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THE NEAR-RING OF GENERALIZED AFFINE TRANSFORMATIONS

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Blackett and Wolfson studied the near-ring *Aff(V)* consisting of all affine transformations of a vector space *V*. This notion is generalized here, and the rear-ring *Aff(G)* consisting of affine-like maps of a nilpotent group *G* is introduced. The ideal structure, and the multiplication rule for *Aff(G)* are determined. Finally a near-ring *S* is introduced which generalized both *Aff(G)*, and Gonshor's abstract affine near-rings. The ideals of *S* are determined.

1. Blackett [1], and then Wolfson [4] studied the near-ring *Aff(V)* consisting of all affine transformations of a vector space *V*. A more general structure, the abstract affine near-ring, was introduced by Gonshor [2]. *Aff(V)* is a subnear-ring of the near-ring *M(V)* consisting of all maps $V \to V$. When viewed as an additive group, the structure of a vector space *V* is very restrictive; either *V* is isomorphic to the direct sum of copies of the additive group of the field of rational numbers, or *V* is the direct sum of cyclic groups of order a fixed prime *p*. In this note a subnear-ring *Aff(G)* of *M(G)* will be considered for *G* an arbitrary nilpotent group. *Aff(G)* consists of affine-like maps $G \to G$, however the ideal structure of *Aff(G)* is much more complicated than that of *Aff(V)*. It will be shown that both the ideal

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structure and the multiplication of $Aff(G)$ are similar to those of Gonshor's abstract affine near-rings. A generalized affine near-ring structure will be defined for which $Aff(G)$ and Gonshor's abstract affine near-rings are special cases.

2. All near-rings are assumed to be associative and right distributive. Terminology follows [3]. The additive group of a near-ring R will be denoted R^+, and the centre of a group H by $Z(H)$. Let G be a group nilpotent of class n, and G_n be the n-th term in the lower central series of G. The subnear-ring of $M(G)$ generated by the endomorphisms of G will be denoted by $E(G)$, and the subnear-ring of $M(G)$ consisting of the constant functions $\hat{c}(x) = c$ for all $x \in G$, $c \in G_n$ by C. Put $Aff(G)$ = the subnear-ring $M(G)$ generated by $E(G)+C$.

LEMMA 1. *For all* $f \in Aff(G)$, $\hat{c} \in C$, $f+\hat{c} = \hat{c}+f$.

Proof. For $x \in G$, $(f+\hat{c})(x) = f(x)+c$. Since $c \in G_n \leq Z(G)$,

$$f(x)+c = c+f(x) - (\hat{c}+f)(x).$$

LEMMA 2. *Let* $f \in Aff(G)$. *Then* $f \in E(G)$ *if and only if* $f(0) = 0$.

Proof. It follows from Lemma 1 that $f = g+\hat{c}$ with $g \in E(G)$ and $\hat{c} \in C$. Therefore $f(0) = g(0)+\hat{c}(0) = c = 0$ if and only if $c = 0$, which occurs if and only if $f = g \in E(G)$.

THEOREM 3. $Aff(G)^+ = E(G)^+ \oplus C^+$, *and multiplication in* $Aff(G)$ *satisfies* $(f_1+\hat{c}_1)(f_2+\hat{c}_2) = f_1f_2 + f_1\hat{c}_2 + \hat{c}_1$ *for all* $f_1, f_2 \in E(G)$, *and* $\hat{c}_1, \hat{c}_2\ C$.

Proof. The fact that $(f_1+\hat{c}_1)(f_2+\hat{c}_2) = f_1f_2+f_1\hat{c}_2+\hat{c}_1$ can be verified by direct calculation. The equality $Aff(G)^+ = E(G)^+ + C^+$ is a simple consequence of Lemma 1. Let $c \in E(G) \cap C$. By Lemma 2, $\hat{c}(0) = 0$. However $\hat{c}(0) = c$, and so $\hat{c} = 0$, that is, $Aff(G)^+ = E(G)^+ = E(G)^+ \oplus C^+$.

LEMMA 4. *For all* $f \in Aff(G)$, *and* $c \in G_n$, $f(c) \in G_n$. *For all* $f \in E(G)$, *and* $\hat{c}_1, \hat{c}_2 \in C$, $f(\hat{c}_1+\hat{c}_2) = f\hat{c}_1+f\hat{c}_2$.

Proof. Let $f \in Aff(g)$. By Theorem 3, $f = g+\hat{d}$ with $g \in E(G)$ and $\hat{d} \in C$. Since G_n is a fully invariant subgroup of G, $g(c) \in G_n$ for all $c \in G_n$. Therefore $f(c) = g(c)+d \in G_n$. Let $f \in E(G)$ and let $\hat{c}_1, \hat{c}_2 \in C$. For $x \in G$, $f(\hat{c}_1+\hat{c}_2)(x) = f(c_1+c_2)$, while $(f\hat{c}_1+f\hat{c}_2)(x) = f(c_1) + f(c_2)$. Since $G_n \leq Z(G)$, the restriction of f to G_n is a homomorphism, and so $f(c_1+c_2) = f(c_1)+f(c_2)$.

THEOREM 5. *A is an ideal in $Aff(G)$ if and only if $A = I \oplus D$ with I an ideal in $E(G)$, and D a subgroup of C^+ satisfying $E(G)D \subseteq D$, and $IC \subseteq D$.*

Proof. Let A be an ideal in $Aff(G)$. By Theorem 3, every $f \in A$ can be uniquely written $f = g+\hat{c}$ with $g \in E(G)$, $\hat{c} \in C$. Let π_1, π_2 be the projections $\pi_1(f) = g$, $\pi_2(f) = \hat{c}$. Clearly $A \subseteq \pi_1(A) \oplus \pi_2(A)$. To prove the Inverse inclusion if suffices to show that $\pi_2(A) \subseteq A$. Let $\hat{c} \in \pi_2(A)$. Then there exists $f \in E(G)$ such that $f+\hat{c} \in A$. Since A is an ideal in $Aff(G)$, $\hat{c} = (f+\hat{c})\hat{0} \in A$, and so $\pi_2(A) \subseteq A$. It is readily seen that $\pi_1(A)$ is a subgroup of $E(G)^+$ and that $\pi_2(A)$ is a subgroup of C^+. Let $f \in \pi_1(A)$, $g \in E(G)$. There exists $\hat{c} \in C$ such that $f+\hat{c} \in A$. The fact that A is an ideal in $Aff(G)$ yields that $h = -g+f+\hat{c}+g \in A$. However $h = (-g+f+g)+\hat{c}$ by Lemma 1. Hence $-g+f+g = \pi_1(h) \in \pi_1(A)$, and so $\pi_1(A)$ is a normal subgroup of $E(G)^+$. Let $f \in \pi_1(A)$, $\hat{c} \in \pi_2(A)$, and let $g_1, g_2 \in E(G)$. Since A is an ideal in $Aff(G)$ it follows that $g_1(g_2+f+\hat{c})-g_1g_2 \in A$. By Lemma 1 and Theorem 3, $g_1(g_2+g+\hat{c})-g_1g_2$ $= g_1(g_2+f)-g_1g_2 + g_1\hat{c}$. Since $g_1(g_2+f)-g_1g_2 \in E(G)$ and $g_1\hat{c} \in C$ it follows that for all $f \in \pi_1(A)$ and all $g_1, g_2 \in E(G)$, $g_1(g_2+f)-g_1g_2 \in \pi_1(A)$, and for all $\hat{c} \in \pi_2(A)$ and $g_1 \in E(G)$, $g_1\hat{c} \in \pi_2(A)$, that is $E(G) \cdot \pi_2(A) \subseteq \pi_2(A)$. For $f \in \pi_1(A)$, and $g \in E(G)$, $f \cdot g \in A \cap E(G) = \pi_1(A)$. Consolidating these results, we have that $A = \pi_1(A) \oplus \pi_2(A)$ with $\pi_1(A)$ an ideal in $E(G)$ and $\pi_2(A)$ a subgroup of C satisfying $E(G) \cdot \pi_2(A) \subseteq \pi_2(A)$. For $f \in \pi_1(A)$ and

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$\pi_2(A)$ a subgroup of C satisfying $E(G)\cdot\pi_2(A) \subseteq \pi_2(A)$. For $f\in\pi_1(A)$ and $\hat{c}\in C$ the fact that A is an ideal in $Aff(G)$ yields that $f\hat{c}\in A$. However $f\hat{c} = \widehat{f(c)} \in C$, and so $f\hat{c}\in A\cap C = \pi_2(A)$, that is $\pi_1(A)\cdot C \subseteq \pi_2(A)$.

Conversely, let $A = I \oplus D$ with I an ideal in $E(G)$ and D a subgroup of C satisfying $E(G)\cdot D \subseteq D$, and $IC \subseteq D$. Clearly A is a normal subgroup of $Aff(G)^+$. Let $f\in I$, $\hat{d}\in D$, $g\in E(G)$, and $\hat{c}\in C$. Then by Theorem 3, $(f+\hat{d})(g+\hat{c}) = fg+f\hat{c}+\hat{d}$. Since I is an ideal in $E(G)$, $fg\in I$, and the fact that $IC \subseteq D$ yields that $f\hat{c}+\hat{d}\in D$. Therefore multiplying an element in A on the right by an element in $Aff(G)$ yields an element in A. To prove that A is an ideal in $Aff(G)$ it suffices to show that for $f\in I$, $\hat{d}\in D$, and $f_1, f_2\in Aff(G)$, $g = f_1(f_2+f+\hat{d})-f_1f_2\in A$. Now $f_i = g_i+\hat{c}_i$ with $g_i\in E(G)$, and $\hat{c}_i\in C$, $i = 1, 2$. Lemma 1 and Theorem 3 yield that $g = g_1(g_2+f)-g_1g_2+g_1(\hat{c}_2+\hat{d})-g_1\hat{c}_2$. Since I is an ideal in $E(G)$, $g_1(g_2+f)-g_1g_2\in I$. By Lemma 4, $g_1(\hat{c}_2+\hat{d}) = g_1\hat{c}_2+g_1\hat{d}$ and so $g_1(\hat{c}_2+\hat{d})-g_1\hat{c}_2 = g_1\hat{d}\in E(G)\cdot D \subseteq D$. Therefore $g\in A$, and so A is an ideal in $Aff(G)$.

Theorems 3 and 5 show that $Aff(G)$ resembles Gonshor's abstract affine near-ring very closely. In fact both these structures are examples of the following: Let R be a near-ring, $(M,+)$ an abelian group, and let $\phi: R \to End(M)$ be a near-ring homomorphism from R into the ring of endomorphisms of M. For $r\in R$, and $m\in M$, the product rm will signify $\phi(r)(m)$. Put $S = R^+ \oplus M$, and define multiplication in S via $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2+m_1)$ for all $(r_1,m_1),(r_2,m_2)\in S$. These products induce a rear-ring structure on S. If R is chosen to be $E(G)$, and M to be C, with $E(G)$ and C as above, then $S = Aff(G)$ with $(f,\hat{c})$ identified with $f+\hat{c}$. If R is a ring, then S is Gonshor's abstract affine near-ring.

An argument similar to that used in proving Theorem 5 yields:

THEOREM 6. *A is an ideal in S if and only if $A = I \oplus N$ with I an ideal in R, and N a subgroup of M satisfying $RN \subseteq N$ and $IM \subseteq N$.*

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