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NEIGHBOURHOOD LATTICES - A POSET APPROACH TO TOPOLOGICAL SPACES

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In this paper neighbourhood lattices are developed as a generalisation of topological spaces in order to examine to what extent the concepts of "openness", "closedness", and "continuity" defined in topological spaces depend on the lattice structure of $\mathcal{P}(X)$, the power set of X.

A general pre-neighbourhood system, which satisfies the poset analogues of the neighbourhood system of points in a topological space, is defined on an $\wedge$-semi-lattice, and is used to define open elements. Neighbourhood systems, which satisfy the poset analogues of the neighbourhood system of sets in a topological space, are introduced and it is shown that it is the conditionally complete atomistic structure of $\mathcal{P}(X)$ which determines the extension of pre-neighbourhoods of points to the neighbourhoods of sets.

The duals of pre-neighbourhood systems are used to generate closed elements in an arbitrary lattice, independently of closure operators or complementation. These dual systems then form the backdrop for a brief discussion of the relationship between pre-neighbourhood systems, topological closure operators, algebraic closure operators, and Čech closure operators.

Continuity is defined for functions between neighbourhood lattices, and it is proved that a function $f: X \to Y$ between topological spaces is continuous if and only if corresponding direct image function between the neighbourhood lattices $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ is continuous in the neighbourhood sense. Further, it is shown that the algebraic character of continuity, that is, the non-convergence aspects, depends only on the properites of pre-neighbourhood systems. This observation leads to a discussion of the continuity properties of residuated mappings. Finally, the topological properties of normality and regularity are characterised in terms of the continuity properties of the closure operator on a topological space.

1. INTRODUCTION

A topology on a set X can be defined in terms of any of four equivalent primitive concepts, namely: openness (interior operators), closedness (closure operators), convergence (either nets or filters), and neighbourhoods (neighbourhood filters). However, the study of topology only begins with the algebraic structure which is introduced on the power set of X by, say, closure operators. The importance of the topological structure lies in its appropriateness for defining continuous functions between topological spaces as a generalisation of "intuitive" continuity. Further, each of the four primitive concepts listed above leads to a significant insight into the nature of continuous functions.

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From a lattice theoretic view-point, a topological space could be considered in the following way: Let (X, T) be a topological space and consider $\mathcal{P}(X)$ the power set of X. The topology T selects a union complete sublattice of "open" elements from the complete, atomistic, completely distributive Boolean lattice $\mathcal{P}(X)$. In addition, the continuous functions between topological spaces are those functions whose associated lattice function, the inverse image function, preserves openness.

This paper is concerned with developing a generalisation of a topological space, called a *neighbourhood lattice.* This topology-like structure, which depends directly on the internal order structure of the poset itself and on poset filters, is used to characterise open and closed elements in terms of neighbourhoods or dual neighbourhoods and to introduce a "natural" generalisation of continuity between neighbourhood lattices. We will show that the theory of neighbourhood lattices includes "reasonable" generalisations of the topological concepts of "openness", "closedness", "neighbourhood", and "continuity". "Reasonable" in this context means that when the lattice is $\mathcal{P}(X)$ for some topological space (X, T), the "new" definitions and theorems agree with the standard topological results.

The observation that a topology was a union complete sublattice of a Boolean lattice motivated the search for a connection between lattices and topological spaces. One starting point in these endeavours, as discussed in the survey article by Johnstone [9], was the consideration of a completely distributive lattice which was the analogue of the lattice of open subsets of a topological space. The objects of study then became the category of frames (or locales) and the associated functors.

By contrast, this paper utilises the order structure of the poset itself and poset filters as a starting point in generalising a topological space. This approach is based on the observation that if X is a poset, $\mathcal{P}(X)$ does not reflect in any way the order relation on X, that is, the lattice structure of $\mathcal{P}(X)$ is cardinality dependent and not order dependent. For example, if all topologies on the chain $\mathbb{R}$ were known, then all topologies on any set X of cardinality c would be known.

2. PRE-NEIGHBOURHOOD MAPPINGS

We will show in this section that if L is an arbitrary lattice, we can define pre-neighbourhood systems for elements of L and these pre-neighbourhood systems give rise to a topology–like structure on L.

We begin with the observation that if (X, T) is a topological space, then T determines for each $x \in X$ a neighbourhood system of the point x with respest to T, denoted by $\eta(x)$, and given by $\eta(x) = \{y\colon y \in \mathcal{P}(X) \text{ and } (\exists g \in T)(x \in g \subseteq y)\}$. Further, $\eta(x)$ is a filter in $\mathcal{P}(X)$. Conversely, it can be shown that if for each $x \in X$, there is an associated filter in $\mathcal{P}(X)$, denoted $\eta(x)$, which satisfies conditions 1–3 listed below, then there is exactly one topology T on X, given by $A \in T$ if and only if

$(\forall x \in A)(A \in \eta(x))$, such that $\eta(x)$ is the neighbourhood system of X. The conditions are:

(1) $N \in \eta(x)$ and $N \subseteq M \Rightarrow M \in \eta(x)$,
(2) $N \in \eta(x)$ and $M \in \eta(x) \Rightarrow N \cap M \in \eta(x)$, and
(3) $N \in \eta(x) \Rightarrow (\exists \theta \subseteq N)(x \in \theta \in \eta(x)$ and $(\forall y \in \theta)(\theta \in \eta(y)))$.

Hence, a topology on a set X is uniquely determined when the neighbourhoods of each of its points are known. It should be noted that the "standard" topological proof of this theorem makes use of the completeness and atomiticity of $\mathcal{P}(X)$, and the fact that in $\mathcal{P}(X)$, the neighbourhoods of a set are determined by the neighbourhoods of its points. We will clarify the rôles that the completeness and atomiticity of $\mathcal{P}(X)$ play in defining topological neighbourhoods in Section 3.

We will make use of the following notation and conventions:
If L is an orthocomplemented lattice, then $'$ will be used to denote complementation; I will be used to denote an arbitrary indexing set; if P is a poset, $\vee$ and $\wedge$ will represent the operations of sup and inf; while $\cup$ and $\cap$ will be used to denote the set theoretic operations of union and intersection, $\bigvee_{\alpha\in\mathsf{I}} x_\alpha$ will be used to represent $\bigvee\{x_\alpha : \alpha \in \mathsf{I}\}$, with similar abbreviations used for $\bigwedge_{\alpha\in\mathsf{I}} x_\alpha$, $\bigcup_{\alpha\in\mathsf{I}} A_\alpha$, and $\bigcap_{\alpha\in\mathsf{I}} A_\alpha$; 1 will represent the greatest element of P and $\emptyset$ will represent the least element of P; $\mathcal{F}(P) = \{F : F \text{ is a filter of } P\}$; $\mathcal{I}(P) = \{I : I \text{ is an ideal of } P\}$; if $x \in P$, then $[x) = \{y : y \in P \text{ and } y \geqslant x\}$ and $(x] = \{y : y \in P \text{ and } y \leqslant x\}$.

DEFINITION 1: Let P be an $\wedge$-semi-lattice. A function $\eta : P \to \mathcal{F}(P)$ is called a *filter mapping*. If η is a filter mapping on P, then $g \in P$ is said to be *neighbourhood open* (*η-open* or simply *open*) if $\eta(g) = [g)$. Further, if η is a filter mapping on P, we will let $G = \{g : g \in P \text{ and } \eta(g) = [g)\}$. Finally, a filter mapping η is said to be a *pre-neighbourhood mapping* if $(\forall x, t \in P)(t \in \eta(x) \iff (\exists g \in G)(x \leqslant g \leqslant t))$.

If η is a pre-neighbourhood mapping, then $\eta(x)$ satisfies the poset analogues of conditions 1–3 for a topological neighbourhood system.

The proofs of those Lemmas which are straightforward computations and which follow directly from the corresponding definitions will be omitted.

LEMMA 2. *If P is an $\wedge$-semi-lattice and η is a pre-neighbourhood mapping on P then*

i) *η is antitone, that is, $x \leqslant y \Rightarrow \eta(y) \subseteq \eta(x)$,*
ii) *$(\forall x \in P)(\eta(x) \subseteq (x])$, and*
iii) *$1 \in P \Rightarrow \eta(1) = [1)$.*

Further, whenever all terms exist in P, we have

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iv) $\eta\Big(\bigvee_{\alpha\in I} x_\alpha\Big) \subseteq \bigcap_{\alpha\in I} \eta(x_\alpha)$,

v) $\bigcap_{\alpha\in I} [x_\alpha) = \Big[\bigvee_{\alpha\in I} x_\alpha\Big)$, and

vi) $\bigcup_{\alpha\in I} \eta(x_\alpha) \subseteq \eta\Big(\bigwedge_{\alpha\in I} x_\alpha\Big)$.

A subset of a topological space (X, T) is open if and only if it is a neighbourhood (that is, a member of the neighbourhood system) of each of its elements. The following theorem is the pre-neighbourhood version of this result.

THEOREM 3. *Let P be an $\wedge$-semi-lattice and η be a pre-neighbourhood mapping on P. $g \in G \Leftrightarrow (\forall x \in P)(x \leqslant g \Rightarrow g \in \eta(x))$.*

PROOF: $g \in G$ and $x \leqslant g \Rightarrow [g) = \eta(g) \subseteq \eta(x)$. Thus, $g \in \eta(x)$. To prove the converse, simply let $x = g$. ∎

Theorem 4 will show how subsets of an $\wedge$-semi-lattice determine pre-neighbourhood mappings.

THEOREM 4. *Let P be an $\wedge$-semi-lattice and $G \subseteq P$. If*

i) $(\forall x \in P)(\exists g \in G)(g \geqslant x)$, *and*

ii) *G is an $\wedge$-semi-lattice of P,*

then there is exactly one pre-neighbourhood mapping $\eta\colon P \to \mathcal{F}(P)$ such that G is the set of open elements of η. Conversely, if $\eta\colon P \to \mathcal{F}(P)$ is a pre–neighbourhood mapping, then G satisfies (i) and (ii).

PROOF: Suppose G satisfies (i) and (ii). Define, $\forall x \in P, \eta(x) = \{t\colon (\exists g \in G)(x \leqslant g \leqslant t)\}$. It is a routine verification to show that η is a pre-neighbourhood mapping. Further, η is unique by Definition 1. Conversely, suppose $\eta\colon P \to \mathcal{F}(P)$ is a pre-neighbourhood mapping. $\eta(x) \neq \emptyset$ implies G satisfies (i). $g_1, g_2 \in G$ and $x \leqslant g_1 \wedge g_2 \Rightarrow g_1, g_2, g_1 \wedge g_2 \in \eta(x)$. Thus, $g_1 \wedge g_2$ is open by Theorem 3, and G satisfies (ii). ∎

COROLLARY 4.1. *If P is an $\wedge$-semi-lattice with 1, $G \subseteq P$, and (i)' $1 \in G$, then in Theorem 4 we have (i) $\Leftrightarrow$ (i)'.*

If P is an $\wedge$-semi-lattice, then $G = P$ generates a *discrete* pre-neighbourhood mapping on P, and, further, if $\emptyset$ and $1 \in P$, then $G = \{\emptyset, 1\}$ determines a *trivial* pre-neighbourhood mapping on P. Finally, if (X, T) is a topological space, then $G = \{A\colon A \in \mathcal{P}(X) \text{ and } A' \in T\}$ determines a pre-neighbourhood mapping on $\mathcal{P}(X)$ but G is not closed under the formation of arbitrary suprema.

3. Neighbourhood mappings and neighbourhood lattices

We will now show that if L is any lattice, which need be neither complete nor atomic, we can introduce a neighbourhood system on L which agrees with topological neighbourhoods when the lattice is $\mathcal{P}(X)$ for a topological space (X, T).

We will find the following definitions useful.

DEFINITION 5: Let L be a lattice and $L_1 \subseteq L$. L_1 is a $\vee$-*semi-complete sublattice of* L if $(\forall A \subseteq L_1)$ ($\bigvee_{a\in A} a$ exists in $L_1 \Leftrightarrow \bigvee_{a\in A} a$ exists in L and they are equal). Similarly, we have an $\wedge$-*semi-complete sublattice of* L.

DEFINITION 6: Let L be a lattice. A pre-neighbourhood mapping $\eta\colon L \to \mathcal{F}(L)$ is a *neighbourhood mapping* if i) G is a $\vee$-semi-complete sublattice of L, and ii) $\emptyset \in L \Rightarrow \eta(\emptyset) = [\emptyset)$. Further, if η is a pre-neighbourhood mapping on L, then the pair (L, η) will be called a *pre-neighbourhood lattice*. Similarly if η is a neighbourhood mapping defined on L, then the pair (L, η) will be called a *neighbourhood lattice*.

We will now show how a neighbourhood mapping can be characterised by the propoerties of its set of open elements.

THEOREM 7. *Let L be a lattice and $G \in \mathcal{P}(L)$. If*

i) $(\forall x \in L)(\exists g \in G)(x \leqslant g)$,
ii) *G is a $\vee$-semi-complete sublattice of L, and*
iii) $\emptyset \in L \Rightarrow \emptyset \in G$,

then G is the set of open elements of L if we define a neighbourhood mapping $\eta\colon L \to \mathcal{F}(L)$ by $\eta(x) = \{t\colon (\exists g \in G)(x \leqslant g \leqslant t)\}$. Conversely, if (L, η) is a neighbourhood lattice, then G satisfied (i), (ii), and (iii).

PROOF: Immediate from Theorem 4. ∎

COROLLARY 7.1. *If L is a lattice with 1, $G \in \mathcal{P}(L)$, and if (i)' $1 \in G$, then in Theorem 7 we have (i) $\Leftrightarrow$ (i)'.*

It is clear from Definition 6 and Theorem 7 that if (X, T) is a topological space, then $\eta\colon \mathcal{P}(X) \to \mathcal{F}(\mathcal{P}(X))$ defined by $\eta(A) = \{N : (\exists g \in T)(A \subseteq g \subseteq N)\}$ is a neighbourhood mapping on $\mathcal{P}(X)$. This neighbourhood mapping η is called the *induced neighbourhood mapping on* $\mathcal{P}(X)$ and the pair $(\mathcal{P}(X), \eta)$ is the *induced neighbourhood lattice of* (X, T). Further, it follows from Theorem 7, that if X is a set, and $(\mathcal{P}(X), \eta)$ is a neighbourhood lattice, then $G = \{g\colon g \in \mathcal{P}(X) \text{ and } \eta(g) = [g)\}$ is a topology on X, and $(\mathcal{P}(X), \eta)$ is the induced neighbourhood lattice of (X, G).

It should also be noted that not only are the η-open elements of the induced neighbourhood lattice $(\mathcal{P}(X), \eta)$ identical with the open elements of (X, T) but also that the neighbourhood system of $\{x\}$ in $\mathcal{P}(X)$ is identical with the neighbourhood system of x in (X, T). Thus, we will identify $\eta(x)$ and $\eta(\{x\})$.

We will now examine the properties of neighbourhood mappings on conditionally complete lattices.

THEOREM 8. *Let (L, η) be a conditionally complete pre-neighbourhood lattice. (L, η) is a neighbourhood lattice $\Leftrightarrow \eta\left(\bigvee_{\alpha\in \mathsf{I}} x_\alpha\right) = \bigcap_{\alpha\in \mathsf{I}} \eta(x_\alpha)$, whenever $\bigvee_{\alpha\in \mathsf{I}} x_\alpha$ exists in L.*

PROOF: Let (L, η) be a neighbourhood lattice. By Lemma 2(iv), $\eta\left(\bigvee_{\alpha\in \mathsf{I}} x_\alpha\right) \subseteq \bigcap_{\alpha\in \mathsf{I}} \eta(x_\alpha)$. In particular, this shows that $\bigcap_{\alpha\in \mathsf{I}} \eta(x_\alpha) \neq \emptyset$. Now, $y \in \bigcap_{\alpha\in \mathsf{I}} \eta(x_\alpha) \Rightarrow \forall\alpha \in \mathsf{I}$, $y \in \eta(x_\alpha) \Rightarrow \forall\alpha \in \mathsf{I}$, $\exists g_\alpha \in G$, such that $x_\alpha \leqslant g_\alpha \leqslant y$ and $\bigvee g_\alpha$ is open. Hence, $y \in \eta\left(\bigvee_{\alpha\in \mathsf{I}} x_\alpha\right)$. Conversely, suppose $\eta\left(\bigvee_{\alpha\in \mathsf{I}} x_\alpha\right) = \bigcap_{\alpha\in \mathsf{I}} \eta(x_\alpha)$. $\forall\alpha \in \mathsf{I}$, $g_\alpha \in G \Rightarrow \eta\left(\bigvee_{\alpha\in \mathsf{I}} g_\alpha\right) = \bigcap_{\alpha\in \mathsf{I}} \eta(g_\alpha) = \bigcap_{\alpha\in \mathsf{I}} [g_\alpha) = \left[\bigvee_{\alpha\in \mathsf{I}} g_\alpha\right)$. If $\emptyset \in L$, let $\mathsf{I} = \emptyset$. ∎

Abbott [1] defined a neighbourhood mapping as, essentially, a pre-neighbourhood mapping satisfying the condition given in Theorem 8.

Theorem 9 will show that in order for a conditionally complete atomistic pre-neighbourhood lattice to be a neighbourhood lattice, we must be able to "extend" the pre-neighbourhood system from the atoms to any element of the lattice. It is this "extension" process which fails for the pre-neighbourhood mapping on a topological space (X, T) determined by $G = \{A\colon A \in \mathcal{P}(X) \text{ and } A' \in T\}$.

THEOREM 9. *Let (L, η) be a conditionally complete, atomistic pre-neighbourhood lattice. (L, η) is a neighbourhood lattice $\Leftrightarrow$ (i) $\eta(\emptyset) = [\emptyset)$ and (ii) for each collection $\{p_\alpha : (\forall\alpha \in \mathsf{I})(p_\alpha \text{ is an atom})\}$, we have $\eta\left(\bigvee_{\alpha\in \mathsf{I}} p_\alpha\right) = \bigcap_{\alpha\in \mathsf{I}} \eta(p_\alpha)$, whenever $\bigvee_{\alpha\in \mathsf{I}} p_\alpha$ exists in L.*

PROOF: Theorem 8 establishes the proof in one direction. To prove the converse. $\forall\beta \in \mathsf{I}$, $g_\beta \in G$ and $\bigvee_{\beta\in \mathsf{I}} g_\beta \in L \Rightarrow g_\beta = \bigvee_{\alpha\in \mathsf{I}_0} p_{\beta_\alpha} \Rightarrow \bigvee_{\beta\in \mathsf{I}} g_\beta = \bigvee_{\beta\in \mathsf{I}} \left(\bigvee_{\alpha\in \mathsf{I}_0} p_{\beta_\alpha}\right)$. Thus,

$$\eta\left(\bigvee_{\beta\in \mathsf{I}} g_\beta\right) = \left[\bigvee_{\beta\in \mathsf{I}} g_\beta\right).$$

Thus, we have shown that the $\vee$-semi-completeness of the open elements in the induced neighbourhood lattice of a topological space simply assures us that for each $A \in \mathcal{P}(X)$, $\eta(A) = \eta\left(\bigcup_{a\in A} \{a\}\right) = \bigcap_{a\in A} \eta(a)$. ∎

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We will now include without proof Theorem 10, which states that if L is a conditionally complete lattice, then a neighbourhood mapping η on L determines, in the usual way, an interior operator on L, and the "open" elements agree.

THEOREM 10. *Let* (L, η) *be a conditionally complete neighbourhood lattice. If we define* $^\circ: L \to L$ *by* $a^\circ = \bigvee\{g: g \in G \text{ and } g \leqslant a\}$, *then*

i) $(\forall a \in L)(a^\circ \leqslant a)$,

ii) $a \in G \Leftrightarrow a = a^\circ$,

iii) $(a^\circ)^\circ = a^\circ$,

iv) $a \leqslant b \Rightarrow a^\circ \leqslant b^\circ$,

v) $(a \wedge b)^\circ = a^\circ \wedge b^\circ$,

vi) $a^\circ \vee b^\circ \leqslant (a \vee b)^\circ$, *and*

vii) $^\circ: L \to L$ *is a dual closure mapping.*

4. PRE-DUAL NEIGHBOURHOOD MAPPINGS, DUAL NEIGHBOURHOOD MAPPINGS, AND DUAL NEIGHBOURHOOD LATTICES

In Section 2, filters and pre-neighbourhood mappings were used to define open elements in an arbitrary lattice. By dualising these definitions, we will derive a parallel theory of "closed" elements in a lattice based on ideals and pre-dual neighbourhood mappings. This dual theory indicates that there is a "natural" lattice theoretic duality between the definitions of open and closed, which is not dependent on complementation. However, we will prove that if L is an orthocomplemented lattice, then an element is closed in the dual theory if and only if it is the complement of an open element.

DEFINITION 11: Let P be a $\vee$-semi-lattice. A function $\gamma: P \to \mathcal{I}(P)$ is called an *ideal mapping*. $h \in P$ is said to be γ-*closed* (or simply *closed*) if $\gamma(h) = (h]$. We will let $H = \{h: h \in P \text{ and } \gamma(h) = (h]\}$. In addition, an ideal mapping γ is said to be a *pre-dual neighbourhood mapping* (or more simply a *pre-dual mapping*) on P if $(\forall a, t \in P)(t \in \gamma(a) \Leftrightarrow (\exists h \in H)(t \leqslant h \leqslant a))$.

We will simply note that for pre-dual $\vee$-semi-lattices the duals of Lemma 2, Theorem 3, Theorem 4 and Corollary 4.1 are valid.

DEFINITION 12: Let L be a lattice. A pre-dual mapping $\gamma: L \to \mathcal{I}(L)$ is a *dual neighbourhood mapping* if (i) H is an $\wedge$-semi-complete sublattice of L and (ii) $1 \in L \Rightarrow \gamma(1) = (1]$. Further, the pair (L, γ) will be called a *dual neighbourhood lattice*.

The duals of Theorem 7 and Corollary 7.1 give alternate characterisations of dual neighbourhood lattices.

THEOREM 13. *Let* $(L,')$ *be an ortholattice. If* (L, η) *is a neighbourhood lattice and* $G = \{g: g \text{ is } \eta\text{-open}\}$, *then* $\gamma: L \to \mathcal{I}(L)$ *given by* $\gamma(x) = \{t: (\exists g \in G)(t \leqslant g' \leqslant x)\}$

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is a dual neighbourhood mapping on L and x is open $\Leftrightarrow x'$ is closed. Conversely, if (L, γ) is a dual neighbourhood lattice and $H = \{h: h$ is γ-closed$\}$, then $\eta: L \to \mathcal{F}(L)$ given by $\eta(x) = \{t: (\exists h \in H)(x \leqslant h' \leqslant t)\}$ is a neighbourhood mapping on L and x is open $\Leftrightarrow x'$ is closed.

PROOF: If (L, η) is a neighbourhood lattice and $G = \{g: g$ is η-open$\}$, then G satisfies (i)', (ii), and (iii) of Corollary 7.1. If we let $H = \{g': g \in G\}$, then (via De Morgan's Laws) H satisfies (i)', (ii) and (iii) of the dual of Corollary 7.1. Thus γ is a dual neighbourhood mapping on L. Now, x is open $\Leftrightarrow x \in G \Leftrightarrow x' \in H \Leftrightarrow x'$ is γ-closed. The proof of the converse is the dual of this proof. ∎

In particular, for the orthocomplemented lattice $(\mathcal{P}(X),')$ associated with a topological space (X, T), the dual mapping γ defined in Theorem 13, is called the *induced dual neighbourhood mapping on* $\mathcal{P}(X)$, and the pair $(\mathcal{P}(X), \gamma)$ is called the *induced dual neighbourhood lattice*. Further, it follows from Theorem 13, that if X is a set and $(\mathcal{P}(X), \gamma)$ is a dual neighbourhood lattice, then $H = \{h: h \in \mathcal{P}(X)$ and $\gamma(h) = (h]\}$ is the set of closed elements determined by a topology T on X and $(\mathcal{P}(X), \gamma)$ is the induced dual neighbourhood lattice of (X, T).

In the "usual" sense of Boolean duality, the ideals generated by the induced dual neighbourhood mapping on $\mathcal{P}(X)$ are the duals of the filters generated by the induced neighbourhood mapping on $\mathcal{P}(X)$.

The dual of Theorem 9 is valid for co-atomistic pre-dual lattices. In particular, if $(P(X), \gamma)$ is a dual neighbourhood lattice, then $A \in \mathcal{P}(X)$ is closed if and only if $(\forall x \in A')(A \in \gamma(\{x\}'))$. This special case of the dual of Theorem 9 indicates that the $\wedge$-semi-complete condition on the closed elements in a topological space allows us to "extend" the induced dual neighbourhood system from co-atoms to any element of $\mathcal{P}(X)$.

A consideration of lattices with both a pre-neighbourhood mapping and a pre-dual mapping suggests the following definition.

DEFINITION 14: Let (L, η, γ) be a lattice with a pre-neighbourhood mapping η, a pre-dual mapping γ, and respective sets G and H. $x \in L$ is *clopen* if $x \in G \cap H$.

We note that if A is a sublattice of a lattice L such that for each $x \in L$, there exists a_1, a_2 elements of A such that $a_1 \leqslant x \leqslant a_2$, then A is the set of clopen elements of L if we define $H = G = A$. In particular, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}'$ are sets of clopen elements of the chain $\mathbb{R}$.

5. Closure mappings

In this section, we will discuss the relationship between dual neighbourhood lattices and closure mappings. In particular, we will show that (algebraic) closure mappings on posets do not give the appropriate information about the "structure" and "location" of the 'closed' elements to determine either pre-dual or dual neighbourhood mappings.

DEFINITION 15: Let P be a poset. A function $^-: P \to P$ is said to be a *closure mapping* on P if (i) $(\forall x, y \in P)(x \leqslant y \Rightarrow x^- \leqslant y^-)$ (isotone), (ii) $(\forall x \in P)$ $\left(x^- = (x^-)^-\right)$ (idempotent), and (iii) $(\forall x \in P)(x \leqslant x^-)$ (extensive). Further, $x \in P$ is said to be *closed with respect to* $^-$ (or simply $^-$ *closed*) if $x = x^-$.

We now note that if (L, γ) is a conditionally complete dual neighbourhood lattice then H, the set of γ-closed elements of L, determines (in the usual way) a closure mapping $^-: L \to L$ given by $a^- = \wedge\{h: h \in H \text{ and } a \leqslant h\}$. In addition, a is γ-closed if and only if a is $^-$closed, and $(a \vee b)^- = a^- \vee b^-$.

This agreement between the closed elements in a conditionally complete dual neighbourhood lattice (L, γ) allows us to show in an unambiguous way that if L is also an ortholattice, then the "usual" characterisation of the interior of an element is valid.

THEOREM 16. *If (L, η) is a conditionally complete, neighbourhood ortholattice, then $a^\circ = a'^{-\prime}$, where closure is taken in the induced dual neighbourhood lattice (L, γ).*

PROOF: $a^\circ = \bigvee\{g: g \in G \text{ and } g \leqslant a\} = \left(\bigwedge\{g': g' \in H \text{ and } a' \leqslant g'\}\right)' = \left((a')^-\right)'$. ∎

It is clear that a closure mapping on an $\wedge$-semi-lattice P determines a pre-neighbourhood mapping on P with $G = \{x: x = x^-\}$. However, if we consider the closure operator defined on $L = \mathcal{P}(\mathbb{R}^2)$ by A^- is the convex hull of A, then $^-$ does not determine a pre-dual mapping on L, even though L is a complete atomistic Boolean lattice and points are closed.

Čech in [4] bases his development of topological spaces on "closure operators" defined on $\mathcal{P}(X)$ which are extensive, preserve unions, and which map $\emptyset$ to $\emptyset$. We will show that a "generalised" Čech closure mapping defined on a lattice with $\emptyset$ determines a dual neighbourhood mapping.

DEFINITION 17: Let P be a $\vee$-semi-lattice. A function $^c: P \to P$ is said to be a *Čech closure mapping* on P if i) $\emptyset \in P \Rightarrow \emptyset^c = \emptyset$, ii) $(\forall x \in P)(x \leqslant x^c)$, and iii) $(\forall x, y \in P)((x \vee y)^c = x^c \vee y^c)$. Further, $x \in P$ is said to be *c-closed* if $x^c = x$. Finally, a Čech closure mapping will be called a *proper Čech closure mapping* (or more simply, a *proper Čech mapping*) if $H = \{h: h = h^c\} \neq \emptyset$.

If P has either $\emptyset$ or 1, then every Čech closure mapping on P is a proper Čech mapping. In particular, every Čech closure mapping on $\mathcal{P}(X)$ is a proper Čech mapping, and $\emptyset$, X are c-closed. It should be noted that the set of elements of a lattice L which are closed with respect to either a proper Čech mapping or a closure mapping

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is an $\wedge$-semi-complete sub-lattice of L, and further, that the c-closed elements form a $\vee$-semi-complete sublattice of L. Thus, we have

THEOREM 18. *Let P be a $\vee$-semi-lattice, let ${}^c\colon P \to P$ be a Čech closure mapping, and let $H = \{h\colon h = h^c\}$. If either (i) $\emptyset \in P$, or (ii) $(\forall x \in P)(\exists h \in H)(h \leqslant x)$, then H determines a pre-dual mapping γ on P. Further, $(\forall x \in P)$ (x is γ-closed $\Leftrightarrow$ x is c-closed). Finally, if P is a lattice and H satisfies (i) or (ii), then H determines a dual neighbourhood mapping γ on P.*

COROLLARY 18.1. *Let L be a conditionally complete lattice with 1. ${}^c\colon L \to L$ be a Čech closure mapping, $H = \{h\colon h = h^c\}$, and let $H_a = \{y\colon y \geqslant a$ and $y \in H\}$.*

- i) *${}^-\colon L \to L$ given by $a^- = \bigwedge H_a$ is a closure mapping on L. In addition, $a^c \leqslant a^-$, and $a^{-c} = a^{c-} = a^-$.*
- ii) *H determines a dual neighbourhood mapping γ on L and a pre-neighbourhood mapping η on L. Further, $a \in L$ is γ-closed $\Leftrightarrow$ a is c-closed $\Leftrightarrow$ a is ${}^-$ closed $\Leftrightarrow$ a is clopen in (L, η, γ).*

If we apply the construction process outlined in Corollary 18.1 to a Čech closure mapping ${}^c\colon \mathcal{P}(X) \to \mathcal{P}(X)$, we will generate a topological closure mapping ${}^-\colon \mathcal{P}(X) \to \mathcal{P}(X)$, and the elements closed with respect to each of these mappings agree. Thus, the conditional completeness of $\mathcal{P}(X)$ effects the link between closure operators, Čech closure mappings, and dual neighbourhood mappings.

To indicate how Corollary 18.1 works in practice, let $F = \{f\colon f\colon \mathsf{R} \to \mathsf{R}\}$, and define ${}^c\colon \mathcal{P}(F) \to \mathcal{P}(F)$ by $A^c = \{f\colon \exists a$ sequence $\{f_n\}$ in A and $f = \lim f_n\}$. ^{c} is a Čech closure mapping on $\mathcal{P}(F)$. By Theorem 18 and the completeness of $\mathcal{P}(F)$, ${}^-\colon \mathcal{P}(F) \to \mathcal{P}(F)$ given by $A^- = \bigcap\{y\colon y \supseteq A$ and $y^c = y\}$ is a closure mapping on $\mathcal{P}(F)$. Thus, ${}^-$ determines a topology T on F. Now, if $A \in \mathcal{P}(F)$ is closed in T_p, the topology of pointwise convergence on F, then A is closed in T. Thus, T is finer than T_p.

We will now show that Čech closure mappings on $\vee$-semi-lattices are determined by $\vee$-homomorphisms that preserve $\emptyset$. Such $\emptyset$ preserving $\vee$-homomorphisms between Boolean algebras, called *hemimorphisms*, were discussed by Jónsson and Tarski [**10**] under the name of 'normal and additive functions'.

DEFINITION 19: Let P_1 and P_2 be $\vee$-semi-lattices. A function $f\colon P_1 \to P_2$ is called a *hemimorphism* if

- i) $(\forall x, y \in P_1)(f(x \vee y) = f(x) \vee f(y))$, and
- ii) $\emptyset_1 \in P_1$ and $\emptyset_2 \in P_2 \Rightarrow f(\emptyset_1) = \emptyset_2$.

Tukey [**16**] uses hemimorphisms, which he calls "closure operators", as the starting point for defining topological spaces. Theorem 20 shows that on $\mathcal{P}(X)$, "Tukey closures" generate Čech closures which in turn determine dual neighbourhood mappings and topological spaces.

Theorem 20. *Let P be a $\vee$-semi-lattice. ${}^c\colon P \to P$ is Čech closure mapping $\Leftrightarrow$ there exists a hemimorphism $f\colon P \to P$ such that $a^c = a \vee f(a)$. Further, a is c-closed $\Leftrightarrow f(a) \leqslant a$.*

Proof: If ${}^c\colon P \to P$ is Čech closure mapping, then define $f\colon P \to P$ by $f(a) = a^c$. Conversely, if $f\colon P \to P$ is a hemimorphism then ${}^c\colon P \to P$ given by $a^c = a \vee f(a)$ is a Čech closure mapping. Finally, a is c-closed $\Leftrightarrow a^c = a = a \vee f(a) \Leftrightarrow f(a) \leqslant a$. ∎

6. Continuity

Let (X, T_1) and (Y, T_2) be topological spaces, and let $f\colon X \to Y$ be a function. The point function f induces two lattice functions, $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$, the *direct image function*, given by $f_*(A) = \{f(a)\colon a \in A\}$, and $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$, the *inverse image function*, given by $f^*(B) = \{x\colon x \in X \text{ and } f(x) \in B\}$. We will define continuous functions on neighbourhood lattices in such a way that $f\colon X \to Y$ is a continuous point function between topological spaces if and only if $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is a continuous lattice function in the induced neighbourhood systems on $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. We should remark that the upper and lower star notation for these induced lattice functions, though non-standard, was used by MacLane and Birkhoff in [**12**], and has many pedagogical advantages over the usual notation. Further, we will write $\eta f(x)$ for $\eta(f(x))$.

The first observation about "global" continuity is that points can be replaced by sets and point functions by lattice functions. For example, if (X, T_1) and (Y, T_2) are topological spaces $f\colon X \to Y$ is defined to be *continuous* if and only if $(\forall A \in T_2)$ $(f^*(A) \in T_1)$.

In order to motivate the definition of neighbourhood continuity and at the same time to indicate why a structure as general as a pre-neighbourhood lattice was introduced in section 2, we will discuss the proof of Theorem 21 which is a standard result from general topology concerning continuous functions, see [**11**].

Theorem 21. *Let (X, T_1) and (Y, T_2) be topological spaces, $(\mathcal{P}(X), \eta_1)$ and $(\mathcal{P}(Y), \eta_2)$ be the induced neighbourhood lattices, and $f\colon X \to Y$ be a function. The following statements are equivalent:*

1) $f\colon X \to Y$ *is continuous;*
2) $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ *is an open function;*
3) $(\forall A \in \mathcal{P}(X))(\forall B \in \eta_2 f^*(A))(\exists Z \in \eta_1(A))(f_*(Z) \subseteq B)$;
4) $(\forall A \in \mathcal{P}(X))(\forall B \in \eta_2 f_*(A))(f^*(B) \in \eta_1(A))$;
5) $(\forall B \in \mathcal{P}(Y))(f_*(B^\circ) \subseteq f^*(B)^\circ)$;
6) $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ *is a closed function;*

7) $(\forall B \in \mathcal{P}(Y))\Big(f^*(B)^- \subseteq f^*(B^-)\Big)$; and

8) $(\forall A \in \mathcal{P}(X))\Big(f_*(A^-) \subseteq f_*(A)^-\Big)$.

An analysis of the proof of Theorem 21 shows that equivalences (1) through (5) and (6) through (8) can be established using only three properties of f_* and f^*, namely, (i) f_* and f^* are each isotone, (ii) $(\forall A \in \mathcal{P}(X))(A \subseteq f^*[f_*(A)])$, and (iii) $(\forall B \in \mathcal{P}(Y))(f_*[f^*(B)] \subseteq B)$. Further, the equivalence of (1) and (6) uses only complementation in $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, and an additional property of f^*, namely, (iv) $(\forall B \in \mathcal{P}(Y))(f^*(B') = [f^*(B)]')$. Surprisingly, the proof makes no use of the fact that the open elements form a $\vee$-semi-complete sublattice of either $\mathcal{P}(X)$ or $\mathcal{P}(Y)$, nor any of the properties of neighbourhood mappings other than the filter property, that is, no properties other than those of pre-neighbourhood mappings.

Of the equivalent statements in Theorem 21, statement (3) alone involves only f_* and the neighbourhood systems of elements. Thus, we will define a continuous function between pre-neighbourhood $\wedge$-semi-lattices by

DEFINITION 22: Let (P_1, η_1) and (P_2, η_2) be pre-neighbourhood $\wedge$-semi-lattices. A function $f\colon P_1 \to P_2$ is said to be *η-continuous at* $a \in P_1$ if $(\forall y \in \eta_2 f(a))$ $(\exists z \in \eta_1(a))(f(a) \leqslant f(z) \leqslant y)$. Further, $f\colon P_1 \to P_2$ is said to be *η-continuous* or *continuous* (on P_1) if f is η-continuous at each element of P_1.

It is clear from Theorem 21 and Definition 22 that if (X, T_1) and (Y, T_2) are topological spaces, then $f\colon X \to Y$ is topologically continuous if and only if $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is continuous in the induced neighbourhood systems on $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Thus, by considering the consequences of replacing point by set in topological neighbourhood systems, not only is one led to pre-neighbourhood posets but also to η-continuity. Further, each of these concepts agrees with the the corresponding topological concept when (X, T) is a topological space and $(\mathcal{P}(X), \eta)$ is the induced neighbourhood lattice.

We will now include the more technical aspects of η-continuity in Lemma 23.

LEMMA 23. *Let (P_1, η_1) and (P_2, η_2) be pre-neighbourhood $\wedge$-semi-lattices, and let G_1 and G_2 be the sets of open elements of P_1 and P_2, respectively.*

(i) *A function $f\colon P_1 \to P_2$ is continuous at $x \in P \Leftrightarrow (\forall g \in G_2)$ $(g \in \eta_2 f(x) \Rightarrow (\exists z \in \eta_1(x))(f(x) \leqslant f(z) \leqslant g))$.*

(ii) *A function $f\colon P_1 \to P_2$ is continuous at each open element of P_1.*

(iii) *An isotone function $f\colon P_1 \to P_2$ is continuous at $x \in P_1 \Leftrightarrow (\forall g \in G_2)$ $(g \in \eta_2 f(x)) \Rightarrow (\exists g_1 \in G_1 \wedge \eta(x))$ and $f(x) \leqslant f(g_1) \leqslant g$.*

(iv) *An antitone function $f\colon P_1 \to P_2$ is continuous at $x \in P_1 \Leftrightarrow$*

$(\exists g_1 \in G_1 \wedge \eta(x))$ *and* $f(x) = f(g_1)$.

(v) *If* $f: P_1 \to P_1$ *is continuous at* x, *and if* $f(x)$ *is open in* P_2, *but* x *is not open in* P_1, *then* $(\exists z \in \eta_1(x))(x < z$ *and* $f(x) = f(z))$.

(vi) *If* (P_1, η_1) *is a neighbourhood lattice, and* $a \in P_1$, *then* $f_a: P_1 \to P_1$, *given by* $f_a(x) = x \vee a$, *is continuous.*

(vii) *If* (P_1, η_1) *is a Boolean neighbourhood lattice, and* $a \in P_1$, *then* $g_a: P_1 \to P_1$, *given by* $g_a(x) = x \wedge a$ *is continuous* $\Leftrightarrow a'$ *is open* $\Leftrightarrow a$ *is closed.*

If we now let (X, T_1) and (Y, T_2) be topological spaces, $(\mathcal{P}(X), \eta_1)$ and $(\mathcal{P}(Y), \eta_2)$ be the induced neighbourhood lattices, and $f: X \to Y$ be continuous, then we can apply Lemma 23 (iii) and (v) to the continuous isotone function f_*, and we obtain the standard result that although f_* need not be an open function, f_* has the property that if $f_*(A)$ is open in Y, then there is g open in X such that $f_*(A) = f_*(g)$. Clearly, if f is injective, then the openness of $f_*(A)$ implies the openness of A. In addition, (iv) proves that $': \mathcal{P}(X) \to \mathcal{P}(X)$ is continuous at $A \in \mathcal{P}(X)$ if and only if A is clopen.

In Theorem 24 we establish necessary and sufficient conditions for f^* to be a continuous function, noting that the proof makes use only of the pre-neighbourhood systems of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

THEOREM 24. *Let* (X, T_1) *and* (Y, T_2) *be topological spaces,* $(\mathcal{P}(X), \eta_1)$ *and* $(\mathcal{P}(Y), \eta_2)$ *be the induced neighbourhood lattices, and* $f: X \to Y$ *be a function.* $f^*: \mathcal{P}(Y) \to \mathcal{P}(X)$ *is continuous* $\Leftrightarrow f_*: \mathcal{P}(X) \to \mathcal{P}(Y)$ *is a closed function.*

PROOF: This follows from the definition of continuity and properties (i) through (iv) of f_* and f^* listed after Theorem 21. ∎

We will introduce the concept of pre-neighbourhood homeomorphisms so that we can characterise topological homeomorphisms in terms of the functions f_* and f^*.

DEFINITION 25: Let (P_1, η_1) and (P_2, η_2) be pre-neighbourhood $\wedge$-semi-lattices. A function $f: P_1 \to P_2$ will be called a *pre-neighbourhood homeomorphism* (or an *η-homeomorphism* or simply a *homeomorphism*) if f and f^{-1} are each continuous.

THEOREM 26. *Let* (X, Y_1) *and* (Y, T_2) *be topological spaces* $(\mathcal{P}(X), \eta_1)$ *and* $(P(Y), \eta_2)$ *be the induced neighbourhood lattices. The following statements are equivalent:*

(i) $f: X \to Y$ *is a homeomorphism;*

(ii) $f^*: \mathcal{P}(Y) \to \mathcal{P}(X)$ *is a homeomorphism; and*

(iii) $f_*: \mathcal{P}(X) \to \mathcal{P}(Y)$ *is a homeomorphism.*

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PROOF: This follows from Definition 25, and the relations $f^* = (f_*)^{-1}$, $f^* = (f^{-1})_*$, $f_* = (f^*)^{-1}$, and $f_* = (f^{-1})^*$. ∎

As one examines the proofs of these 'topological' results, it becomes clear that only the properties of pre-neighbourhood mappings are used, and that, unless closure is involved, only properties (i) through (iii) of f_* and f^* listed after Theorem 21 are needed. These observations lead one to consider the 'natural' algebraic analogues of f_* and f^*, namely, residuated and residual functions. Our immediate objective of paralleling these topological proofs seems better served by this direct approach of utilising selected pairs of isotone functions, rather than the 'traditional' approach using as 'primitives' pairs of antitone functions, called Galois connections, to construct isotone functions.

DEFINITION 27: Let P_1 and P_2 be posets. A function $f\colon P_1 \to P_2$ is said to be *residuated* if there exists a function $f^+\colon P_2 \to P_1$, called the *residual of* $\mathbf{f}$, such that

(1) f and f^+ are each isotone,
(2) $(\forall y \in P_2)(f(f^+(y)) \leqslant y)$, and
(3) $(\forall x \in P_1)(x \leqslant f^+(f(x)))$.

It is clear that if $f\colon X \to Y$ is a function, then f_* is a residuated function with residual f^*. Further, since the discussion of the proof of Theorem 21 indicates that only the pre-neighbourhood structure and conditions (1), (2) and (3) of Definition 25 are used in the proof we will state without proof the corresponding results for residuated functions.

THEOREM 28. *Let (P_1, η_1) and (P_2, η_2) be pre-neighbourhood $\wedge$-semi-lattices, and $f\colon P_1 \to P_2$ be a residuated function. The following statements are equivalent:*

(i) $f^+\colon P_2 \to P_1$ *is an open function;*
(ii) $f\colon P_1 \to P_2$ *is continuous. (η-continuous); and*
(iii) $(\forall x \in P_1)(\forall y \in \eta_2 f(x))(f^+(y) \in \eta_1(x))$.

Further, if P_1 and P_2 are conditionally complete lattices, then (i), (ii) and (iii) are equivalent to

(iv) $(\forall y \in P_2)\big(f^+(y^\circ) \leqslant f^+(y)^\circ\big)$.

THEOREM 29. *Let (L_1, γ_1) and (L_2, γ_2) be pre-dual conditionally complete lattices, and $f\colon L_1 \to L_2$ be a residuated function. The following statements are equivalent:*

(i) $f^+\colon L_2 \to L_1$ *is an closed function;*
(ii) $(\forall y \in L_2)\Big(f^+(y)^- \leqslant f^+(y^-)\Big)$*; and*
(iii) $(\forall x \in L_1)\Big(f(x^-) \leqslant f(x)^-\Big)$.

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It is quite easy to construct an example to show that the composition of continuous functions need not be continuous. However, Theorem 30 shows that the composition of continuous residuated functions is continuous.

THEOREM 30. *Let* (P_1, η_1), (P_2, η_2) *and* (P_3, η_3) *be pre-neighbourhood* $\wedge$*-semilattices. If* $f_1 : P_1 \to P_2$, $f_2 : P_2 \to P_3$ *are each continuous functions, and* f_2 *is isotone, then* $f_2 \circ f_1$ *is continuous.*

PROOF: Let $x \in P_1$ and $g \in G_3$ such that $g \in \eta_3(f_2 \circ f_1)(x)$, and let $b = f_1(x)$. $g \in \eta_3 f_2(b) \Rightarrow (\exists z_1 \in \eta_2(b))(f_2(b) \leqslant f_2(z_1) \leqslant g)$. Now, the isotoneness of f_2 proves the continuity of $f_2 \circ f_1$ at x. ∎

Theorem 30 and the relation $(f_2 \circ f_1)_* = f_{2*} \circ f_{1*}$ proves that the composition of topological continuous functions is continuous and shows that the proof of this result depends only on the isotoneness of f_*.

We will now show that both regularity and normality reflect the continuity of the closure operator.

THEOREM 31. *Let* (X, T) *be a topological space and* $(\mathcal{P}(X), \eta)$ *be the induced neighbourhood lattice.*

(1) *A* T_1 *topological space* X *is regular* $\Leftrightarrow$ $^- : \mathcal{P}(X) \to \mathcal{P}(X)$ *is continuous at each point of* $\mathcal{P}(X)$.

(2) X *is normal* $\Leftrightarrow$ $^- : \mathcal{P}(X) \to \mathcal{P}(X)$ *is continuous.*

PROOF: (1) This follows from the topological result which states that X is regular $\Leftrightarrow$ the family of closed neighbourhoods of each point is a base for the neighbourhood system of the point and the fact that in a T_1 space $\eta(\{x\}^-) = \eta(\{x\})$.

(2) This follows from the topological result which states that X is normal $\Leftrightarrow$ the family of closed neighbourhoods of each set is a base for the neighbourhood system of the set. ∎

A few comments of Theorem 31. Regularity always implies continuity at points of $\mathcal{P}(X)$. Further, since a regular T_1 space is Hausdorff, Theorem 31 has as an obvious corollary the statement that in a T_1 space the continuity of $^- : \mathcal{P}(X) \to \mathcal{P}(X)$ at points implies Hausdorffness.

The agreement between topological continuity and neighbourhood continuity illustrated in the statement that $f : X \to Y$ is topologically continuous if and only if $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ is continuous in the induced neighbourhood systems on $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, suggests that a 'generalised' continuity can be defined for a function from a set X to a set Y, which will agree with topological continuity when either X and Y are topological spaces, or $(\mathcal{P}(X), \eta_1)$ and $(\mathcal{P}(Y), \eta_2)$ are neighbourhood lattices.

DEFINITION 32: Let X and Y be sets and $f\colon X \to Y$ be a function. If $(\mathcal{P}(X), \eta_1)$ and $(\mathcal{P}(Y), \eta_2)$ are pre-neighbourhood lattices, then f is *η-continuous* on X if $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is continuous.

It follows from Definition 32 that if (X_i, μ_i) are measure spaces for $i = 1, 2$ and η_i is the pre-neighbourhood mapping on $\mathcal{P}(X_i)$ determined by $G_i = \{A\colon A$ is a measurable subset of $X_i\}$, then $f\colon X_1 \to X_2$ is a measurable function if and only if f is η-continuous. Further, if (Y, T) is a topological space, then $f\colon X_1 \to Y$ is a measurable function if and only if f is η-continuous in the appropriate pre-neighbourhood systems on $\mathcal{P}(X_1)$ and $\mathcal{P}(Y)$.

We will conclude this paper with an example of a non-topological application of pre-neighbourhood continuity and η-continuity. For $i = 1, 2$, let X_i be a group with identity element e_i, let η_i be the pre-neighbourhood mapping on $\mathcal{P}(X_i)$ determined by $\mathcal{S}(X_i) = G_i = \{A\colon A$ is a subgroup of $X_i\}$, let $\alpha\colon X_1 \to X_2$ be a homomorphism with kernel K, and let H be a normal subgroup of X_1. Further, if $A \in \mathcal{P}(X_i)$, then $\langle A\rangle$ will denote the subgroup generated by A.

(a) $f\colon (\mathcal{P}(X_1), \eta_1) \to (\mathcal{P}(X_1), \eta_1)$ given by $f(A) = \langle A\rangle$ is a isotone function that is both open and continuous.

(b) $g\colon (\mathcal{P}(X_1), \eta_1) \to (\mathcal{P}(X_1), \eta_1)$ given by $g(A) = \langle A\rangle H$ is a continuous open function.

(c) $\alpha_*\colon (\mathcal{P}(X_1), \eta_1) \to (\mathcal{P}(X_2), \eta_2)$ is an open continuous function. Thus, α is η-continuous and open. Further, if α is an isomorphism, then α_* is a homeomorphism. Thus, α is an η-homeomorphism.

(d) Let η_3 be the neighbourhood mapping on $\mathcal{P}(X_1)$ determined by $G_3 = \{A\colon A \in \mathcal{P}(X_1)$ and $K \subseteq A\}$, and let η_4 be the neighbourhood mapping on $\mathcal{P}(X_2)$ determined by $G_4 = \{B\colon B \in \mathcal{P}(X_2)$ and $e_2 \in B\}$. $\alpha_*\colon (\mathcal{P}(X_1), \eta_3) \to (\mathcal{P}(X_2), \eta_4)$ is an open continuous function. Thus α is a (topological) homeomorphism. Finally, $\alpha_*\colon (\mathcal{P}(X_1), \eta_3) \to (\mathcal{P}(X_2), \eta_2)$ is a continuous function. Thus, in this case α_* is an η-continuous function.

(e) Let $(\mathcal{S}(X_i), \vee, \wedge)$ be the lattice of subgroups of X_i, let $\tilde{\eta}_i$ be the pre-neighbourhood mapping on $\mathcal{S}(X_i)$ determined by $\tilde{G}_i = \{A\colon A \trianglelefteq X_i\}$, and let $\tilde{\alpha}_*\colon (\mathcal{S}(X_1), \tilde{\eta}_1) \to (\mathcal{S}(X_2), \tilde{\eta}_2)$ be the restriction of α_* to $\mathcal{S}(X_1)$. $\tilde{\alpha}_*$ is continuous. If α is onto, then $\tilde{\alpha}_*$ is a continuous open function. Thus, if α is a isomorphism, $\tilde{\alpha}_*$ is an η-homeomorphism. The function $h\colon (\mathcal{S}(X_1), \eta) \to (\mathcal{S}(X_2), \eta)$ given by $h(A) = AH$ is a "naturally" continuous function, in the sense that if η is any pre-neighbourhood mapping on $\mathcal{S}(X_1)$, then h is continuous. Further, h is an open function if H is η-open.

Conclusion

In this paper an introduction to pre-neighbourhood posets was presented. The development of this topic was motivated by questions which are not addressed in the "usual" treatment of topological spaces, in which the Boolean properties of the orthocomplemented lattice $\mathcal{P}(X)$, and the properties of the function f_* and f^* are used whenever convenient. For example, we have shown that even if distributivity and orthocomplementation are not available in a lattice, 'openness' can be defined in such a way that 'closedness' is in a real sense a dual lattice theoretic concept. Further, the topological continuity of a function $f\colon X \to Y$ is characterised in terms of the neighbourhood continuity of $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$. Thus, one is led in a natural way to continuous residuated functions defined on pre-neighbourhood $\wedge$-semi-lattices, and to η-continuity. It is hoped that neighbourhood lattices have been shown to be a useful structure for examining the way in which topology depends on the Boolean properties of $\mathcal{P}(X)$.

Finally, this lattice theoretic view point of topological spaces suggests the following 'natural' questions: "Since continuous functions are characterised by the behaviour of f^*, a function between Boolean lattices, why is there no dual continuity in topology?", "Do the T_i properties of topological spaces generalise to neighbourhood lattices?", "Is there any reasonable concept of convergence in neighbourhood lattices, and if so, how does it interact with continuity?","Can proximity structures be defined in neighbourhood lattices?", "If it possible to 'lift' topologies from $\mathcal{P}(X)$ to $\mathcal{P}(\mathcal{P}(X))$ in such a way that f, f_*, and f_{**} are each continuous?", "If P_1 and P_2 are posets, are there pairs of compatible pre-neighbourhood systems and topologies on P_1 and P_2 so that η-continuity implies or is implied by topological continuity?", and finally, "Can neighbourhood continuity be applied to relations, by using the lower star function associated with a relation?".

These questions have been answered by the author in work that is in preparation. It may be noted that in the theory of neighbourhood convergence, the limits of η-convergent nets are unique, and a residuated function f defined on conditionally complete T_1 lattices is continuous if and only if f preserves the limits of convergent nets. Thus, each of the four equivalent primitive concepts from topology, namely: openness, closedness, convergence, and neighbourhoods is available in neighbourhood lattices, and each is relevant to neighbourhood continuity.

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