$ meeting a fixed subspace $W \subseteq \mathbb{C}^{d+1}$ of dimension $d-a$. We also recall that σ_{1^r} denotes the class of the special Schubert cycle of codimension r consisting of those $(r+1)$-planes $V \in \mathrm{Gr}(r+1, d+1)$ whose intersection with a fixed codimension 2 linear subspace $U \subseteq \mathbb{C}^{d+1}$ has dimension at least r. Our main result is as follows:

Theorem 1.2. *Suppose that either:*

- $d \geq rg + r$, *(i.e. the same hypothesis as in Theorem 1.1),*
- $d = r + \frac{rg}{r+1}$ *(in which case,* $n = r+2$*), or*
- $r = 1$.

In each of these cases,

$$L_{g,r,d} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0 + \cdots + \alpha_r = (r+1)(d-r) - rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right].$$

In particular, comparing Theorem 1.1 with 1.2 when $d \geq rg + r$ yields a nontrivial combinatorial identity[3] .

In the second case, in which d is as small as possible, we have $(r+1)(d-r) = gr$, so the second term is interpreted to be 1, and Theorem 1.2 recovers Castelnuovo's formula for $s = g/(r+1)$. On the other hand, when $gr > (r+1)(d-r)$ (equivalently, $n < r+2$), the summation is interpreted to be zero, so that $L_{g,r,d} = 0$. Indeed, this corresponds to the case $\dim G^r_d(C) = \dim(P^n_r) < 0$, in which we find no such morphisms $f: C \to \mathbb{P}^r$.

The case of intermediate d when $r > 1$ is the most subtle, and will be addressed in later work.

For $r = 1$, Theorem 1.2, via Giambelli's formula, yields the following explicit formulas for $L_{g,1,d}$, the last of which agrees with the results of [5, Theorem 6], see Proposition 3.7 for details.

$$\begin{aligned} L_{g,1,d} &= \sum_{\alpha_0 + \alpha_1 = 2d-2-g} \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \cdot \sigma_{\alpha_0} \cdot \sigma_{\alpha_1} \\ &= \sum_{i=0}^{\left\lfloor \frac{2d-g-2}{2} \right\rfloor} \frac{(2d-g-2i-1)^2}{g+1} \binom{g+1}{d-i} \\ &= 2^g - 2 \sum_{i=0}^{g-d-1} \binom{g}{i} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}. \end{aligned}$$

[3] A combinatorial proof of this identity has been given by Gillespie, Reimer and Berg [10] after our paper appeared on arXiv, see §4 for a discussion.

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Here, we adopt the convention that $\binom{g}{j} = 0$ when $j < 0$, so in particular, we have again that $L_{g,1,d} = 2^g$ when $d \geq g+1$.

To prove Theorems 1.1 and 1.2, we proceed via a standard degeneration to a flag curve consisting of a rational spine and $g+1$ tails, one of which being rational and on which all the marked points specialise, the remaining g tails being elliptic curves. We reduce to a concrete problem in genus zero in §2, and then handle this problem via Schubert calculus in §3; care needs to be taken to avoid degenerate solutions, particularly excess contributions from constant maps $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ obtained from linear series with base points at some of the points x_i. Because such excess contributions in our setup persist when the hypotheses of Theorem 1.2 are not satisfied (see Remark 3.6), the general computation of $L_{g,r,d}$ remains open.

Our method in the case $r = 1$ also allows us to recompute the more general counts of Cela, Pandharipande and Schmitt [5], where some points of the source curve are constrained to have the same image. If $r = 1$ and $1 \leq k \leq d,n$, let $L'_{g,d,k}$ be the number of morphisms $f\colon C \to \mathbb{P}^1$ as before, but where we take $y_1 = y_2 = \cdots = y_k$, and the y_i otherwise general. (Note that our indexing differs from that of [5], where d is written as $g+1+\ell$ for some $\ell \in \mathbb{Z}$ and k is called r, whereas we have reserved the variable r to denote the dimension of the target projective space.) We find:

Theorem 1.3.

$$L'_{g,d,k} = \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \sigma_{k-1} \cdot \left[\sum_{i+j=2(d-1)-g-(k-1)} \sigma_i \sigma_j \right] - \int_{\mathrm{Gr}(2,d)} \sigma_1^g \sigma_{k-2} \cdot \left[\sum_{i+j=2(d-2)-g-(k-2)} \sigma_i \sigma_j \right].$$

The second term is taken to be zero when $k = 1$. Note that $L'_{g,d,1} = L_{g,1,d}$, so Theorem 1.3 agrees with Theorem 1.2 in the case $r = 1$. From here, the formulas of [5, Theorem 6] can be recovered by recursion, see Corollary 5.1. We sketch the proof of Theorem 1.3 in §5.1; a more general statement with detailed proofs is given in [4, §6].

Finally, we remark that the degeneration technique also allows one to impose ramification conditions at additional fixed points $p_1,\ldots,p_m \in C$, see §5.2.

Relation to other work. We discuss results related to this circle of ideas that appeared after our paper was published on arXiv. The count of Theorem 1.1 agrees with a virtual count of maps $C \to \mathbb{P}^r$ in Gromov-Witten theory as computed by Buch and Pandharipande [2], the so-called *virtual Tevelev degrees* of $\mathbb{P}^r$. We consider the map $\tau\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^r)^n$ be the map remembering $[(C,x_1,\ldots,x_n)]$ and the points $y_i = f(x_i)$. Then, under assumption (2), we have [2, §1.3]:

$$\tau_*([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)]^{\mathrm{vir}}) = (r+1)^g \cdot [\overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^r)^n].$$

for *all* d. When $d < rg + r$, the virtual count includes excess contributions we wish to exclude in our counts $L_{g,r,d}$.

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More generally, when $\mathbb{P}^r$ is replaced by an arbitrary target variety X, the corresponding virtual degrees are expressed in terms of the quantum cohomology of X (see [2, Theorem 1.3]). It is expected that for all Fano varieties X, the virtual count of maps of sufficiently large degree is enumerative, as in Theorem 1.1, see [13] for partial results in this direction.

2. Reduction to genus zero

In this section, we reduce the enumerative problem to genus zero via a standard limit linear series degeneration, see for example [12, 18]. We begin by recalling some notation, while assuming throughout some familiarity with basics of the theory of limit linear series [7].

We recall the usual notation for Schubert cycles in the Grassmannian $\mathrm{Gr}(r+1,d+1) := \mathrm{Gr}(r+1,V)$, where V is a $(d+1)$-dimensional vector space, following [9, §4]. For a nonincreasing sequence $\mu := (\mu_0 \geq \mu_1 \geq \cdots \geq \mu_r)$ and a flag $\mathbf{F} : V = V_{d+1} \supset V_d \supset \ldots \supset V_1 \supset V_0 = 0$, we introduce the Schubert cycle:

$$\Sigma_\mu = \Sigma_\mu(\mathbf{F}) := \Big\{\Lambda \in \mathrm{Gr}(r+1,d+1) : \dim\ \big(\Lambda \cap V_{d-r+1+i-\mu_i}\big) \geq i+1, \ \text{ for } i = 0,\ldots,r\Big\}.$$

Note that $\mathrm{codim}\big(\sigma_\mu, \mathrm{Gr}(r+1,d+1)\big) = |\mu| = \mu_0 + \cdots + \mu_r$. If $\mu = (1,\ldots,1,0) =: 1^r$, in projective terms, Σ_{1^r} consists of r-dimensional subspaces $L = \mathbb{P}(\Lambda) \subset \mathbb{P}(V) \cong \mathbb{P}^d$ intersecting a fixed codimension 2 subspace along an r-dimensional locus. We set $\sigma_\mu := [\Sigma_\mu] \in \mathrm{CH}^{|\mu|}\big(\mathrm{Gr}(r+1,g+1)\big)$.

For a smooth curve C and a linear series $\ell = (L,V) \in G^r_d(C)$, we denote by:

$$\alpha^\ell(p) := \big(0 \leq \alpha^\ell_0(p) \leq \alpha^\ell_1(p) \leq \cdots \leq \alpha^\ell_r(p) \leq d-r\big)$$

the *ramification sequence* at a point $p \in C$. Keeping with the tradition of [7] or [11], we write ramification indices of linear series nondecreasingly, whereas indices indexing Schubert cycles are written nonincreasingly. We formalise this practice as follows:

Definition 2.1. For any partition $\mu = (\mu_0 \geq \mu_1 \geq \cdots \geq \mu_r)$, denote by $\overline{\mu}$ the tuple of components of μ in reverse (increasing) order, that is, $\overline{\mu} = (\mu_r,\ldots,\mu_0)$.

We introduce the proper stack of limit linear series of type g^r_d:

$$\sigma : \widetilde{\mathcal{G}}^r_d \to \mathcal{M}^{\mathrm{ct}}_g$$

over the moduli space $\mathcal{M}^{\mathrm{ct}}_g$ of curves of compact type. For a curve C of compact type, we denote by $\overline{G}^r_d(C)$ the variety of limit linear series on C. For pairwise distinct smooth points $p_1,\ldots,p_n \in C_{\mathrm{reg}}$ and Schubert indices $\alpha^i = \big(0 \leq \alpha^i_0 \leq \cdots \leq \alpha^i_r \leq d-r\big)$, where $i = 1,\ldots,n$, we set:

$$\overline{G}^r_d\Big(C,(p_1,\alpha^1),\ldots,(p_n,\alpha^n)\Big) := \big\{\ell \in \overline{G}^r_d(C) : \alpha^\ell(p_i) \geq \alpha^i, \text{ for } i = 1,\ldots,n\big\},$$

viewed as a generalised degeneracy locus of expected dimension:

$$\rho(g,r,d,\alpha^1,\ldots,\ldots,\alpha^n) := g-(r+1)(g-d+r) - \sum_{i=1}^n \sum_{j=0}^r \alpha^i_j. \qquad (3)$$

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We now consider a degeneration to the following flag curve of genus g, already considered in [7]. Let $[C_0,x_1,\dots,x_n]$ be the n-pointed genus g curve of compact type consisting of a rational spine $R_{\rm sp}$ to which general elliptic tails $E_1,\dots,E_g$ are attached at general points $p_1,\dots,p_g\in R_{\rm sp}$, respectively, and a further rational component R_0 also attached at a general point x_0 of $R_{\rm sp}$. The marked points $x_1,\dots,x_n$ specialise to general points of R_0.

Let $\mathcal{C}\to(B,b_0)$ be the versal deformation space of $[C_0,x_1,\dots,x_n]$ and denote by $\tau_1,\dots,\tau_n\colon B\to\mathcal{C}$ the sections corresponding to the n marked points. We may assume that each point of B parametrises an n-pointed curve of genus g of compact type. We further consider the induced moduli map $B\to\mathcal{M}_g^{\rm ct}$ forgetting the markings and let:

$$\sigma_B\colon \widetilde{\mathcal{G}}^r_d/B := \widetilde{\mathcal{G}}^r_d\times_{\mathcal{M}_g^{\rm ct}} B\to B$$

be the corresponding family of limit linear series and consider the evaluation map:

$$\mathrm{ev}\colon \widetilde{\mathcal{G}}^r_d/B \dashrightarrow B\times P^n_r,\quad (C_b,\ell)\mapsto \Big(b,\Big(\phi_\ell(\tau_1(b)),\dots,\phi_\ell(\tau_n(b))\Big)\Big), \tag{4}$$

where ϕ_ℓ denotes the rational map to $\mathbb{P}^r$ corresponding to the aspect of the limit linear series ℓ on the component of C_b on which all the marked points $\tau_1(b),\dots,\tau_n(b)$ lie.

Using [8, Theorem 1.1], it follows that $\widetilde{\mathcal{G}}^r_d/B$ is smooth of dimension $3g-3+n+\rho(g,r,d)$ over B and every limit linear series on C_0 smooths to a linear series on a neighboring smooth curve. It follows that $\deg(\mathrm{ev})=L_{d,g,r}$. We will determine this degree by looking at the scheme-theoretic fibre $\mathrm{ev}^{-1}(b_0,y_1,\dots,y_n)$, where $y_1,\dots,y_n$ are general points in $\mathbb{P}^r$ considered up to projective equivalence. We will show in Lemma 3.2 that every point $[C_0,\ell]\in\mathrm{ev}^{-1}(b_0,y_1,\dots,y_n)$ corresponds to a limit linear series $\ell\in\overline{G}^r_d(C_0)$ which is base point free at each point $x_1,\dots,x_n$. In particular, $\mathrm{ev}^{-1}(b_0,y_1,\dots,y_n)$ is disjoint from the indeterminacy locus of the map ev defined in (4).

To that end, we wish to count limit linear series ℓ on C_0 of degree d and rank r, subject to the condition that, after twisting down base points on the R_0-aspect, the points $x_1,\dots,x_n$ have prescribed images in $\mathbb{P}^r$. For a limit linear series ℓ on C_0, we denote by $\ell_{R_0},\ell_{R_{\rm sp}}$ and ℓ_{E_i} its corresponding aspects. By the additivity of the Brill-Noether number for ℓ encoded in the very definition of a limit linear series, we have the following inequality:

$$\rho(g,r,d)\ge\rho\Big(\ell_{R_0},\alpha^{\ell_{R_0}}(x_0)\Big)+\rho\Big(\ell_{R_{\rm sp}},\alpha^{\ell_{R_{\rm sp}}}(x_0),\alpha^{\ell_{R_{\rm sp}}}(p_1),\dots,\alpha^{\ell_{R_{\rm sp}}}(p_g)\Big)$$
$$+\sum_{i=1}^g\rho\Big(\ell_{E_i},\alpha^{\ell_{E_i}}(p_i)\Big).$$

Since over the curve $[C_0,x_1,\dots,x_n]$ the map ev evaluates the R_0-aspect of each limit linear series, it follows that we must only consider the components of $\overline{G}^r_d(C_0)$ in which ℓ_{R_0} varies in a family of dimension $\rho(g,r,d)=\dim P^n_r$. This happens when the remaining aspects of ℓ satisfy $\rho\big(\ell_{E_i},\alpha^{\ell_{E_i}}(p_i)\big)=0$ for $i=1,\dots,g$ and $\rho\Big(\ell_{R_{\rm sp}},\alpha^{\ell_{R_{\rm sp}}}(x_0),\alpha^{\ell_{R_{\rm sp}}}(p_1),\dots,\alpha^{\ell_{R_{\rm sp}}}(p_g)\Big)=0$.

This implies that on each elliptic tail E_i, the ramification sequence at the node p_i must be equal to $(d-r-1,\dots,d-r-1,d-r)$. Indeed, we need $\alpha_r^{\ell_{E_i}}(p_i)=d-r$, or else:

$$\alpha_0^{\ell_{E_i}}(p_i)+\cdots+\alpha_r^{\ell_{E_i}}(p_i)\le r(d-r-1),$$

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but also $\alpha_{r-1}^{\ell_{E_i}}(p_i) \leq d-r-1$, or else E_i would carry a linear series of rank 1 and degree 1. We therefore have a unique choice of the E_i-aspect, precisely $\ell_{E_i} = (d-r-1)p_i + \left|(r+1)p_i\right|$, for $i = 1,\ldots,g$. By compatibility of the aspects of limit linear series, we find that $\alpha^{\ell_{R_{\mathrm{sp}}}}(p_i) = (0,1,\ldots,1)$ for $i = 1,\ldots,g$, that is, $\ell_{R_{\mathrm{sp}}}$ has a simple cusp at each of the points $p_i,\ldots,p_g$.

From here, on R_{sp}, the ramification sequence of ℓ at the point x_0:

$$\left(\alpha_0^{\ell_{R_{\mathrm{sp}}}}(x_0),\ldots,\alpha_r^{\ell_{R_{\mathrm{sp}}}}(x_0)\right)$$

must satisfy the equality:

$$\sum_{j=0}^{r} \alpha_j^{\ell_{R_{\mathrm{sp}}}}(x_0) = (r+1)(d-r) - rg,$$

whereas the R_0-aspect of ℓ satisfies:

$$\sum_{j=0}^{r} \alpha_j^{\ell_{R_0}}(x_0) = rg. \tag{5}$$

Let $\mu = (\mu_0 \geq \cdots \geq \mu_r) := \overline{\alpha_r^{\ell_{R_{\mathrm{sp}}}}(x_0)}$, that is, we write the partition $\left(\alpha_r^{\ell_{R_{\mathrm{sp}}}}(x_0),\ldots,\alpha_0^{\ell_{R_{\mathrm{sp}}}}(x_0)\right)$, where the ramification indices are given in descending order, and let λ be the complement of μ in $(d-r)^{r+1}$, that is, $\lambda_j = d-r-\mu_j$, for $j = 0,1,\ldots,r$. Summarising the discussion so far, for each limit linear series ℓ on C_0 contributing towards the degree of the map ev, one has:

$$\alpha^{\ell_{R_0}}(x_0) = \overline{\lambda}. \tag{6}$$

The number of possible aspects $\ell_{R_{\mathrm{sp}}}$ on R_{sp} with ramification sequence $\overline{\mu}$ at x_0 and cusps at $p_1,\ldots,p_g$ is given by:

$$\beta_\lambda := \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \sigma_\mu.$$

The transversality of the intersection follows from [6], see also [15], that is, for a general choice of the points $p_1,\ldots,p_g$ and x_0, one has precisely β_λ distinct linear series on R_{sp} with these property. Since $\sigma_{\lambda'} \cdot \sigma_\mu = 0$, for any Schubert index $\lambda' \neq \lambda$ with $|\lambda'| = rg$, whereas $\sigma_\lambda \cdot \sigma_\mu = 1$, we can write:

$$\sigma_{1^r}^g = \sum_{|\lambda|=rg} \beta_\lambda \cdot \sigma_\lambda \in \mathrm{CH}^g\big(\mathrm{Gr}(r+1,d+1)\big). \tag{7}$$

Definition 2.2. Given a partition $\lambda = \left(\lambda_0 \geq \cdots \geq \lambda_r\right)$ with $|\lambda| = rg$ and general points $y_1,\ldots,y_n \in \mathbb{P}^r$, we define $L_{g,r,d,\lambda}$ to be the number of maps $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ of degree $d-\lambda_r$ sending x_i to y_i for $i = 1,\ldots,n$ and with ramification sequence given by $\overline{\lambda}$ at x_0.

Such maps are obtained by twisting the R_0-aspect of each limit linear series ℓ on C_0 by the order λ_r of its base point x_0. Our degeneration shows:

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Proposition 2.3. *For a general n-pointed curve* $[C,x_1,\dots,x_n]$ *of genus g, the degree of the map* $\mathrm{ev}_{(x_1,\dots,x_n)}$ *is given by the formula:*

$$L_{g,r,d} = \sum_{|\lambda|=rg} \beta_\lambda L_{g,r,d,\lambda}.$$

Proof. We have already explained that $L_{g,r,d}$ is the degree of the map $\mathrm{ev}\colon \widetilde{\mathcal{G}}^r_d/B \dashrightarrow B\times P^n_r$. Having fixed general points $y_1,\dots,y_n\in\mathbb{P}^r$, the fibre over $(b_0,y_1,\dots,y_n)\in B\times P^n_r$ of the map ev is then *scheme-theoretically* isomorphic to the variety of limit linear series $\ell\in\overline{G}^r_d(C_0)$, whose R_0-aspect maps the marked points x_i to y_i for $i=1,\dots,n$. From the discussion above, it follows that $\overline{G}^r_d\big(C_0\big)$ contains β_λ components all isomorphic to the variety $G^r_d\big(R_0,(x_0,\overline{\lambda})\big)$; the remaining components of $\overline{G}^r_d(C_0)$ do not contribute to the degree of ev. Finally, observe that $L_{g,r,d,\lambda}$ is precisely the contribution to the degree of the map ev corresponding to the component $G^r_d\big(R_0,(x_0,\overline{\lambda})\big)$. □

3. Counting linear series with assigned incidences on $\mathbb{P}^1$

Having reduced both Theorems 1.1 and 1.2 to a question on rational curves, we use Schubert calculus to complete their proofs.

Let us first sketch the argument. The set of maps $\mathbb{P}^1\to\mathbb{P}^r$ counted by the number $L_{g,r,d,\lambda}$ naturally sits inside the projective space $\mathbb{P}^{(r+1)(d+1)-1}$ parametrising morphisms $f=[f_0,\dots,f_r]$ of degree d, as given by the intersection of the conditions:

(i) $f(x_i)=y_i$ for $i=1,\dots,n$,

(ii) f has ramification at least $\overline{\lambda}$ at x_0.

The conditions $f(x_i)=y_i$ cut out linear subspaces, while, upon summing over all λ with the multiplicities β_λ, the ramification conditions at x_0 cut out an intersection of g subvarieties of degree $r+1$. The expected degree of the intersection is therefore $(r+1)^g$, and we show in the proof of Theorem 1.1 that this intersection is indeed transverse when $d\geq rg+r$.

In general, however, the intersection described above has many excess components. Under the conditions of Theorem 1.2, we remove these excess contributions by passing to a certain incidence correspondence dominating $\mathbb{P}^{(r+1)(d+1)-1}$ to compute $L_{g,r,d}$.

3.1. Proof of Theorem 1.1

For a complex polynomial $u=a_0+\cdots+a_dt^d$, we denote by $c(u)$ the column vector of its coefficients. Let $\mathbb{P}^{(r+1)(d+1)-1}$ be the projective space parametrising $(r+1)$-tuples $(f_0,\dots,f_r)$ of polynomials of degree d in one variable viewed as sections of $\mathcal{O}_{\mathbb{P}^1}(d)$, up to simultaneous scaling, and not all zero. When not all polynomials f_i are zero and have no common zeroes, they define a map $f=[f_0,\dots,f_r]$ of degree d from $\mathbb{P}^1$ to $\mathbb{P}^r$.

We introduce the map:

$$\pi\colon \mathbb{P}^{(r+1)(d+1)-1} \dashrightarrow \mathrm{Gr}(r+1,d+1), \tag{8}$$

remembering the linear series spanned by $f_0,\dots,f_r$, whenever they are linearly independent. The indeterminacy locus of this map is irreducible of codimension $d-r+1$, for an

$(r+1)$-tuple of polynomials $(f_0,\dots,f_r)$ lies in the indeterminacy locus of π if and only if the $(r+1)\times(d+1)$-matrix of coefficients $\big(c(f_0),\dots,c(f_r)\big)$ has rank at most r.

For a Schubert variety $\Sigma_\lambda = \Sigma_\lambda(\mathbf{F}) \subseteq \mathrm{Gr}(r+1,d+1)$ of codimension at most rg in $\mathrm{Gr}(r+1,d+1)$, let $\widetilde{\Sigma}_\lambda := (\pi_1)_*\big(\pi_2^*(\Sigma_\lambda)\big)$ be the closure of its pullback under $\mathbb{P}^{(r+1)(d+1)-1}$. Because the codimension of Σ_λ is lower than that of the indeterminacy locus of π (by our assumption, $d-r+1>rg$), the cycle $\widetilde{\Sigma}_\lambda$ has the expected codimension of $|\lambda|=rg$ and defines a well-defined class $\widetilde{\sigma}_\lambda \in \mathrm{CH}^{rg}\big(\mathbb{P}^{(r+1)(d+1)-1}\big)$. Using (7) we have the formula:

$$\widetilde{\sigma}_{1^r}^g = \sum_{|\lambda|=rg} \beta_\lambda \widetilde{\sigma}_\lambda.$$

Recall that we have fixed n general points $y_1,\dots,y_n \in \mathbb{P}^r$. The condition on maps $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ that $f(x_i)=y_i$ for $i=1,\dots,n$ impose nr linear conditions on the matrix of coefficients $\big(c(f_0),\dots,c(f_r)\big)$. Observe that this condition is automatically satisfied for those i for which x_i is a base point of f. The points $y_1,\dots,y_n$ having been chosen to be general, these linear conditions are independent. Since $(r+1)(d+1)-1-nr=rg$, the conditions $f(x_i)=y_i$ give rise to a linear subspace:

$$\mathbf{L} \cong \mathbb{P}^{rg} \subseteq \mathbb{P}^{(r+1)(d+1)-1}.$$

Now, let $\Sigma_\lambda(x_0) = G^r_d\big(\mathbb{P}^1,(x_0,\overline{\lambda})\big)$ be the Schubert variety of $\mathrm{Gr}(r+1,d+1)$ parametrising linear series on $\mathbb{P}^1$ with ramification sequence at least $\overline{\lambda}$ at x_0. We wish to intersect its pullback $\widetilde{\Sigma}_\lambda(x_0)$ with $\mathbf{L}$ on $\mathbb{P}^{(r+1)(d+1)-1}$. We call a point $[f_0,\dots,f_r]$ in this intersection *generic* if $\langle f_0,\dots,f_r\rangle$ is a linear series of rank r with ramification sequence exactly $\overline{\lambda}$, and which defines a (nondegenerate) morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ after twisting down the base points at x_0 with $f(x_i)=y_i$ (in particular, $\langle f_0,\dots,f_r\rangle$ has no base points away from x_0).

Remark 3.1. We have already seen above that the condition $d \geq rg+r$ ensures that the classes $\widetilde{\sigma}_{1^r}^g$ and $\widetilde{\sigma}_\lambda$ live in codimension strictly smaller than that of the indeterminacy locus of π. However, as we will see in Lemma 3.2, the same condition $d \geq rg+r$ also ensures that $\mathbf{L}$ contains no points corresponding to degenerate maps $f\colon \mathbb{P}^1 \to \mathbb{P}^r$. In fact, this is already evident in the case of constant maps; indeed, suppose instead that $d \geq n-1$. Then, we may take the nonzero polynomials $f_0,\dots,f_r$ to vanish at $x_1,\dots,x_{n-1}$, and after twisting away all base points, the resulting map $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ to be the constant map with image y_n. Then, $f=[f_0,\dots,f_r]$ lies on the one hand in $\mathbf{L}$, and on the other hand in the indeterminacy locus of π.

Lemma 3.2. *The intersection points of $\widetilde{\Sigma}_\lambda(x_0)$ with $\boldsymbol{L}$ are generic in the previous sense. In particular, the intersection occurs away from the indeterminacy locus of π.*

Proof. We construct the locus $\mathbf{L}$ 'relatively', allowing the points $y_1,\dots,y_n$ to vary, and show that, for dimension reasons, the locus where $\mathbf{L}\cap\widetilde{\Sigma}_\lambda(x_0)$ contains nongeneric points cannot dominate the space of choices of the y_i. In particular, if the y_i are chosen to be general, we obtain the desired conclusion.

More precisely, let $V \subseteq (\mathbb{P}^r)^n$ be the open subset of collections of points $y_1,\dots,y_n \in \mathbb{P}^r$, where the y_i are in linearly general position, that is, no m of the y_i lie on a linear space of dimension $m-2$ if $2 \leq m \leq r+1$. Consider the product $V \times \mathbb{P}^{(r+1)(d+1)-1}$, where

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the second factor parametrises maps $f = [f_0,\dots,f_r]$ as before, and the closed subscheme $V \times \widetilde{\Sigma}_\lambda(x_0)$ of the expected dimension rg as defined above. We then define the locus, abusively denoted $\mathbf{L} \subseteq V \times \mathbb{P}^{(r+1)(d+1)-1}$, of maps f satisfying $f(x_i) = y_i$ for $i = 1,2,\dots,n$, by relativising the above construction.

We have a forgetful map $\psi\colon \mathbf{L} \cap \widetilde{\Sigma}_\lambda(x_0) \to V$, and wish to show that the locus of nongeneric points of source does not dominate V; to do so, we show that the locus of nongeneric points has dimension strictly less than that of V.

First, consider the locus on $\mathbf{L} \cap \widetilde{\Sigma}_\lambda(x_0)$ of nongeneric $f = [f_0,\dots,f_r] \in \mathbf{L} \cap \widetilde{\Sigma}_\lambda(x_0)$ away from the indeterminacy locus of π. Suppose that f has base points of total order k away from $x_0,\dots,x_n$ and order k' on $x_1,\dots,x_n$, and that $k+k' > 0$. We see upon twisting down by these base points that the locus of such f has the expected codimension $(r+1)(k+k')$ in $\widetilde{\Sigma}_\lambda(x_0)$, and the incidence conditions $f(x_i) = y_i$ impose at least $(n-k')r$ additional conditions inside $V \times \widetilde{\Sigma}_\lambda(x_0)$. In total, we find that the locus of possible f has codimension strictly greater than $rg + rn$ in $V \times \widetilde{\Sigma}_\lambda(x_0)$, and therefore cannot dominate V. Similarly, a parameter count shows that f cannot have ramification sequence strictly more than $\overline{\lambda}$ at x_0.

Consider now a point of $\mathbf{L} \cap \widetilde{\Sigma}_\lambda(x_0)$, for which $\dim\langle f_0,\dots,f_r\rangle \le r$. We show again by counting parameters that no such f can exist. By twisting away base points at x_0 (which decreases the number of moduli and the number of conditions by the same amount), we may assume that f is base point free at x_0. We may also assume that f has no base points away from $x_1,\dots,x_n$. Suppose now that f has k (simple) base points among these x_i, we label them as $x_{n-k+1},\dots,x_n$; we twist down our linear series to have degree $d-k$, and lose the corresponding k linear conditions. Note that in this case, the ramification condition at x_0 can no longer be imposed in terms of $f_0,\dots,f_r$ alone, since by assumption, the resulting map $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ is degenerate, that is, the corresponding linear series has dimension $r' < r$. Note, however, that if the remaining y_i do not themselves live in a linear subspace of $\mathbb{P}^r$ of dimension r', then this is impossible; we therefore need $n-k \le r'+1$.

Then, it must be true that if $x_1,\dots,x_{n-k}$ are general points of $\mathbb{P}^1$, there exists a map $f\colon \mathbb{P}^1 \to \mathbb{P}^{r'}$ of degree $d-k$ with $f(x_i) = y_i$ for $i = 1,\dots,n-k$. Therefore, we have:

$$(d-k+1)(r'+1)-1 \ge r'(n-k).$$

Rearranging yields:

$$k \le d - r'(n-d-1).$$

On the other hand, because $n-k \le r'+1$, we find:

$$(d-n+1) \ge r'(n-d-2).$$

However, by assumption, we have $n \ge d+2$ and $r' \ge 0$, so we have reached a contradiction. □

Lemma 3.3. *For a general choice of the points $x_1,\dots,x_n \in \mathbb{P}^1$ and $y_1,\dots,y_n \in \mathbb{P}^r$, the intersection of $\widetilde{\Sigma}_\lambda(x_0)$ and $\boldsymbol{L}$ is transverse.*

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Proof. Let $\mathcal{M}_{n,d,r}$ be the open subscheme of the space $\mathrm{Hom}_d(\mathbb{P}^1,\mathbb{P}^r)\times(\mathbb{P}^1)^n$ parametrising elements $([f\colon \mathbb{P}^1\to\mathbb{P}^r],x_1,\dots,x_n)$, where f is a nondegenerate morphism of degree d and the x_i are pairwise distinct points that in addition are distinct from a fixed point $x_0\in\mathbb{P}^1$.

One may construct $\mathcal{M}_{n,d,r}$ as an open subset of a $(\mathbb{P}^1)^n$-bundle over $\mathbb{P}^{(r+1)(d+1)-1}$. We have a smooth, regular map $\chi\colon \mathcal{M}_{n,d,r}\to \mathrm{Gr}(r+1,d+1)$, from which we can pull back the smooth, open Schubert cycle of linear series with ramification exactly $\overline{\lambda}$ at x_0 to obtain the smooth subscheme $Y_{n,d,r}$ parametrising the morphisms we wish to count. Finally, the projection $\phi\colon Y_{n,d,r}\to(\mathbb{P}^1)^{n+1}\times(\mathbb{P}^r)^n$ remembering the marked points and their images on the source is generically unramified of finite degree.

By construction, any nonzero tangent vector to the intersection $\widetilde{\Sigma}_\lambda(x_0)$ and $\mathbf{L}$ in the generic locus yields a nonzero relative tangent vector of ϕ. Thus, when the points x_i,y_i are general, there are no such tangent vectors, and the intersection is transverse. □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By the above discussion summarised in Proposition 2.3, it suffices to intersect nr linear conditions with $\widetilde{\sigma}^g_{1^r}$ on $\mathbb{P}^{(r+1)(d+1)-1}$ and compute the degree, that is:

$$L_{g,r,d}=\sum_{|\lambda|=rg}\beta_\lambda\mathbf{L}\cdot\widetilde{\sigma}_\lambda=\mathbf{L}\cdot\widetilde{\sigma}^g_{1^r}=\deg\left(\widetilde{\sigma}^g_{1^r}\right)=\deg\left(\sigma^g_{1^r}\right),$$

where the last two degrees are computed on $\mathrm{Gr}(r+1,d+1)$ and on $\mathbb{P}^{(r+1)(d+1)-1}$, respectively.

Theorem 1.1 then follows from the fact that the degree of $\widetilde{\sigma}_{1^r}$ is $r+1$. To see this, note that on $\mathrm{Gr}(r+1,d+1)$, the Schubert cycle Σ_{1^r} is the locus of $(r+1)$-planes intersecting a fixed codimension 2 subspace $P\subseteq H^0(\mathbb{P}^1,\mathcal{O}(d))$ in a subspace of dimension at least r. Identifying $\mathbb{P}^{(r+1)(d+1)-1}$ with the space of $(r+1)\times(d+1)$ matrices, whose entries are taken up to simultaneous scaling, the pullback of Σ_{1^r} may be identified with the determinantal locus of matrices, such that the $(r+1)\times 2$ submatrix formed by the first two columns has rank 1. This, in turn, is the pullback under linear projection from $\mathbb{P}^{(r+1)(d+1)-1}$ of the Segre embedding $\mathbb{P}^1\times\mathbb{P}^r\to\mathbb{P}^{2r+1}$. Denoting by h_1 and h_2 the pullbacks to $\mathbb{P}^1\times\mathbb{P}^{r+1}$ of the hyperplane classes of $\mathbb{P}^1$ and $\mathbb{P}^{r+1}$, observe that:

$$\deg\left(\mathbb{P}^1\times\mathbb{P}^r\right)=(h_1+h_2)^{r+1}=\binom{r+1}{1}h_1h_2^r=r+1.$$

This completes the proof. □

Remark 3.4. The inequality $d\geq rg+r$ in Theorem 1.1 is sharp. Indeed, the largest possible value of d outside of this range is $d=rg$, corresponding to $d=n-1$. In this case, following the proof of Theorem 1.1 shows that our intersection of cycles inside $\mathbb{P}^{(r+1)(d+1)-1}$ contains an additional zero-dimensional locus of constant maps $[f_0,\dots,f_r]$, where each f_i is a constant multiple of the degree $d=n-1$ polynomial vanishing at all of the points $x_1,\dots,x_n$ except one, x_i, and the image of f is the point y_i. There is

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one such map for each of the marked points x_i, so we find that $L_{g,r,d} = (r+1)^g - n = (r+1)^g - (d+1)$.

Remark 3.5. It is interesting to observe that on a smooth curve C of genus g, a general stable vector bundle E of rank $r+1$ and degree d has precisely $(r+1)^g$ line subbundles of maximal degree d', where $d-(r+1)d' = r(g-1)$ (see [16] or [17]). The reinterpretation of the numbers $L_{g,r,d}$ from this point of view will be pursued elsewhere.

3.2. Proof of Theorem 1.2

We recast the calculation of the previous section in the following light: we consider the incidence correspondence on $\mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1)$ of $(r+1)$-tuples of degree d polynomials, spanning a $r+1$-dimensional subspace of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$, then pullback Schubert cycle conditions on the Grassmannian side and linear conditions on the projective space side. This incidence correspondence is defined by pulling back the diagonal under the map:

$$(\pi, \mathrm{id})\colon \mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1) \dashrightarrow \mathrm{Gr}(r+1,d+1) \times \mathrm{Gr}(r+1,d+1),$$

and the condition $d \geq rg + r$ is needed in order to prevent the indeterminacy locus from being too large. In this section, we obtain formulas for $L_{g,r,d}$ in the cases $r = 1$ and $g \geq 1$ by shrinking this base locus.

More precisely, for $j = 0,1,\ldots,r$, let $\rho_j\colon \mathbb{P}^{(r+1)(d+1)-1} \dashrightarrow \mathbb{P}^d$ be the linear projection remembering $f_j \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, where we recall that $(f_0,\ldots,f_r)$ is the $(r+1)$-tuple of polynomials, whose coefficients are parametrised by $\mathbb{P}^{(r+1)(d+1)-1}$. We now consider the following incidence correspondence:

$$\begin{array}{ccc} & Z := \Big\{ \big([u],\Lambda\big) \in \mathbb{P}^d \times \mathrm{Gr}(r+1,d+1) : u \in \Lambda \Big\} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{P}^d & & \mathrm{Gr}(r+1,d+1) \end{array}$$

If $\mathcal{Q}$ denotes the rank $d-r$ tautological quotient bundle on $\mathrm{Gr}(r+1,d+1)$, then Z can be realised as the degeneracy locus of the composition:

$$\pi_1^*\big(\mathcal{O}_{\mathbb{P}^d}(-1)\big) \longrightarrow \mathcal{O}^{d+1}_{\mathbb{P}^d \times \mathrm{Gr}(r+1,d+1)} \longrightarrow \pi_2^*\big(\mathcal{Q}\big),$$

and thus has class:

$$\Big\{ c\big(\pi_2^*\mathcal{Q}\big) \cdot c\big(\pi_1^*\mathcal{O}_{\mathbb{P}^d}(1)\big)\Big\}_{d-r} = \sum_{i+j=d-r} \pi_2^*\big(\sigma_i\big) \cdot \pi_1^*\big(H^j\big) \in \mathrm{CH}^{d-r}\big(\mathbb{P}^d \times \mathrm{Gr}(r+1,d+1)\big),$$

where H is the hyperplane class on $\mathbb{P}^d$, and where we have also used that $c_j(\mathcal{Q}) = \sigma_j$.

Because the codimension of the base locus of ρ_j is $d+1 > d-r$, the closure:

$$Z_j := \big(\rho_j \times \mathrm{id}_{\mathrm{Gr}(r+1,d+1)}\big)^{-1}(Z)$$

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of the pullback of the correspondence Z has the same class, that is, $\sum_{i+j=d-r} \pi_2^*(\sigma_i) \cdot \pi_1^*(H^j)$, where this time $\pi_2 \colon \mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1) \to \mathrm{Gr}(r+1,d+1)$ denotes the second projection.

Proof of Theorem 1.2. We wish to compute the intersection inside $\mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1)$ of the nr linear conditions pulled back from $\mathbb{P}^{(r+1)(d+1)-1}$ given by the equations $f(x_i) = y_i$ for $i = 1,\ldots,n$, the pullback under π_2 of the Schubert cycles σ_λ, where λ is a Schubert index with $|\lambda| = rg$, and the classes of the cycles $Z_0,\ldots,Z_r$ defined above. We proceed as in Lemmas 3.2 and 3.3.

First, we introduce the incidence correspondence:

$$\mathcal{X} := \left\{([f_0,\ldots,f_r],\Lambda) \in \mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1) : f_i \in \Lambda\right\}$$

$$\mathbb{P}^{(r+1)(d+1)-1} \xleftarrow{\ \pi_1\ } \mathcal{X} \xrightarrow{\ \pi_2\ } \mathrm{Gr}(r+1,d+1)$$

We first claim that the intersection $\pi_2^*(\sigma_\lambda) \cdot \pi_1^*(\mathbf{L})$ in question is supported away from the locus of $(f,\Lambda) \in \mathcal{X}$, where f defines a degenerate map $f \colon \mathbb{P}^1 \to \mathbb{P}^{r'}$, for some $r' < r$. When $d \geq rg + r$, the same proof as in Lemma 3.2 applies.

Suppose that either $r = 1$ or $n = d+2$, there is such a $(f,\Lambda) \in Z$ in our intersection, and that k of the points $x_1,\ldots,x_n$ are base points of the r'-dimensional linear system Λ_f spanned by $f_0,\ldots,f_r$. As in the proof of Lemma 3.2, it must be the case that $k \geq n - r' - 1$.

Denote the total ramification of Λ_f at x_0 by t. Then,

$$\begin{aligned} t &\leq \dim \mathrm{Gr}(r'+1, d-k+1) \\ &\leq \dim \mathrm{Gr}(r'+1, d-n+r'+2) \\ &= (r'+1)(d-n+1). \end{aligned}$$

Thus,

$$\begin{aligned} d &\geq \frac{t}{r'+1} + n - 1 \\ &= \frac{t}{r'+1} + d + \frac{d}{r} - g, \end{aligned}$$

whence

$$d \leq rg - \frac{rt}{r'+1}.$$

On the other hand, we require that $\Lambda \in \sigma_\lambda$, where $|\lambda| = rg$, and $\Lambda_f \subseteq \Lambda$. Such a Λ can only exist if:

$$t + (r-r')(d-r) \geq rg,$$

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as $\dim(\Lambda/\Lambda_f) = r - r'$, and each dimension can contribute at most $d - r$ to the ramification of Λ at x_0. Since $r > r'$, we obtain:

$$t + (r - r')\left(rg - \frac{rt}{r'+1} - r\right) \geq rg$$
$$t\left(1 - \frac{(r-r')r}{r'+1}\right) + (r-r')(rg - r) \geq rg.$$

When $r = 1$, and thus $r' = 0$, we obtain a contradiction. It remains to consider the case $n = r + 2$, in which case:

$$rg = (d - r)(r + 1).$$

Then, comparing the inequalities:

$$t \leq (r'+1)(d - n + 1) = (r'+1)(d - r - 1)$$
$$t \geq rg - (r - r')(d - r) = (d - r)(r + 1) - (r - r')(d - r)$$

also yields a contradiction.

Therefore, we are back in the situation of Lemma 3.2, in which all intersection points occur where f is nondegenerate, and in particular, (f, Λ) lies away from the indeterminacy of the ρ_j. The same parameter counts show that f indeed defines a map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree d with vanishing exactly $\overline{\lambda}$ at x_0.

Furthermore, the intersection in question is transverse by exactly the same argument as in Lemma 3.3, so it suffices to compute the degree of the intersection cycle on $\mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1, d+1)$. This equals:

$$\int_{\mathbb{P}^{(r+1)(d+1)-1} \times \mathrm{Gr}(r+1,d+1)} \pi_2^*(\sigma_{1^r}^g) \cdot \pi_1^*(H^{nr}) \left(\sum_{i+j=d-r} \pi_2^*(\sigma_i) \cdot \pi_1^*(H^j)\right)^{r+1}$$

$$= \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r = (r+1)(d-r)-rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i}\right)\right],$$

as desired. □

Remark 3.6. While the proof of Theorem 1.2 shows that our refined incidence correspondence avoids the constant maps of Remark 3.1 when $r = 1$ or $n = r + 2$, this is not the case in general. Indeed, suppose that $r \geq 2$, $n \geq r + 3$ and $d \geq n - 1$. Then, take $f_0, \ldots, f_r$ to have simple zeroes at $x_1, \ldots, x_{n-1}$ and an order $d - (n-1)$ zero at x_0, such that $f = [f_0, \ldots, f_r]$ defines the constant map with image $y_n \in \mathbb{P}^r$. If λ is a Schubert index with $|\lambda| = rg$, then the condition that $f_0, \ldots, f_r \in \Lambda$, where $\Lambda \in \Sigma_\lambda(x_0) \subseteq \mathrm{Gr}(r+1, d+1)$, may be satisfied as long as $rg \leq (d - n + 1) + \dim \mathrm{Gr}(r, d) = (d - n + 1) + r(d - r)$. Substituting $rg = dr + r + d - rn$, this is equivalent to $n \geq \frac{r^2+r-1}{r-1} = r + 2 + \frac{1}{r-1}$. When $r \geq 2$ and $n \geq r + 3$, this is immediate.

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For $r=1$, Theorem 1.2 can be used to recover (via simple manipulations) the explicit formulas in terms of binomial coefficients for the degrees $L_{g,1,d}$. These numbers are also determined in [5] using excess intersection on Hurwitz spaces of admissible covers.

Proposition 3.7. *For $d \geq \frac{g+2}{2}$, we have:*

$$\begin{aligned} L_{g,1,d} &= \sum_{\alpha_0+\alpha_1=2d-2-g} \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \cdot \sigma_{\alpha_0} \cdot \sigma_{\alpha_1} \\ &= \sum_{i=0}^{\left\lfloor \frac{2d-g-2}{2} \right\rfloor} \frac{(2d-g-2i-1)^2}{g+1} \binom{g+1}{d-i} \\ &= 2^g - 2\sum_{i=0}^{g-d-1} \binom{g}{i} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}, \end{aligned}$$

where, in the last line, we take $\binom{g}{j} = 0$ when $j<0$.

Proof. We use Giambelli's formula $\sigma_{a,b} = \sigma_a \cdot \sigma_b - \sigma_{a+1} \cdot \sigma_{b-1} \in \mathrm{CH}^{a+b}\big(\mathrm{Gr}(2,g+1)\big)$ for $a \geq b$, coupled with the formula (see e.g. [11, p. 269]

$$\begin{aligned} \int_{\mathrm{Gr}(2,d+1)} \sigma_{a,b} \cdot \sigma_1^g &= \frac{a-b+1}{g+1} \cdot \binom{g+1}{d-b} \\ &= \binom{g}{d-b-1} - \binom{g}{d-b} \end{aligned}$$

for all $a \geq b$ with $a+b = 2d-2-g$. Substituting in the formula provided by Theorem 1.2 yields the claims. □

3.3. Degrees of determinantal Schubert cycles

We note here that comparison of the incidence correspondences given above in the proofs of Theorems 1.1 and 1.2 allows one to compute the degrees of pullbacks of Schubert cycles of low codimension on $\mathrm{Gr}(r+1,d+1)$ to $\mathbb{P}^{(r+1)(d+1)-1}$.

Proposition 3.8. *Let Σ_λ be a Schubert cycle of codimension $|\lambda| \leq d-r$ in $\mathrm{Gr}(r+1,d+1)$, and let $\widetilde{\Sigma}_\lambda$ be the closure of its pullback under the rational map $\pi\colon \mathbb{P}^{(r+1)(d+1)-1} \dashrightarrow \mathrm{Gr}(r+1,d+1)$. Then, the degree of $\widetilde{\Sigma}_\lambda$ is:*

$$\int_{\mathrm{Gr}(r+1,d+1)} \sigma_\lambda \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r-|\lambda|=(r+1)(d-r)-rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right].$$

Proof. Let $N = (r+1)(d+1)-1-|\lambda|$. Recall that the codimension of Σ_λ is strictly smaller that the codimension of the indeterminacy locus of π. Accordingly, adopting the

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notation of the previous two sections, we have:

$$\begin{aligned}\deg(\widetilde{\Sigma}_\lambda) &= \int_{\mathbb{P}^{(r+1)(d+1)-1}} H^N \cdot [\widetilde{\Sigma}_\lambda] \\ &= \int_{\mathbb{P}^{(r+1)(d+1)-1}\times \mathrm{Gr}(r+1,d+1)} \pi_1^*\big(H^N \cdot [\widetilde{\Sigma}_\lambda]\big)\cdot [Z_0]\cdots[Z_r] \\ &= \int_{\mathbb{P}^{(r+1)(d+1)-1}\times \mathrm{Gr}(r+1,d+1)} \pi_1^*\big(H^N\big)\cdot \pi_2^*(\Sigma_\lambda)\cdot [Z_0]\cdots[Z_r] \\ &= \int_{\mathbb{P}^{(r+1)(d+1)-1}\times \mathrm{Gr}(r+1,d+1)} \pi_1^*\big(H^N\big)\cdot \pi_2^*(\Sigma_\lambda)\cdot \left(\sum_{i+j=d-r} \pi_2^*(\sigma_i)\cdot \pi_1^*\big(H^j\big)\right)^{r+1} \\ &= \int_{\mathrm{Gr}(r+1,d+1)} \sigma_\lambda \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg-|\lambda|} \left(\prod_{i=0}^r \sigma_{\alpha_i}\right)\right],\end{aligned}$$

where we have used the equality:

$$\pi_1^*(\widetilde{\Sigma}_\lambda)\cap Z_0\cap\cdots\cap Z_r = \pi_2^*(\Sigma_\lambda)\cap Z_0\cap\cdots\cap Z_r$$

as *subschemes* of the incidence correspondence $\mathcal{X}$. □

4. Young tableaux interpretation

Comparison of Theorems 1.1 and 1.2 yields the following purely combinatorial statement.

Proposition 4.1. *Suppose that $g\ge 0, r\ge 1, d\ge rg+r$, and d is divisible by r. Then,*

$$\int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^r \sigma_{\alpha_i}\right)\right] = (r+1)^g.$$

Indeed, both sides are equal to $L_{g,d,r}$ whenever $n = d-g+1+\frac{d}{r}$ is an integer. However, when $d\ge g+r$, both sides are independent of d; for the left-hand side, this can be seen in terms of Schubert calculus but will also be made transparent in the combinatorial interpretation that follows. In particular, Proposition 4.1 holds under the weaker inequality $d\ge g+r$ with no condition on the divisibility by r.

We give a combinatorial interpretation of the left-hand side in terms of a Young tableaux. Consider a filling of the boxes of a $(r+1)\times(d-r)$ grid with:

- rg red integers among $1,2,\ldots,g$, with each appearing exactly r times, and
- $(r+1)(d-r)-rg$ blue integers among $0,1,\ldots,r$, with each appearing any number of times,

subject to the following conditions:

- the red integers are top- and left- justified, i.e. they appear above blue integers in the same column and to the left of blue integers in the same row,

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- the red integers are strictly increasing across rows and weakly increasing down columns, and
- the blue integers are weakly increasing across rows and strictly increasing down columns.

An example filling is given in the case $(g,d,r) = (6,15,2)$ below.

1	2	3	4	6	0	0	0	0	0	0	0	0
1	3	5	6	0	1	1	1	1	1	1	1	1
2	4	5	0	2	2	2	2	2	2	2	2	2

Note that the rightmost $d-r-g$ columns must be filled with the blue integers $0,1,\ldots,r$ in order, so a filling as above is determined by the leftmost g columns, which are those that may contain red integers. In particular, the number of such fillings is independent of d when $d \geq r+g$. Now, we claim that this number of fillings is given exactly by the intersection number on the left-hand side of Proposition 4.1. Indeed, by the Pieri rule, the term σ_{1^r} corresponds to the transposed semistandard Young tableau given by the red integers, and the broken strips formed by the blue entries equal to i correspond to the Schubert cycle σ_{α_i}.

Proposition 4.1 therefore implies:

Proposition 4.2. *Suppose $d \geq r+g$. Then, the number of fillings of a $(r+1)\times(d-r)$ grid satisfying the above conditions is equal to $(r+1)^g$.*

A combinatorial proof of Proposition 4.2 via the Robinson-Schrensted-Knuth (RSK) algorithm has been given by Gillespie, Reimer and Berg [10].

5. Variants

5.1. Linear series with fixed incidences and secancy conditions

We briefly explain how our methods also recover the more general *Tevelev degrees* of [5], where some of the points x_i are constrained to lie in the same fibre of f. Recall from §1 that, if $1 \leq k \leq n$, we defined $L'_{g,d,k}$ to be the number of morphisms $f\colon C \to \mathbb{P}^1$ of degree d sending general points $x_1,\ldots,x_n \in C$ to points $y_1,\ldots,y_n \in \mathbb{P}^1$, where $y_1 = y_2 = \cdots = y_k$, but the y_i are otherwise general.

More generally, we may fix integers $0 \leq a \leq k \leq d$, a general n-pointed curve $(C,x_1,\ldots,x_k,x_{k+1},\ldots,x_n)$ of genus g, where n is given by (2), and consider the variety:

$$G^{r,k-a}_{d,k}(C,x_1,\ldots,x_k) := \Big\{\ell \in G^r_d(C) : \dim\, \ell(-x_1-\cdots-x_k) \geq r-k+a\Big\},$$

parametrising linear systems ℓ whose induced map $\phi_\ell\colon C \dashrightarrow \mathbb{P}^r$ has the property that:

$$\big\langle \phi_\ell(x_1),\ldots,\phi_\ell(x_k)\big\rangle \cong \mathbb{P}^{k-a-1}.$$

Then $G^{r,k-a}_{d,k}(C,x_1,\ldots,x_k)$ is a determinantal variety of dimension:

$$\rho(g,r,d) - a(r+1-k+a).$$

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Fixing points $y_1,\dots,y_n \in \mathbb{P}^r$ general with the property that:

$$\dim \langle y_1,\dots,y_k\rangle = k-1-a,$$

one can ask for the number of maps $f\colon C \to \mathbb{P}^r$ of degree d, such that $f(x_i)=y_i$ for $i=1,\dots,n$. For any such map, the corresponding linear series $\ell := f^*\left|\mathcal{O}_{\mathbb{P}^r}(1)\right|$ lies in $G^{r,k-a}_{d,k}(C,x_1,\dots,x_k)$.

In the interest of simplicity, we deal only with the case:

$$r=1,\ a=k-1,$$

in which case, this number equals $L'_{g,d,k}$. We only sketch the proof; we refer the reader to [4, §6] for detailed proofs and more general statements.

Proof of Theorem 1.3. Consider a linear series V on our general curve C satisfying the needed incidence conditions. We employ a further degeneration after that of §2, allowing $x_1,\dots,x_k$ to coalesce onto a bubbled rational component R_k, attached to R_0 at x, and consider the resulting limit V_0 on this bubbled curve.[4] We find that the R_k-aspect of V_0 must have ramification sequence $(d-k,d-1)$ at x, and sends $x_1,\dots,x_k$ to the same point after twisting down the base points at x.

It now suffices to count linear series on R_0 with the aggregate ramification condition σ_1^g at x_0, the new ramification condition σ_{k-1} at x, an additional linear incidence condition at x (with image $y_1=\cdots=y_k$) and linear incidence conditions at $x_{k+1},\dots,x_n$. The computation of §3.2 yields the count:

$$\int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \sigma_{k-1}\cdot \left[\sum_{i+j=2(d-1)-g-(k-1)} \sigma_i\sigma_j\right].$$

However, we find the following extraneous solutions: if the linear series in question has a base point at x, then we twist down, so that the new ramification sequence is $(0,k-2)$, and d decreases by 1; in addition, we lose the linear incidence condition at x. Therefore, we see a (zero-dimensional) excess contribution of:

$$\int_{\mathrm{Gr}(2,d)} \sigma_1^g \sigma_{k-2}\cdot \left[\sum_{i+j=2(d-2)-g-(k-2)} \sigma_i\sigma_j\right].$$

Subtracting the above yields the formula for $L'_{g,d,k}$. One needs to check that there are no additional degenerate contributions, and that the intersections are transverse as before, but we omit the details. □

Applying the Pieri rule to the formula of Theorem 1.3 yields the following recursions, recovering [5, Proposition 7] after the change of coordinates $\mathrm{Tev}_{g,\ell,r} = L'_{g,g+\ell+1,r}$. These recursions are then used in [5] to obtain explicit formulas in terms of binomial coefficients.

[4]As explained in [4, §6], one should more precisely consider the degeneration of the data of both V and two (possibly linear-dependent) sections of V defining a map $f : C \to \mathbb{P}^1$. We do not discuss the details here.

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Corollary 5.1. *We have:*

$$L'_{g,d,1} = L'_{g-1,d-1,1} + L'_{g-1,d,2}$$

and

$$L'_{g,d,k} = L'_{g-1,d-1,k-1} + L'_{g-1,d,k+1}$$

for $k > 1$.

Remark 5.2. The proof of Theorem 1.1 may also be employed to show that $L'_{g,d,k} = 2^g$ whenever $n \geq d+k+1$. For general r, the number of linear series in question is $(r+1)^g$ whenever $n \geq d+a+2$.

However, even when $r=1$, the proof of Theorem 1.2 breaks down as soon as $k>1$, as we will see contributions from constant maps with value $y_1 = \cdots = y_k$ and base points at $x_{k+1} = \cdots = x_n$. Thus, the additional degeneration as above is needed to obtain the general formula for $L'_{g,d,k}$.

5.2. Linear series with imposed incidences and prescribed ramification

We fix a general pointed curve $[C, p_1, \ldots, p_m, x_1, \ldots, x_n] \in \mathcal{M}_{g,m+n}$, general points $y_1, \ldots, y_n \in \mathbb{P}^r$, as well as m partitions $\lambda_1, \ldots, \lambda_m$ of length $r+1$. We may consider morphisms $f\colon C \to \mathbb{P}^r$ of degree d satisfying $f(x_i) = y_i$ for $i = 1, \ldots, n$, and f has ramification sequence at least $\overline{\lambda_j}$ at p_j for $j = 1, \ldots, m$. Suppose, for simplicity, that the $(r+1)$-st part of each λ_j is zero, so that f has no base points. Equivalently, like in (1) we can consider the evaluation map:

$$\mathrm{ev}_{(x_1,\ldots,x_n)}\colon G^r_d\Big(C, (p_1, \overline{\lambda}_1), \ldots, (p_m, \overline{\lambda}_m)\Big) \dashrightarrow P^n_r, \tag{9}$$

and ask for its degree when the dimension of the two varieties in question are equal. Using (3), we expect a finite number of such maps $f\colon C \to \mathbb{P}^r$ whenever $\rho(g, r, d, \overline{\lambda}_1, \ldots, \overline{\lambda}_m) = rn - (r^2+2r)$, that is, when:

$$n = \frac{dr + d + r - \lambda_{\mathrm{tot}} - gr}{r}, \tag{10}$$

where $\lambda_{\mathrm{tot}} := |\lambda_1| + \cdots + |\lambda_m|$ is the total size of the partitions λ_j. Let $L^{\lambda_1,\ldots,\lambda_m}_{g,r,d}$ be this number, that is, the degree of the map given by (9).

Degenerating the general genus g curve C to a flag curve as in §2 so that the points $p_1, \ldots, p_m$ specialise to general points on the component R_{sp}, whereas $x_1, \ldots, x_n$ specialise, as before, to general points of the rational component R_0, we reduce the computation to the numbers $L_{g,r,d,\lambda}$, as defined in Definition 2.2, where now $|\lambda| = rg + \lambda_{\mathrm{tot}}$. Following the proof of Theorem 1.1, we obtain the following result.

Proposition 5.3. *Suppose that* $d \geq rg + r + \lambda_{\mathrm{tot}}$, *or equivalently,* $n \geq d+2$. *Then,*

$$L^{\lambda_1,\ldots,\lambda_m}_{g,r,d} = (r+1)^g \cdot \prod_{j=1}^{m} \deg(\widetilde{\Sigma}_{\lambda_j}),$$

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where $\deg(\widetilde{\Sigma}_{\lambda_j})$ *is the degree of the cycle* $\widetilde{\Sigma}_{\lambda_j}$ *on* $\mathbb{P}^{(r+1)(d+1)-1}$ *obtained by taking the closure of the pullback of* $\Sigma_{\lambda_j}(x_0) \subseteq \mathrm{Gr}(r+1,d+1)$ *under the rational map* $\pi\colon \mathbb{P}^{(r+1)(d+1)-1} \dashrightarrow \mathrm{Gr}(r+1,d+1)$ *(see Proposition 3.8).*

Similarly, closely following the proof of Theorem 1.2, we obtain:

Proposition 5.4. *Suppose that:*

- $d \geq rg + r + \lambda_{\mathrm{tot}}$*,*
- $n = r+2$*, or*
- $r = 1$*.*

Then,

$$L_{g,r,d}^{\lambda_1,\dots,\lambda_m} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \prod_{j=1}^m \sigma_{\lambda_j} \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg-\lambda_{\mathrm{tot}}} \left(\prod_{i=0}^r \sigma_{\alpha_i} \right) \right].$$

Indeed, in both results, the only significant modification is that the total ramification imposed at x_0 after degeneration is $rg+\lambda_{\mathrm{tot}}$, instead of rg. However, this number is equal to $dr+d+r-nr$ in both cases, and from here, the proofs go through without change.

Acknowledgments. This problem was brought to our attention by Rahul Pandharipande. We thank him as well as Alessio Cela and Johannes Schmitt for interesting discussions related to this circle of ideas. We also thank Laurent Manivel and Maria Gillespie for pointing out an error in §4 in an earlier version and communicating alternate proofs of Proposition 4.1, and Benjamin Gunby for help with combinatorial manipulation. Finally, we thank the anonymous referee for numerous improvements.

Gavril Farkas has been supported by the DFG Grant *Syzygien und Moduli* and by the European Research Council (ERC) Advanced Grant SYZYGY. This project has received funding from the ERC under the European Union Horizon 2020 research and innovation programme (grant agreement No. 834172). Carl Lian was supported by an NSF Postdoctoral Fellowship, grant DMS-2001976.

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