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Calabi–Yau Threefolds and Moduli of Abelian Surfaces I

MARK GROSS[1]★ and SORIN POPESCU[2]★★
[1]*Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.*
e-mail: mgross@maths.warwick.ac.uk
[2]*Department of Mathematics, Columbia University, New York, NY 10027, U.S.A.*
e-mail: psorin@math.columbia.edu

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Abstract. We describe birational models and decide the rationality/unirationality of moduli spaces $\mathcal{A}_d$ (and $\mathcal{A}_d^{\mathrm{lev}}$) of $(1, d)$-polarized Abelian surfaces (with canonical level structure, respectively) for small values of d. The projective lines identified in the rational/unirational moduli spaces correspond to pencils of Abelian surfaces traced on nodal threefolds living naturally in the corresponding ambient projective spaces, and whose small resolutions are new Calabi–Yau threefolds with Euler characteristic zero.

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Let $\mathcal{A}_d$ denote the moduli space of polarized Abelian surfaces of type $(1,d)$, and let $\mathcal{A}_d^{\mathrm{lev}}$ be the moduli space of polarized Abelian surfaces with canonical level structure. Both are (possibly singular) quasi-projective threefolds, and $\mathcal{A}_d^{\mathrm{lev}}$ is a finite cover of $\mathcal{A}_d$. We will also denote by $\widetilde{\mathcal{A}}_d$ and $\widetilde{\mathcal{A}}_d^{\mathrm{lev}}$ nonsingular models of suitable compactifications of these moduli spaces. We will use in the sequel definitions and notation as in [GP1, GP2]; see also [Mu1], [LB] and [HKW] for basic facts concerning Abelian varieties and their moduli. Throughout the paper, the base field will be $\mathbf{C}$.

The main goal of this paper, which is a continuation of [GP1] and [GP2], is to describe birational models for moduli spaces of these types for small values of d. Since the Kodaira dimension is a birational invariant, thus independent of the chosen compactification, we can decide the uniruledness, unirationality or rationality of nonsingular models of (any) compactifications of these moduli spaces.

Motivation for our project has come from several directions:

Tai, Freitag and Mumford have proved that moduli spaces of principally polarized Abelian varieties are of general type when the dimension g of the Abelian varieties is large enough (in fact $\geqslant 7$). However, for small dimensions, they are

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rational or unirational and have nice projective models: see the work of Katsylo [Kat] for $g = 3$, van Geemen [vG] and Dolgachev and Ortland [DO] for $g = 3$ with level 2 structure, Clemens [Cle] for $g = 4$, and Donagi [Do], Mori and Mukai [MM] and Verra [Ver] for $g = 5$. However, it is an open problem to determine the Kodaira dimension of the moduli space for $g = 6$.

Using a version of the Maass–Kurokawa lifting, Gritsenko [Gri1, Gri2] has recently proved the existence of weight 3 cusp forms with respect to the paramodular group Γ_t, for almost all values of t. Since one knows the dimension for the space of Jacobi cusp forms one deduces in this way lower bounds for the dimensions of the spaces of cusp forms with respect to the paramodular group Γ_t, and thus for the plurigenera of the corresponding moduli spaces. More precisely, he has shown that

$$\mathcal{A}_d \text{ is not unirational (in fact } p_g(\widetilde{\mathcal{A}}_d) \geqslant 1) \text{ if } d \geqslant 13 \text{ and}$$
$$d \neq 14, 15, 16, 18, 20, 24, 30, 36.$$

In fact it is pointed out in a note by Gritsenko and Hulek [Gri1] that the same method shows that $\mathcal{A}_p^{\mathrm{lev}}$ is of general type for all primes $p \geqslant 37$. See also [Bori], [HS1] and [HS2] for related results.

A few other moduli spaces have known descriptions:

- $\mathcal{A}_1 \cong \overline{\mathcal{M}_2} \setminus \Delta_0$ via the Jacobian map. $\overline{\mathcal{M}_2}$ is the moduli space of stable curves of genus 2 and Δ_0 stands for the divisor of the curves with at least one nondisconnecting node. In particular $\mathcal{A}_1$ is rational [I].
- $\mathcal{A}_2$ and $\mathcal{A}_3$ are rational [BL]. It follows from results of [Ba] that $\mathcal{A}_{(2,4)}^{\mathrm{lev}}$, the moduli space of (2,4)-polarized Abelian surfaces with canonical level structure, is birational to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. On the other hand it doesn't appear to be known whether $\mathcal{A}_3^{\mathrm{lev}}$ is unirational.
- $\mathcal{A}_4^{\mathrm{lev}}$ is rational. See [BLvS] for a proof of this and for the geometry of (1,4)-polarized Abelian surfaces.
- $\overline{\mathcal{A}_5^{\mathrm{lev}}} \cong \mathbf{P}(H^0(\mathcal{F}_{\mathrm{HM}}(3)))$, where bar stands for the toroidal compactification for the Voronoi or Igusa decomposition, and $\mathcal{F}_{\mathrm{HM}}$ is the Horrocks–Mumford bundle on $\mathbf{P}^4$. See [HM] and [HKW] for details, and Section 3 for a brief review of the relevant facts.
- $\mathcal{A}_7^{\mathrm{lev}}$ is birational to V_{22}, a prime Fano threefold of index 1 and genus 12 [MS]. We will give a short proof of this result in Section 5.
- $\mathcal{A}_9$ is rational (see [O'G] for unirationality and [GP2]). In fact $\mathcal{A}_9^{\mathrm{lev}}$ is naturally birational to $\mathbf{P}^3$ ([GP2]). O'Grady shows also that $\widetilde{\mathcal{A}}_{p^2}$ is a threefold of general type for all prime numbers $p \geqslant 11$.
- $\mathcal{A}_{11}^{\mathrm{lev}}$ is birational to the Klein cubic threefold

$$\mathcal{K} = \left\{ \sum_{i=0}^{4} x_i^2 x_{i+1} = 0 \right\} \subset \mathbf{P}^4$$

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See [GP2] for details. This is the unique $PSL_2(\mathbf{Z}_{11})$-invariant cubic in $\mathbf{P}^4$, and $PSL_2(\mathbf{Z}_{11})$ is its full automorphism group [Ad]. The Klein cubic being smooth is unirational but not rational. It would be interesting also in this case to compare $\mathcal{K}$ with the toroidal compactification.

In this paper and its sequel [GP3], we will focus on (most of) the moduli spaces which Gritsenko has not shown to have nonnegative Kodaira dimension. In particular, we will give details in this paper as to the structure of $\mathcal{A}_d^{\mathrm{lev}}$, $d=6$, 8 and 10, and in [GP3] will discuss $\mathcal{A}_{12}^{\mathrm{lev}}$ and $\mathcal{A}_d$ for $d=14$, 16, 18 and 20. In this paper we prove the following results:

THEOREM 0.1

(a) $\mathcal{A}_6^{\mathrm{lev}}$ *is birational to a nonsingular quadric hypersurface in* $\mathbf{P}^4$.
(b) $\mathcal{A}_8^{\mathrm{lev}}$ *is birational to a rational conic bundle over* $\mathbf{P}^2$.
(c) $\mathcal{A}_{10}^{\mathrm{lev}}$ *is birational to a quotient* $\mathbf{P}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$, *where the action on* $\mathbf{P}^3$ *is given by*

$$(x_1 : x_2 : x_3 : x_4) \longmapsto (x_4 : x_3 : x_2 : x_1) \quad \text{and} \quad (x_1 : x_2 : x_3 : x_4) \longmapsto (x_1 : -x_2 : x_3 : -x_4).$$

This quotient is rational and isomorphic to a (singular) prime Fano threefold of genus 9, *index* 1 *in* $\mathbf{P}^{10}$.

Of these, the description of $\mathcal{A}_{10}^{\mathrm{lev}}$ is the easiest, following immediately from [GP1]. The description of $\mathcal{A}_6^{\mathrm{lev}}$ follows from the fact that a (1,6)-polarized Abelian surface $A \subseteq \mathbf{P}^5$ is determined (but not completely cut out by) the cubics vanishing on A. Since the structure of the ideal of a (1,6)-polarized Abelian surface was not previously known, we will provide details in Section 4.

The structure of $\mathcal{A}_8^{\mathrm{lev}}$ is the most difficult to analyze because it lies at the boundary between those surfaces determined by cubics and those determined by quadrics. In fact we will show that if $A \subseteq \mathbf{P}^7$ is a general (1,8)-polarized Abelian surface, then its homogeneous ideal is generated by 4 quadrics and 16 cubics. The 4 quadrics cut out a complete intersection threefold $X \subset \mathbf{P}^7$ containing A, which in general has as singular locus 64 ordinary double points. We show that such an X in fact contains a pencil of (1,8)-polarized Abelian surfaces, and since the family of such Heisenberg invariant threefolds turns out to be an open subset of $\mathbf{P}^2$, we obtain a description of $\mathcal{A}^8_{\mathrm{lev}}$ as a $\mathbf{P}^1$-bundle over an open subset of $\mathbf{P}^2$.

In the above outline, the threefold X plays a special role. It is a Calabi–Yau threefold since there is a small resolution $\pi\colon \tilde{X} \longrightarrow X$ with $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$. Furthermore, as X contains a pencil of Abelian surfaces, this small resolution can be chosen so that $\tilde{X}$ possesses an Abelian surface fibration. Such fibrations are of independent interest in the study of Calabi–Yau threefolds. Historically there have been very few examples; in fact, for a long time the only examples of a Calabi–Yau threefolds with a fibration whose general fibre is a simple Abelian surface have been the Horrocks–Mumford quintics, whose geometry we review in Section 3. The Horrocks–Mumford quintic in $\mathbf{P}^4$ is defined as the zero locus of the wedge of two independent sections in $H^0(\mathcal{F}_{\mathrm{HM}}(3))$, where $\mathcal{F}_{\mathrm{HM}}$ is the Horrocks–Mumford bundle. Since a (general) section of this bundle vanishes along a (1,5)-polarized Abelian surface in $\mathbf{P}^4$, a Horrocks–Mumford quintic contains a pencil of such surfaces.

It turns out that in every case considered in this paper, there exist similar examples of Calabi–Yau threefolds. This is the second focus of this paper. We find similarities in many of these examples. In particular, many of them contain second pencils of Abelian surfaces of a different type. The classic example of this occurs in the case of Horrocks–Mumford quintics, where a surface obtained via liaison starting from a (1,5)-polarized Abelian surface is a nonminimal (2,10)-polarized Abelian surface. We will see similar phenomena also occur in many of the other examples. Another remark is that all the Calabi–Yau threefolds we discuss in this paper have Euler characteristic zero.

There is a natural reason for the existence of such Calabi–Yau threefolds. In fact, this is motivated by the existence of two series of degenerate 'Calabi–Yau' threefolds. The first series, which occurs in $\mathbf{P}^{n-1}$ for $n \geqslant 5$, are the secant varieties of elliptic normal curves in $\mathbf{P}^{n-1}$. These were studied in [GP1], § 5. Many of the examples of Calabi–Yau threefolds given here are partial smoothings of these secant varieties. This also works naturally in case $n = 4$, if one thinks of the secant variety as a double cover of $\mathbf{P}^3$. As a result, we discuss in Section 2 of this paper the (1,4) case. We also include a section on (1,7)-polarized Abelian surfaces, where the Calabi–Yau threefold which arises in this fashion (first noted by Alf Aure and Kristian Ranestad, unpublished) plays an important role in the description of the moduli. In particular, we give in Section 5 an alternative approach to rationality of $\mathcal{A}_7^{\mathrm{lev}}$ to that given in [MS].

Another series of degenerate 'Calabi–Yau' threefolds appear in $\mathbf{P}^{2n-1}$, $n \geqslant 3$, as the join of two elliptic normal curves of degree n, lying in disjoint linear subspaces of $\mathbf{P}^{2n-1}$. These are threefolds of degree n^2 in $\mathbf{P}^{2n-1}$. We discuss the geometry of these singular threefolds briefly in Section 1. This series gives rise to other classes of Calabi–Yau threefolds, as partial smoothings, in the (1,8) and (1,10) cases. Much of this paper is devoted to describing the geometry of these threefolds.

In brief, the Calabi–Yau threefolds discussed in this paper can be described as follows. Partial smoothings of secant varieties are

- Double covers of $\mathbf{P}^3$ branched over certain nodal octics, see Theorem 2.2. These double covers contain pencils of (1,4)-polarized Abelian surfaces.
- Horrocks–Mumford quintics, containing both a pencil of (1,5) and a pencil of (2,10)-polarized Abelian surfaces, see Theorem 3.2.
- Nodal complete intersections of two cubics in $\mathbf{P}^5$, containing both a pencil of (1,6) and a pencil of (2,6)-polarized Abelian surfaces, see Theorem 4.10.
- Calabi–Yau threefolds defined by the 6×6 Pfaffians of certain 7×7-skew-symmetric matrix of linear forms in $\mathbf{P}^6$, see Proposition 5.2. These contain both a pencil of (1,7) and a pencil of (1,14)-polarized Abelian surfaces. Some details concerning the geometry of these threefolds are delayed until [GP3] in the discussion of (1,14)-polarized Abelian surfaces.

- Calabi–Yau threefolds defined by the 3×3 minors of a 4×4 matrix of linear forms in $\mathbf{P}^7$, which contain a pencil of (1,8)-polarized Abelian surfaces, see Definition 6.11.
- Calabi–Yau threefolds defined by the 3×3 minors of a 5×5 symmetric matrix of linear forms in $\mathbf{P}^9$, which contain a pencil of (1,10)-polarized Abelian surfaces, see Theorem 7.4.

The reader may note that there is a gap corresponding to the partial smoothing of the secant variety of a degree 9 elliptic normal curve, which should contain a pencil of (1,9)-polarized Abelian surfaces. This does not mean that such a Calabi–Yau threefold does not exist, but we have not been able to find it! There is also a nice analogy between the sequence of Calabi–Yau threefolds described above and Del Pezzo surfaces. Del Pezzo surfaces of degree 2 through 6 can be described as a double cover of $\mathbf{P}^2$, a cubic hypersurface in $\mathbf{P}^3$, a complete intersection of two quadrics in $\mathbf{P}^4$, a surface defined by the 4×4-Pfaffians of a 5×5 skew-symmetric matrix of linear forms in $\mathbf{P}^5$, and a surface defined by the 2×2 minors of a 3×3 matrix of linear forms in $\mathbf{P}^6$, respectively. Furthermore, the Del Pezzo surface of degree 8 in $\mathbf{P}^8$ isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$ can be described by the 2×2 minors of a 4×4 symmetric matrix of linear forms. While we do not explore this analogy further, it is a curious one! From this perspective the missing Calabi–Yau, which should contain a pencil of (1,9)-polarized Abelian surfaces, would be a ladder determinantal variety of degree 27 defined by those 3×3-minors of a 5×5 symmetric matrix of linear forms in $\mathbf{P}^8$ which do not involve the lower right corner of the matrix.

Partial smoothings of the join of two elliptic curves naturally occurring in this paper are described as

- Complete intersections of two cubics in $\mathbf{P}^5$, containing a pencil of (1,6)-polarized Abelian surfaces.
- Complete intersections of 4 quadrics in $\mathbf{P}^7$, containing a pencil of (1,8) and a pencil of (2,8)-polarized Abelian surfaces, see Theorem 6.5, Theorem 6.9 and Remark 6.10.
- The proper intersection of 'two copies' of the Plücker embedding of the Grassmannian $\mathrm{Gr}(2,5)\subseteq\mathbf{P}^9$, containing a pencil of (1,10) and a pencil of (3,15)-polarized Abelian surfaces, see Theorem 7.4 and Remark 7.5.

Further examples of such Calabi–Yau threefolds will appear in [GP3].

1. Preliminaries

We review our notation and conventions concerning Abelian surfaces; more details can be found in [GP1].

Let $(A,\mathcal{L})$ be a general Abelian surface with a polarization of type $(1,d)$. If $d\geqslant 5$, then $|\mathcal{L}|$ induces an embedding of $A\subset\mathbf{P}^{d-1}=\mathbf{P}(H^0(\mathcal{L})^\vee)$ of degree $2d$. The line bundle $\mathcal{L}$ induces a natural map from A to its dual Â, $\phi_{\mathcal{L}}\colon A\longrightarrow\hat{A}$, given by

$x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where $t_x\colon A \longrightarrow A$ is the morphism given by translation by $x \in A$. Its kernel $K(\mathcal{L})$ is isomorphic to $\mathbf{Z}_d \times \mathbf{Z}_d$, and is dependent only on the polarization $c_1(\mathcal{L})$.

For every $x \in K(\mathcal{L})$ there is an isomorphism $t_x^*\mathcal{L} \cong \mathcal{L}$. This induces a projective representation $K(\mathcal{L}) \longrightarrow \mathrm{PGL}(H^0(\mathcal{L}))$, which lifts uniquely to a linear representation of $K(\mathcal{L})$ after taking a central extension of $K(\mathcal{L})$

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{G}(\mathcal{L}) \longrightarrow K(\mathcal{L}) \longrightarrow 0,$$

whose Schur commutator map is the Weil pairing. $\mathcal{G}(\mathcal{L})$ is the *theta group* of $\mathcal{L}$ and is isomorphic to the abstract Heisenberg group $\mathcal{H}(d)$, while the above linear representation is isomorphic to the Schrödinger representation of $\mathcal{H}(d)$ on $V = \mathbf{C}(\mathbf{Z}_d)$, the vector space of complex-valued functions on $\mathbf{Z}_d$. An isomorphism between $\mathcal{G}(\mathcal{L})$ and $\mathcal{H}(d)$, which restricts to the identity on centers induces a symplectic isomorphism between $K(\mathcal{L})$ and $\mathbf{Z}_d \times \mathbf{Z}_d$. Such an isomorphism is called a *level structure of canonical type* on $(A, c_1(\mathcal{L}))$. (See [LB], Chapter 8, § 3 or [GP1], § 1.)

A decomposition $K(\mathcal{L}) = K_1(\mathcal{L}) \oplus K_2(\mathcal{L})$, with $K_1(\mathcal{L}) \cong K_2(\mathcal{L}) \cong \mathbf{Z}_d$ isotropic subgroups with respect to the Weil pairing, and a choice of a characteristic c ([LB], Chapter 3, § 1) for $\mathcal{L}$, define a unique natural basis $\{\vartheta_x^c | x \in K_1(\mathcal{L})\}$ of *canonical theta functions* for the space $H^0(\mathcal{L})$ (see [Mu2] and [LB], Chapter 3, § 2). This basis allows an identification of $H^0(\mathcal{L})$ with V via $\vartheta_\gamma^c \mapsto x_\gamma$, where x_γ is the function on $\mathbf{Z}_d$ defined by

$$x_\gamma(\delta) = \begin{cases} 1 & \gamma = \delta, \\ 0 & \gamma \neq \delta, \end{cases} \quad \text{for } \gamma, \delta \in \mathbf{Z}_d.$$

The functions $x_0, \ldots, x_{d-1}$ can also be identified with coordinates on $\mathbf{P}(H^0(\mathcal{L})^\vee)$. Under this identification, the representation $\mathcal{G}(\mathcal{L}) \longrightarrow \mathrm{GL}(H^0(\mathcal{L}))$ coincides with the Schrödinger representation $\mathcal{H}(d) \longrightarrow \mathrm{GL}(V)$. We will only consider the action of $\mathbf{H}_d$, the finite subgroup of $\mathcal{H}(d) \longrightarrow \mathrm{GL}(V)$ generated in the Schrödinger representation by σ and τ, where

$$\sigma(x_i) = x_{i-1}, \qquad \tau(x_i) = \xi^{-i} x_i,$$

for all $i \in \mathbf{Z}_d$, and where $\xi = \mathrm{e}^{\frac{2\pi i}{d}}$ is a primitive root of unity of order d. Notice that $[\sigma, \tau] = \xi$, thus $\mathbf{H}_d$ is a central extension

$$1 \longrightarrow \mu_{\mathbf{d}} \longrightarrow \mathbf{H}_d \longrightarrow \mathbf{Z}_d \times \mathbf{Z}_d \longrightarrow 0.$$

Therefore, the choice of a canonical level structure means that if A is embedded in $\mathbf{P}(H^0(\mathcal{L})^\vee)$ using as coordinates $x_\gamma = \vartheta_\gamma^c$, $\gamma \in \mathbf{Z}_d$, then the image of A will be invariant under the action of the Heisenberg group $\mathbf{H}_d$ via the Schrödinger representation. (See [LB], Chapter 6, § 7 for details.)

If moreover the line bundle $\mathcal{L}$ is chosen to be symmetric (and there are always finitely many choices of such an $\mathcal{L}$ for a given polarization type), then the embedding via $|\mathcal{L}|$ is also invariant under the involution ι, where $\iota(x_i) = x_{-i}$, $i \in \mathbf{Z}_d$. This involution restricts to A as the involution $x \mapsto -x$. We will denote the $\mathbf{P}_+$ and

$\mathbf{P}_-$ the $(+1)$ and (-1)-eigenspaces of the involution ι, respectively. We will also denote as usual by $\mathbf{H}_n^e := \mathbf{H}_n \rtimes \langle \iota \rangle$.

We also recall a key result from [GP1]: In that paper, on $\mathbf{P}^{2d-1} \times \mathbf{P}^{2d-1}$, we have introduced a matrix

$$M_{\mathrm{d}}(x, y) = (x_{i+j}y_{i-j} + x_{i+j+d}y_{i-j+d})_{0 \leqslant i,j \leqslant d-1},$$

where the indices of the variables x and y above are all modulo $2d$. This matrix has the property that if $A \subseteq \mathbf{P}^{2d-1}$ is a Heisenberg invariant $(1, 2d)$-polarized Abelian surface, then M_d has rank at most two on $A \times A \subseteq \mathbf{P}^{2d-1} \times \mathbf{P}^{2d-1}$. Similarly, if $A \subseteq \mathbf{P}^{2d}$ is a Heisenberg invariant $(1, 2d+1)$-polarized Abelian surface, then the (Moore) matrix

$$M'_{2d+1}(x, y) = (x_{d(i+j)}y_{d(i-j)})_{i \in \mathbf{Z}_{2d+1}, j \in \mathbf{Z}_{2d+1}},$$

on $\mathbf{P}^{2d} \times \mathbf{P}^{2d}$ has rank at most four on $A \times A \subseteq \mathbf{P}^{2d} \times \mathbf{P}^{2d}$. These matrices will prove to be ubiquitous!

We include a brief discussion of Abelian surface fibrations on Calabi–Yau threefolds at the end of this section. Throughout this paper we will use the following terminology:

DEFINITION 1.1. A Calabi–Yau threefold is a nonsingular projective threefold satisfying $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$.

LEMMA 1.2. *Let X be a Calabi–Yau threefold, and let $A \subseteq X$ be a minimal Abelian surface. Then A is a member of a base-point free linear system of Abelian surfaces which induces a fibration $\pi\colon X \longrightarrow \mathbf{P}^1$ with A as a fibre.*

Proof. On X, we have $0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(A) \longrightarrow \omega_A \longrightarrow 0$, from which we obtain $\dim|A| = 1$. It then follows from [Og] that $|A|$ must be a base-point free linear system inducing an Abelian surface fibration $X \longrightarrow \mathbf{P}^1$. $\square$

The Calabi–Yau threefolds we will consider in this paper will be partial smoothings of two types of singular threefolds. Recall first [GP1], Proposition 5.1:

PROPOSITION 1.3. *Let $E \subseteq \mathbf{P}^{n-1}$ be an elliptic normal curve of degree n. Then*

(1) $\mathrm{Sec}(E)$ *is an irreducible threefold of degree* $n(n-3)/2$.
(2) $\mathrm{Sec}(E)$ *is nonsingular outside* E.
(3) $\omega_{\mathrm{Sec}(E)} \cong \mathcal{O}_{\mathrm{Sec}(E)}$ *and* $h^1(\mathcal{O}_{\mathrm{Sec}(E)}) = 0$.

Thus, form a numerical point of view, $\mathrm{Sec}(E)$ is a degenerate 'Calabi–Yau' threefold, containing a pencil of type II degenerations of $(1, n)$-polarized Abelian surfaces. Namely, assume that $E \subset \mathbf{P}^{n-1}$ is Heisenberg invariant under the

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Schrödinger representation. Then $\mathrm{Sec}(E)$ contains the pencil of Heisenberg invariant translation scrolls $S_{E,\rho} := \bigcup_{P\in E}\langle P, P+\rho\rangle$, where $\langle P, P+\rho\rangle$ denotes the line spanned by P and $P+\rho$ and $\rho\in E$ is not a 2-torsion point. The fibers of the pencil for ρ a 2-torsion point on E are multiplicity two structures on the corresponding smooth elliptic scrolls S_ρ.

A second series of degenerate 'Calabi–Yau' threefolds is given as follows:

PROPOSITION 1.4. *Let $L_1, L_2 \subseteq \mathbf{P}^{2n-1}$ be two disjoint linear subspaces of dimension $n-1$, and let $E_1 \subseteq L_1$, $E_2 \subseteq L_2$ be two elliptic normal curves of degree n. Put* $\mathrm{Join}(E_1, E_2) := \bigcup_{e_1\in E_1, e_2\in E_2}\langle e_1, e_2\rangle$, *where $\langle e_1, e_2\rangle$ denotes the linear span of e_1 and e_2. Then*

(1) $\mathrm{Join}(E_1, E_2)$ *is an irreducible threefold of degree n^2 in $\mathbf{P}^{2n-1}$.*
(2) $\mathrm{Join}(E_1, E_2)$ *is nonsingular outside of* E_1 and E_2.
(3) $\omega_{\mathrm{Join}(E_1,E_2)} \cong \mathcal{O}_{\mathrm{Join}(E_1,E_2)}$ *and* $h^1(\mathcal{O}_{\mathrm{Join}(E_1,E_2)}) = 0$.

Proof. (1) Irreducibility is obvious. The degree of $\mathrm{Join}(E_1, E_2)$ is the product of the degrees of E_1 and E_2, hence the join is of degree n^2.

(2) Let $Y \subseteq E_1 \times E_2 \times \mathbf{P}^{2n-1}$ be defined by $Y = \{(e_1, e_2, y) \mid y \in \langle e_1, e_2\rangle\}$.

The projection $p_{12}\colon Y \longrightarrow E_1 \times E_2$ clearly gives Y the structure of a $\mathbf{P}^1$-bundle, while the projection $p_3\colon Y \longrightarrow \mathbf{P}^{2n-1}$ gives a resolution of singularities of $\mathrm{Join}(E_1, E_2)$. Note that $p_3\colon Y\setminus p_3^{-1}(E_1 \cup E_2) \longrightarrow \mathbf{P}^{2n-1}$ is an embedding. Indeed, p_3 is one-to-one away from $p_3^{-1}(E_1 \cup E_2)$; otherwise there exist distinct points $e_1, e_1' \in E_1$ and $e_2, e_2' \in E_2$ such that $\langle e_1, e_2\rangle \cap \langle e_1', e_2'\rangle \neq \emptyset$. But then these two lines span only a $\mathbf{P}^2$, so that $\langle e_1, e_1'\rangle \cap \langle e_2, e_2'\rangle \neq \emptyset$, contradicting the assumption that $L_1 \cap L_2 = \emptyset$. To see that p_3 is an immersion, a local calculation suffices. Consider an affine patch $\mathbf{C}^{2n-1}$ of $\mathbf{P}^{2n-1}$, with coordinates $y_1, \dots, y_{2n-1}$, in which L_1 is defined by $y_n = \cdots = y_{2n-1} = 0$ and L_2 is defined by $y_1 = \cdots = y_{n-1} = 0$, and $y_n = 1$. If E_1 is locally parametrized by $t \mapsto (\alpha_1(t), \dots, \alpha_{n-1}(t), 0, \dots 0)$ and E_2 is parametrized by $u \mapsto (0, \dots, 0, 1, \beta_1(u), \dots, \beta_{n-1}(u))$, then Y has local coordinates (t,u,s) in which p_3 is given by

$$(t, u, s) \mapsto (s\alpha_1(t), \dots, s\alpha_{n-1}(t), (1-s), (1-s)\beta_1(u), \dots, (1-s)\beta_{n-1}(u)).$$

Computing the differential of this map, one sees easily that it is injective for $s \neq 0,1$, given that E_1 and E_2 are nonsingular. Thus $p_3\colon Y\setminus p_3^{-1}(E_1 \cup E_2) \longrightarrow \mathbf{P}^{2n-1}$ is an embedding, and so $\mathrm{Join}(E_1, E_2)\setminus(E_1 \cup E_2)$ is nonsingular.

(3) First note that $\mathrm{Join}(E_1, E_2)$ is normal. Indeed, locally the singularities of $\mathrm{Join}(E_1, E_2)$ look like curve$\times$(cone over elliptic normal curve), hence are normal. Now the map $p_3\colon Y \longrightarrow \mathrm{Join}(E_1, E_2)$ contracts two sections σ_1 and σ_2 of the $\mathbf{P}^1$-bundle $p_{12}\colon Y \longrightarrow E_1 \times E_2$ to E_1 and E_2, respectively. Since p_{12} is a $\mathbf{P}^1$-bundle, we can write $K_Y = -\sigma_1 - \sigma_2 + p_{12}^*D$ for some divisor D on $E_1 \times E_2$. But σ_1 and σ_2 are both

isomorphic to $E_1 \times E_2$ and are disjoint, so by adjunction

$$\begin{aligned} 0 = K_{\sigma_1} &= (K_Y + \sigma_1)|_{\sigma_1} \\ &= (-\sigma_2 + p_{12}^* D)|_{\sigma_1} \\ &= D. \end{aligned}$$

Thus $K_Y = -\sigma_1 - \sigma_2$. On the other hand, on $\mathrm{Join}(E_1, E_2)$, both E_1 and E_2 are curves of simple elliptic singularities, so $\mathrm{Join}(E_1, E_2)$ is Gorenstein and

$$K_Y = p_3^* K_{\mathrm{Join}(E_1,E_2)} - \sigma_1 - \sigma_2,$$

from which we conclude that $p_3^* K_{\mathrm{Join}(E_1,E_2)} = 0$. Since $p_3^*\colon \mathrm{Pic}(\mathrm{Join}(E_1, E_2)) \longrightarrow \mathrm{Pic}(Y)$ is injective, this shows $K_{\mathrm{Join}(E_1,E_2)} = 0$, as desired.

Just as in the proof of [GP1], Proposition 5.1, to show that $h^1(\mathcal{O}_{\mathrm{Join}(E_1,E_2)}) = 0$, it is enough to show that $\mathrm{Pic}^0(\mathrm{Join}(E_1, E_2))$ is discrete. Note that the image of $p_3^*\colon \mathrm{Pic}^0(\mathrm{Join}(E_1, E_2)) \longrightarrow \mathrm{Pic}^0(Y)$ is contained in the subgroup $\mathcal{P}$ of $\mathrm{Pic}^0(Y)$ given by

$$\mathcal{P} = \{\mathcal{L} \in \mathrm{Pic}^0(Y) \mid \mathcal{L}|_{\sigma_1} = p_1^*\mathcal{M}_1 \quad \text{and} \quad \mathcal{L}|_{\sigma_2} = p_2^*\mathcal{M}_2 \text{ for some } \mathcal{M}_i \in \mathrm{Pic}(E_i)\}.$$

Thus it is enough to show $\mathcal{P}$ is discrete. Suppose $D \in \mathcal{P}$ is a divisor algebraically equivalent to zero. Then we can write $D \sim n\sigma_1 + p_{12}^* C$ for some $n \in \mathbf{Z}$, $C \in \mathrm{Pic}(E_1 \times E_2)$. Then $n = 0$ and C is algebraically equivalent to zero on $E_1 \times E_2$. Since $p_{12}^* C \in \mathcal{P}$, it follows by restriction to σ_1 and σ_2 that in fact $C = p_1^* F_1$ and $C = p_2^* F_2$ for divisors F_1 and F_2 on E_1 and E_2, respectively. Thus $F_1 = F_2 = C = 0$, so $D = 0$ in $\mathrm{Pic}(Y)$. Thus $\mathcal{P}$ is a discrete group, as is $\mathrm{Pic}^0(\mathrm{Join}(E_1, E_2))$, allowing us to conclude that $h^1(\mathcal{O}_{\mathrm{Join}(E_1,E_2)}) = 0$. □

In fact, in a suitable context, these degenerate 'Calabi–Yau' threefolds contain a pencil of degenerate Heisenberg invariant (1,2n)-polarized Abelian surfaces. Fix the standard Schrödinger action of $\mathbf{H}_{2n}$ on $\mathbf{P}^{2n-1}$, and consider the subgroup $\mathbf{H}'' \subseteq \mathbf{H}_{2n}$ generated by σ^2 and τ. Then $\mathbf{H}''/\langle\tau^n\rangle \cong \mathbf{H}_n$ acts as the Schrödinger representation of $\mathbf{H}_n$ on the subspaces $L_1, L_2 \subseteq \mathbf{P}^{2n-1}$, where

$$L_1 = \{x_0 = x_2 = \cdots = x_{2n-2} = 0\} \quad \text{and} \quad L_2 = \{x_1 = x_3 = \cdots = x_{2n-1} = 0\}.$$

Let $E \subseteq L_1 \cong \mathbf{P}^{n-1}$ be an $\mathbf{H}_n$-invariant elliptic normal curve (under the Schrödinger representation). Then $\sigma(E) \subseteq L_2$ is also an $\mathbf{H}_n$-invariant elliptic normal curve. Let $0 \in E$ denote the origin of E. Define, for $\eta \in E$,

$$S_\eta := \bigcup_{P \in E} \langle P, \sigma(P + \eta)\rangle \subseteq \mathrm{Join}(E, \sigma(E)) \subset \mathbf{P}^{2n-1}.$$

PROPOSITION 1.5

(1) *S_η is a nonsingular elliptic scroll of degree $2n$ in $\mathbf{P}^{2n-1}$.*

(2) *$A_\eta := S_\eta \cup \sigma(S_\eta) = S_\eta \cup S_{-\sigma^2(\eta)}$ is an $\mathbf{H}_{2n}$-invariant surface of degree $4n$ and sectional arithmetic genus $2n$+1. Moreover, $A_\eta = A_{\eta'}$ if and only if η=η' or*

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$\eta' = -\sigma^2(\eta)$. Thus the set $\{A_\eta | \eta \in E\}$ forms a linear pencil of surfaces in $\mathrm{Join}(E_1, E_2)$.

(3) *For $\eta \in E$ general, there exists a flat family $\mathcal{A} \longrightarrow \Delta$, a point $0 \in \Delta$, along with a $\mathbf{H}_{2n}$-invariant embedding $\mathcal{A} \subseteq \mathbf{P}^{2n-1}_\Delta$ such that $\mathcal{A}_0 \cong A_\eta$, and $\mathcal{A}_t$ is a nonsingular* (1,2n)*-polarized Abelian surface for $t \in \Delta$, $t \neq 0$.*

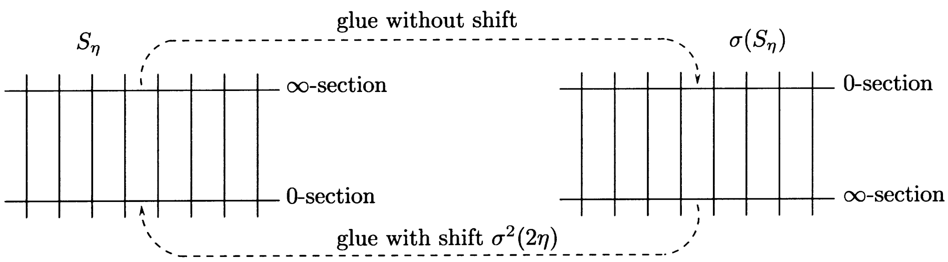
Proof. (1) Nonsingularity of S_η is straightforward and can be proved in much the same way as the non-singularity of $\mathrm{Join}(E_1, E_2)$ away from E_1 and E_2, see Proposition 1.4. To compute the degree of S_η, choose a general hyperplane $H_1 \subseteq L_1$, and let $H = \mathrm{Join}(H_1, L_2)$. Then $H \cap S_\eta = \sigma(E) \cup l_1 \cup \cdots \cup l_n$, where $l_1, \ldots, l_n$ are lines passing through the points of $E \cap H_1$. From this we see that S_η is of degree $2n$.

(2) Straightforward.

(3) The elliptic curves E and $\sigma(E)$ are two disjoint sections of S_η, thus $S_\eta \cong S = \mathbf{P}(\mathcal{O} \oplus \mathcal{L})$, with $\mathcal{L} \in \mathrm{Pic}^0(E)$. This isomorphism can be chosen so that the section E of S_σ corresponds to the 0-section of S, i.e. the section corresponding to the subbundle $\mathcal{O}$ of $\mathcal{O} \oplus \mathcal{L}$, while the section $\sigma(E)$ of S_η corresponds to the ∞-section of S, i.e. the section corresponding to the subbundle $\mathcal{L}$ of $\mathcal{O} \oplus \mathcal{L}$. Then $\mathcal{O}_{S_\eta}(E)|_E = \mathcal{L}$ and $\mathcal{O}_{S_\eta}(\sigma(E))|_{\sigma(E)} = \mathcal{L}^{-1}$. Since E maps to an $\mathbf{H}_n$-invariant elliptic normal curve in L_1, the identification of S with S_η is induced by the complete linear system $|\sigma(E) + n \cdot f_o|$. Now $\sigma(E) + n \cdot f_o \sim E + n \cdot f_p$ for some $p \in E$, and restricting both of these divisors to E, we see that

$$\mathcal{L} = \mathcal{O}_E(-n([p] - [o])).$$

Now as an abstract surface A_η is a type II degeneration, namely a 2-cycle of elliptic ruled surfaces as in the following figure:



One starts with the surface S_η with E and $\sigma(E)$ as its 0 and ∞-sections, respectively, then glues with no shift the ∞-section of S_η to the 0-section $\sigma(E)$ of the surface $\sigma(S_\eta) \cong \mathbf{P}(\mathcal{O} \oplus \mathcal{L})$, and then finally glues with shift $\sigma^2(2\eta)$ the ∞-section E of $\sigma(S_\eta)$ to the 0-section of the S_∞. The restrictions of $\mathcal{O}_{S_\eta}(1)$ and $\mathcal{O}_{\sigma(S_\eta)}(1)$ to E coincide after a shift by $\sigma^2(2\eta)$, which yields $n([o] - [p]) = n([\sigma^2(2\eta)] - [o])$. In other words $p \in E$ is such that $np = -n\sigma^2(2\eta)$ in the group law of E, and $\mathcal{L} = \mathcal{O}_E(-n([p] - [o]))$. As in [HKW], part II, especially Theorem 3.10 and Proposition 4.1, and [DHS], §3, one may show that such type II degenerations actually occur, and in fact all elliptic curves E and all general shifts η can be realized. $\square$

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2. Moduli of (1,4)-Polarized Abelian Surfaces

Even though the geometry of the moduli space of (1,4)-polarized Abelian surfaces is well understood (see [BLvS], we wish to partially review it here as this will lead to another example of a Calabi–Yau threefold with an Abelian surface fibration. However, we will not provide many details, leaving further investigations to the interested reader.

The moduli space $\mathcal{A}_4^{\mathrm{lev}}$ was studied in [BLvS]. Let $(A, \mathcal{L})$ be an Abelian surface with $\mathcal{L}$ an ample line bundle of type (1,4). It is proved in [BLvS] that if A is sufficiently general, then the morphism $\psi_{|\mathcal{L}|}\colon A \longrightarrow \mathbf{P}^3$ induced by $|\mathcal{L}|$ is birational onto its image, an octic surface. Furthermore, [BLvS] give explicitly the defining equation of such an octic surface. Namely, let $(x_0\colon \ldots\colon x_3)$ be the coordinates on $\mathbf{P}^3$, on which the Heisenberg group $\mathbf{H}_4$ acts via the Schrödinger representation

$$\sigma\colon x_i \longmapsto x_{i-1} \quad \text{and} \quad \tau\colon x_i \longmapsto \xi^{-i} x_i,$$

where ξ is a fixed primitive fourth root of unity. We now change coordinates to

$$\begin{aligned} z_0 &= x_0 + x_2, \qquad & z_2 &= x_3 + x_1, \\ z_1 &= x_0 - x_2, \qquad & z_3 &= x_3 - x_1. \end{aligned}$$

Then the image of $\psi_{|\mathcal{L}|}$ is defined in $\mathbf{P}^3$ by the equation $f=0$, where $f(z_0, \ldots, z_3) = \lambda N^t \lambda$, with $N =$

$$\begin{pmatrix} z_0^2z_1^2z_2^2z_3^2 & 0 & 0 & 0 \\ 0 & (z_0^4z_1^4 + z_2^4z_3^4) & (z_0^2z_1^2 + z_2^2z_3^2)(-z_0^2z_2^2 + z_1^2z_3^2) & (z_0^2z_1^2 - z_2^2z_3^2)(z_0^2z_3^2 - z_1^2z_2^2) \\ 0 & (z_0^2z_1^2 + z_2^2z_3^2)(-z_0^2z_2^2 + z_1^2z_3^2) & (z_0^4z_2^4 + z_1^4z_3^4) & (z_0^2z_2^2 - z_2^2z_3^2)(z_0^2z_3^2 - z_1^2z_2^2) \\ 0 & (z_0^2z_1^2 - z_2^2z_3^2)(z_0^2z_3^2 - z_1^2z_2^2) & (z_0^2z_2^2 + z_1^2z_3^2)(z_0^2z_3^2 + z_1^2z_3^2) & (z_0^4z_3^4 + z_1^4z_2^4) \end{pmatrix}$$

and for some value of the parameter $\lambda = (\lambda_0\colon \ldots\colon \lambda_3) \in \mathbf{P}^3$.

For a fixed value of λ we will denote by A_λ the octic surface defined by $\{f = 0\} \subset \mathbf{P}^3$.

Note that $(\lambda_0\colon \ldots\colon \lambda_3)$ and $(-\lambda_0\colon \lambda_1\colon \lambda_2\colon \lambda_3)$ give the same equation f=0 and, conversely, two points $\lambda, \lambda' \in \mathbf{P}^3$ yield the same equation only if they are related in this manner. Thus, if $\mathbf{Z}_2$ acts on $\mathbf{P}^3$ via

$$(\lambda_0\colon \ldots\colon \lambda_3) \longmapsto (-\lambda_0\colon \lambda_1\colon \lambda_2\colon \lambda_3),$$

then there is a birational map

$$\Theta_4\colon \mathcal{A}_4^{\mathrm{lev}} \dashrightarrow \mathbf{P}^3/\mathbf{Z}_2.$$

This latter quotient is isomorphic to the cone over the Veronese surface in $\mathbf{P}^5$.

We recall briefly from [BLvS] the structure of the singularities of $A_\lambda \subset \mathbf{P}^3$ for general λ. The singular locus of A_λ is contained in the union of the coordinate planes $\{z_0z_1z_2z_3 = 0\}$. In fact, $\mathrm{Sing}(A_\lambda) \cap \{z_i = 0\}$ is a quartic curve with double points at the three coordinate vertices on the plane $\{z_i = 0\}$. Now A_λ has generically ordinary double points along this curve, with 12 pinch points in the smooth locus of the quartic, while A_λ has quadruple points at the coordinate vertices (with tangent cone a union of four planes).

DEFINITION 2.1. Let $l \subseteq \mathbf{P}^3$ be a line. Define $V^1_{4,l}$ to be the normalization of the hypersurface $X_l \subseteq \mathbf{P}^3 \times l \subseteq \mathbf{P}^3 \times \mathbf{P}^3$ given by the equation

$$\{f(z_0, \ldots, z_3; \lambda_0, \ldots, \lambda_3) = 0 \mid (\lambda_0, \ldots, \lambda_3) \in l\}$$

with $(z_0 : \ldots : z_3)$ coordinates on the first $\mathbf{P}^3$ and $(\lambda_0 : \ldots : \lambda_3)$ coordinates on l on the second $\mathbf{P}^3$. Let $X_l \xrightarrow{\psi} Y_l \longrightarrow \mathbf{P}^3$ be the Stein factorization of the projection $X_l \longrightarrow \mathbf{P}^3$ on the first factor, and define $V_{4,l}$ to be the normalization of Y_l.

Because this case is not of great relevance to the main thrust of the paper, we state basic properties of these threefolds and only sketch their proof.

THEOREM 2.2. *For a general line l in* $\mathbf{P}^3$,

(1) *$V_{4,l}$ is a double cover of $\mathbf{P}^3$ branched over an octic surface $B \subseteq \mathbf{P}^3$.*
(2) *The singular locus of B consists of* 148 *ordinary double points, all of which are contained in A_λ, $\lambda \in l$. These are of three types*:
   (A) 128 *of them are contained in the smooth locus of A_λ*,
   (B) 16 *others are contained in the double point locus of A_λ* for *$\lambda \in l$ general*,
   (C) *The remaining four occur at the coordinate vertices* (*at the quadruple points of A_λ*).
(3) *The map $\psi' : V^1_{4,l} \longrightarrow V_{4,l}$ induced by the map $\psi : X_l \longrightarrow Y_l$ is a small resolution of $V_{4,l}$. In particular, $V^1_{4,l}$ is a nonsingular Calabi–Yau threefold which contains a base-point free pencil $\widetilde{A}_\lambda$ of nonsingular* (1,4)-*polarized Abelian surfaces mapping to the pencil A_λ, $\lambda \in l$.*
(4) $\chi(V^1_{4,l}) = 0$ *and* $h^{1,1}(V^1_{4,l}) = h^{1,2}(V^1_{4,l}) = 8$.

Proof (Sketch). (1) Let $l \subset \mathbf{P}^3$ be parametrized by $\lambda_j = a_{0j}\mu_0 + a_{1j}\mu_1$, $j \in \{0,1,2,3\}$, where $(\mu_0 : \mu_1) \in \mathbf{P}^1$, and $A = (a_{ij})_{0 \leqslant i \leqslant 1, 0 \leqslant j \leqslant 3}$ is a 2×4 matrix. Then

$$f(z_0, \ldots, z_3, a_{00}\mu_0 + a_{10}\mu_1, \ldots, a_{03}\mu_0 + a_{13}\mu_1)$$

is a quadratic polynomial in μ_0, μ_1 whose Hessian is the 2×2 matrix $2AN^tA$. Thus the discriminant of this quadratic polynomial is $\det(AN^tA)$. By inspection, one finds that every 2×2 minor of the matrix N is divisible by $z_0^2z_1^2z_2^2z_3^2$, and thus so is $\det(AN^tA)$, being a linear combination of minors of N. Hence, one can view Y_l as the double cover of $\mathbf{P}^3$ branched over $\{\det(AN^tA) = 0\}$ Since this determinant contains the factor $z_0^2z_1^2z_2^2z_3^2$, we see Y_l is not normal. Now the surface

$$B := \{(\det AN^tA)/z_0^2z_1^2z_2^2z_3^2 = 0\} \subset \mathbf{P}^3$$

can be seen to have only 148 ordinary double points. One may essentially check this, for a random choice of $l \subset \mathbf{P}^3$ via a straightforward computation in

Macaulay/Macaulay2. Thus $V_{4,l}$, the normalization of Y_l, can be obtained by taking a double cover branched over the octic surface B.

The claims in (2) can be verified directly via *Macaulay/Macaulay2*.

(3) Let $f\colon V_{4,l} \longrightarrow \mathbf{P}^3$, $g\colon Y_l \longrightarrow \mathbf{P}^3$ be the double covers. For $\lambda \in l$, the surface $A_\lambda \times \{\lambda\} \subseteq X_l$ maps isomorphically to $A_\lambda \subseteq \mathbf{P}^3$ via the first projection, and hence $g^{-1}(A_\lambda)$ splits as the union of two surfaces, each being isomorphic to A_λ. Thus $f^{-1}(A_\lambda)$ also splits as $S_\lambda \cup S'_\lambda$, where S_λ is the proper transform of $A_\lambda \times \{\lambda\}$.

A local analysis now shows that for general λ, S_λ is nonsingular except at the ordinary double points of types (B) and (C). At nodes of type (B), S_λ has an improper double point (i.e. the tangent cone is the union of two planes meeting at a point), while at the nodes of type (C) the surface S_λ has tangent cone a union of four planes meeting at a point. Let $V^2_{4,l} \longrightarrow V_{4,l}$ be the blow-up of $V_{4,l}$ along S_λ. Then $V^2_{4,l}$ is a small resolution of $V_{4,l}$, and if all exceptional curves are flopped simultaneously, we obtain a small resolution $Z_l \longrightarrow V_{4,l}$ in which the family of surfaces S_λ forms a base-point free pencil, thus yielding an Abelian surface fibration $Z_l \longrightarrow \mathbf{P}^1$. This map along with the natural map $Z_l \longrightarrow \mathbf{P}^3$ yields a map $\phi\colon Z_l \longrightarrow \mathbf{P}^3 \times \mathbf{P}^1$ whose image is clearly the hypersurface X_l. Hence, via the universal property of the normalization, we get a map ϕ' in the diagram

$$\begin{tikzcd}
Z_l \arrow[d, "\phi'"'] \arrow[dr, "\phi"] & \\
V^1_{4,l} \arrow[r] \arrow[d, "\psi'"'] & X_l \arrow[d, "\psi"] \\
V_{4,l} \arrow[r] & Y_l
\end{tikzcd}$$

The morphism ϕ' is a birational map between normal varieties, and it is also clear that it doesn't contract any positive dimensional components. Thus it is an isomorphism, and (3) follows, with $\widetilde{A}_\lambda$ the proper transform of S_λ in $V^l_{4,l}$.

(4) Now $\chi(V^1_{4,l}) = 0$ follows from the fact that the Euler characteristic of a nonsingular double cover of $\mathbf{P}^3$ branched along a smooth octic is -296, and that $V_{4,l}$ has 148 ordinary double points (each ordinary double point increases by two the Euler characteristic with respect to that of a double solid branched over a smooth octic). The calculation of the Hodge numbers may be done in a general example via *Macaulay/Macaulay2* using standard techniques (see [Scho] and [We], or Remark 4.11 below for details). $\square$

Remark 2.3. We note here that $X_{4,l}$ can be viewed as a partial smoothing of the secant variety of an elliptic normal curve in $\mathbf{P}^3$, in the sense that through the general point of $\mathbf{P}^3$ pass precisely two secants of an elliptic normal curve, so we can think of

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the secant variety as a double cover of $\mathbf{P}^3$. The branch locus is in fact the union of the four quadric cones containing the elliptic curve.

3. Moduli of (1,5)-Polarized Abelian Surfaces

We review here certain aspects of the geometry of (1,5)-polarized Abelian surfaces. We will see in many ways that this case is a paradigm for many of the higher-degree cases. We first review briefly the well-known description of $\mathcal{A}_5^{\mathrm{lev}}$. See [HKW] and references therein for proofs and details.

The main result of [HM] is that every (1,5)-polarized Heisenberg invariant Abelian surface in $\mathbf{P}^4$ is the zero locus of a section of the (twisted) Horrocks–Mumford bundle $\mathcal{F}_{\mathrm{HM}}(3)$, and conversely, when the zero locus is smooth. Here, $H^0(\mathcal{F}_{\mathrm{HM}}(3))$ is a four-dimensional vector space. Thus, if $U \subseteq \mathcal{A}_5^{\mathrm{lev}}$ is the open set consisting of triples (A,H,α) with α the level structure and H very ample on A, we obtain a morphism $\Theta_5\colon U \longrightarrow \mathbf{P}(H^0(\mathcal{F}_{\mathrm{HM}}(3)))$ defined by

$$(A, H, \alpha) \longmapsto [s] \in \mathbf{P}(H^0(\mathcal{F}_{\mathrm{HM}}(3))),$$

where s is a section (uniquely determined up to scalar multiple) of $\mathcal{F}_{\mathrm{HM}}(3)$ vanishing on A. The results of [HKW] then show that this morphism extends to a morphism (denoted exactly as the previous one)

$$\Theta_5\colon \overline{\mathcal{A}_5^{\mathrm{lev}}} \longrightarrow \mathbf{P}(H^0(\mathcal{F}_{\mathrm{HM}}(3))),$$

where $\overline{A_5^{\mathrm{lev}}}$ is the Igusa toroidal compactification of $\mathcal{A}_5^{\mathrm{lev}}$. Θ_5 is a birational morphism, and [HKW] gives a complete description of its structure, as well as a description of $\mathcal{A}_5^{\mathrm{lev}} \subseteq \overline{\mathcal{A}_5^{\mathrm{lev}}}$. In theory, for many of the cases in this paper, one could produce a similar fine structure theory, and thus provide a biregular description of $\overline{\mathcal{A}_n^{\mathrm{lev}}}$, but we shall not even begin to attempt this. See also the introduction for comments regarding the (1,7) and (1,11)-polarizations.

DEFINITION 3.1. Paraphrasing [Moo], (see also [Au], [ADHPR1] and [ADHPR2]) we define for $y \in \mathbf{P}^4$ the Horrocks–Mumford quintic $X_{5,y} := \{\det(M_5'(x, y)) = 0\} \subseteq \mathbf{P}^4$, whenever this determinant does not vanish identically, where $M_5'(x, y) = (x_{3(i+j)}y_{3(i-j)})_{i,j\in\mathbf{Z}_5}$ as in Section 1.

This is not the usual definition of the Horrocks–Mumford quintics: more standard is to choose two independent sections $s, s' \in H^0(\mathcal{F}_{\mathrm{HM}}(3))$ and consider the vanishing locus of $s \wedge s' \in H^0(\wedge^2\mathcal{F}_{\mathrm{HM}}(3)) \cong H^0(\mathcal{O}_{\mathbf{P}^4}(5))^{\mathbf{H}_5}$. However, these two definitions coincide. An argument is given in Remark 4.1 and preceding discussion of [ADHPR2]. Summarizing that argument, we define a rational map

$$\overline{\Theta}\colon \mathbf{P}^4 \dashrightarrow \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5))^{\mathbf{H}_5}) \cong \mathbf{P}(\wedge^2 H^0(\mathcal{F}_{\mathrm{HM}}(3)))$$

by taking $y \in \mathbf{P}^4$ to the $\mathbf{H}_5$-invariant quintic $\det(M_5'(x, y))$. This map is defined outside of the so-called Horrocks–Mumford lines, the $\mathbf{H}_5$-orbit of $\mathbf{P}^1_-$. The image of this map is the Plücker quadric of decomposable tensors in $\mathbf{P}(\wedge^2 H^0(\mathcal{F}_{\mathrm{HM}}(3)))$. This is worked out explicitly for instance in Remark 4.1 of [ADHPR2].

THEOREM 3.2. *For a general* $y \in \mathbf{P}^4$,

(1) $X_{5,y}$ *is a quintic hypersurface in* $\mathbf{P}^4$ *whose singular locus consists of* 100 *ordinary double points.*
(2) *There is a small resolution* $X^1_{5,y} \longrightarrow X_{5,y}$ *such that* $X^1_{5,y}$ *is a Calabi–Yau threefold with* $\chi(X^1_{5,y}) = 0$, $h^{1,1}(X^1_{5,y}) = h^{1,2}(X^1_{5,y}) = 4$. *In addition,* $X^1_{5,y}$ *is fibred in* (1,5)-*polarized Abelian surfaces.*
(3) $X^1_{5,y}$ *also contains a pencil of Abelian surfaces with a polarization of type* (2,10), *blown up in* 25 *points.*

Proof. These are all well-known. Sketching the ideas here, for (1), two general sections of the Horrocks–Mumford bundle $\mathcal{F}_{\mathrm{HM}}(3)$ vanish along two smooth Abelian surfaces that meet transversally in 100 points, the nodes of the corresponding quintic (see [HM], [Au] and [Hu2] for details). For (2), first note that if $A \subseteq \mathbf{P}^4$ is an Abelian surface and $y \in A$, then $M_5'(x, y)$ has rank at most 4 on A by [GP1], Corollary 2.8. Thus $A \subseteq X_{5,y}$. Blowing up A produces a small resolution, and flopping the 100 exceptional curves, we obtain $X^1_{5,y}$, in which A moves in a base-point free pencil, by Lemma 1.2. The invariants of $X^1_{5,y}$ are well known, see [Au] for details.

For (3), it is well known that if $A \subseteq X_{5,y}$ as above, and X' is another general quintic hyper-surface containing A, then $X_{5,y} \cap X' = A \cup A'$, where $A' \subseteq \mathbf{P}^4$ is a nonsingular surface of degree 15, in fact an Abelian surface with a (2,10)-polarization blown up in 25 points (Ellingsrud–Peskine unpublished, see [Au] or [ADHPR1, ADHPR2] for details). It follows from Lemma 1.2 that there is a pencil of such surfaces on $X_{5,y}$. □

Remark 3.3. (1) The Horrocks–Mumford quintics $X_{5,y}$ can be viewed as partial smoothings of the secant varieties of elliptic normal curves in $\mathbf{P}^4$, as [GP1], Theorem 5.3 or [Hu1], p. 109 shows that such secant varieties are also Horrocks–Mumford quintics.

(2) The Käler and moving cones of $X^1_{5,y}$ are well-studied: see [Bor1], [Bor2], [Scho], [Scho2] and [Fry]. In particular there are an infinite number of minimal models, and $X^1_{5,y}$ contains an infinite number of pencils of (birationally) Abelian surfaces, of both types.

We now collect a number of results about Horrocks–Mumford quintics and (2,10)-polarized Abelian surfaces we will need later which do not appear to be in the literature. We first describe a certain family of Horrocks–Mumford (HM) quintic hypersurfaces $X_{5,y} \subset \mathbf{P}^4$, where the parameter point y lies in $\mathbf{P}^2_+ \subseteq \mathbf{P}^4$. This result will be needed for understanding the singularity structure of Calabi–Yau threefolds arising in the (1,10) case (see Theorem 7.4):

PROPOSITION 3.4. *Let $M'_5(x,y)=(x_{3(i+j)}y_{3(i-j)})_{i\in\mathbf{Z}_5}$, and let $X_{5,y}$ be the (symmetric) HM-quintic hypersurface given by $\{\det M'_5(x,y)=0\}\subset\mathbf{P}^4(x)$, for a parameter point $y\in\mathbf{P}^2_+$ (here and in the proof of the proposition $\pm$ are again with respect to the Heisenberg involution ι acting in $\mathbf{P}^4$).*

(1) If $y\in B\subset\mathbf{P}^2_+$, *where B is the Brings curve (the curve swept by the nontrivial 2-torsion points of $\mathbf{H}_5$ invariant elliptic normal curves in $\mathbf{P}^4$, see* [BHM]*), then $X_{5,y}$ is the secant variety of an elliptic normal curve in $\mathbf{P}^4$.*

(2) *If $y\in C_+\subset\mathbf{P}^2_+$, where $C_+=\{y_0^2+4y_1y_2=0\}$ is the modular conic (cf.* [BHM]*), then $X_{5,y}$ is the trisecant variety of an elliptic quintic scroll in $\mathbf{P}^4$.*

(3) *For a general $y\in\mathbf{P}^2_+\setminus(C_+\cup B)$, the singular locus of $X_{5,y}$ is the union of two elliptic curves of degree* 10, *meeting along the $\mathbf{H}_5$-orbit of the parameter point y. See* (5) *below for the nature of the singularities.*

(4) Let $\tilde{X}_{5,y}\subseteq\mathbf{P}^4(x)\times\mathbf{P}^4(z)$ *be defined by*

$$\tilde{X}_{5,y}=\{(x,z)\in\mathbf{P}^4\times\mathbf{P}^4\mid M'_5(x,y)z=0\}.$$

Then for general $y\in\mathbf{P}^2_+\setminus(C_+\cup B)$, $\tilde{X}_{5,y}$ has 50 *ordinary double points, and* $\operatorname{Sing}(\tilde{X}_{5,y})$ *maps to the Heisenberg orbit of y under projection to $\mathbf{P}^4(x)$.*

(5) *For general $y\in\mathbf{P}^2_+\setminus(C_+\cup B)$, $X_{5,y}$ has only cA_1 singularities away from the Heisenberg orbit of y, while at a point of the Heisenberg orbit of y, $X_{5,y}$ has a cA_3 singularity.*

(6) *If $p_1,p_2\colon\mathbf{P}^4\times\mathbf{P}^4\longrightarrow\mathbf{P}^4$ are the two projections, then $X'_{5,y}:=p_2(\tilde{X}_{5,y})\subseteq\mathbf{P}^4$ is a $\mathbf{H}_5$-invariant quintic which, for general $y\in\mathbf{P}^2_+\setminus(C_+\cap B)$, has singular locus a union of two elliptic quintic normal curves in $\mathbf{P}^4$. Furthermore, $X'_{5,y}\subset\mathbf{P}^4(z)$ is defined by the equation $\{\det L(z,y)=0\}$, where $L(z,y)$ is the 5×5 matrix given by $L(z,y):=(z_{2i-j}y_{i-j})_{i,j\in\mathbf{Z}_5}$.*

Proof. First we note that (1) follows immediately from [GP1]. Theorem 5.3. On the other hand, (2) follows from [ADHPR1], Proposition 24.

For the rest, recall that $\Gamma=\wedge^2H^0(\mathcal{F}_{\mathrm{HM}}(3))=H^0(\mathcal{O}_{\mathbf{P}^4}(5))^{\mathbf{H}_5}$ is the six-dimensional space of $\mathbf{H}_5$-invariant quintics in $\mathbf{P}^4$. The decomposable vectors in Γ correspond to the Horrocks–Mumford quintics, namely the quintic hypersurfaces in $\mathbf{P}^4$ whose equations are the determinants of matrices of type M'_5, see Definition 3.1 and the discussion thereafter. We will need the following standard facts concerning HM-quintics and $\mathbf{H}_5^e=\mathbf{H}\rtimes\langle\iota\rangle$-invariant elliptic quintic scrolls in $\mathbf{P}^4$, most of which can be found in [BHM]:

(I) There exists a one-dimensional family X_t, $t\in\mathbf{P}^1_-$, of $\mathbf{H}_5^e$-invariant elliptic quintic scrolls in $\mathbf{P}^4$; the smooth ones correspond to $t\in\mathbf{P}^1_-\setminus\{(0:1),(1:0),((1\pm\sqrt{5})\xi^k:2),k\in\mathbf{Z}_5\}$, the singular are cycles of planes. The ruling of a smooth elliptic scroll X_t over the origin of the base curve maps to a line $l\subset\mathbf{P}^2_+$, which is tangent to the conic C_+ at a point which corresponds to the point $t\in X(5)\cong C_+$ via the standard identification [BHM], or [ADHPR2], Proposition 4.3. The scroll

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X_t is embedded in $\mathbf{P}^4$ by the linear system $|C_0 + 2l|$, where C_0 is the (unique) section of the scroll with self-intersection 1 meeting $\mathbf{P}^1_-$.

(II) For any smooth elliptic scroll X_t, there exist exactly three pairs (E_i, τ_i), $i = 0,1,2$, of $\mathbf{H}_5^e$-invariant elliptic normal curves in $\mathbf{P}^4$ and 2-torsion points, such that X_t is the τ_i-translation scroll of E_i, see the discussion after Proposition 1.3 and [Hu2] for exact definitions.

(III) The linear system $|2E_i| = |-2K_{X_t}| = |4C_0 - 2l|$ is a base-point free elliptic pencil, whose only singular fibres are the double curves $2E_i$.

(IV) The linear system $|C_0 + 2E_i| = |5C_0 - 2l|$ on X_t, which has as base locus the three origins (on $\mathbf{P}^1_-$) of the elliptic normal curves E_i, defines a $2:1$ rational map from X_t onto $\mathbf{P}^2$. The map is branched along l and the three exceptional lines (over the base points), it contracts C_0 and the elliptic normal curves E_i to p and p_i, respectively, and maps the other elements of the pencil $|-2K_{X_t}|$ to the pencil of lines through p (cf. [BHM], Propositions 5.4 and 5.5).

(V) By [BHM], proof of Proposition 5.9, the restriction of Γ, the space of $\mathbf{H}_5$-invariant quintics, to a $\mathbf{H}_5^e$-invariant elliptic quintic scroll $X_t \subset \mathbf{P}^4$ is always three-dimensional. Furthermore, the kernel of this restriction is $s_t \wedge H^0(\mathcal{F}_{\mathrm{HM}}(3))$, where $s_t \in H^0(\mathcal{F}_{\mathrm{HM}}(3))$ is the unique section of the Horrocks–Mumford bundle vanishing on a double structure on the elliptic scroll X_t. The sections $s_t \in H^0(\mathcal{F}_{\mathrm{HM}}(3))$ vanishing on a double structure on the elliptic scroll X_t are parameterized by a smooth rational sextic curve $C_6 \subset \mathbf{P}^3 = \mathbf{P}(H^0(\mathcal{F}_{\mathrm{HM}}(3)))$ (cf. [BM], 1.2).

(VI) The linear system Γ induces a rational map Θ: $\mathbf{P}^4 \dashrightarrow \Omega \subset \mathbf{P}^5$, where Ω is a smooth quadric, which is $100:1$ and is not defined exactly on the so called Horrocks–Mumford lines (the $\mathbf{H}_5$ orbit of $\mathbf{P}^1_-$). In this setting, Horrocks–Mumford quintics correspond to pullbacks via Θ of the tangent hyperplanes to the Plüker quadric Ω.

(VII) Θ restricted to X_t factors as Θ: $X_t \longrightarrow X_t/\mathbf{Z}_5 \times \mathbf{Z}_5 \cong X_t \dashrightarrow \Omega$, and the latter map is induced by the linear system $|C_0 + 2E_i|$, and thus by (IV) above, maps the scroll onto a linear subspace $\mathbf{P}^2 \subset \Omega \subset \mathbf{P}^5$.

Next we identify the quintics in part (3) of the statement of Proposition 3.4:

LEMMA 3.5. *Symmetric HM-quintics $X_{5,y} \subset \mathbf{P}^4$, for a parameter $y \in \mathbf{P}^2_+ \setminus C_+ \cup B$, correspond to wedge products $s_1 \wedge s_2$, where $s_i \in H^0(\mathcal{F}_{\mathrm{HM}}(3))$, $i = 1,2$, are two sections of the Horrocks–Mumford bundle each vanishing on a $\mathbf{H}_5^e$-invariant elliptic quintic scroll $X_i \subset \mathbf{P}^4$. Such quintic hypersurfaces can also be characterized as the unique quintics in $\mathbf{P}^4$ containing the union of the two elliptic quintic scrolls X_i: If l_i are the rulings over the origin of X_i, $i = 1,2$, then we may take as parameter y of the matrix $M_5'(x, y)$ the point $\{y\} = l_1 \cap l_2 \in \mathbf{P}^2_+$. In particular, this allows us to identify* $\mathrm{Sym}^2(C_6)$ *with $\mathbf{P}^2_+$.*

Proof. Let $y \in \mathbf{P}^2_+ \setminus (C_+ \cup B)$, and let l_1 and l_2 be the two tangent lines to the modular conic C_+ that pass through the point y. By fact (I) above l_1 and l_2 are each rulings over the origin for two distinct $\mathbf{H}_5^e$-invariant quintic elliptic scrolls $X_i \subset \mathbf{P}^4$, $i=1,2$. On the other hand, for a fixed point $x \in B$, the quintic $\{\det M_5'(x,y)=0\} \subset \mathbf{P}^4(y)$ is the secant variety of the elliptic normal curve in $\mathbf{P}^4(y)$ passing through x. Now if $E_1^i, E_2^i, E_3^i \subseteq X_i$ are the three elliptic normal curves of fact (II) above, then $E_j^i \cap l_i$ consists of two distinct points, and thus $B \cap l_i = \bigcup_{j=1}^3 (E_j^i \cap l_i)$ consists of six distinct points (see [BHM], §6). Thus l_i is a secant to each E_j^i, and thus $\det M_5'(x,y)$ vanishes at each $x \in B \cap l_i$. By Bézout's theorem, it follows that $X_{5,y}$ must vanish along both rulings l_1 and L_2. Further, by Heisenberg invariance, $X_{5,y}$ must then vanish on the $\mathbf{H}_5$ orbits of l_1 and l_2, and thus on both elliptic scrolls X_1 and X_2 again by Bézout's theorem. The first claim in the lemma follows now from fact (V) above. For the second claim, observe that by fact (VII) above, the map $\Theta\colon \mathbf{P}^4 \dashrightarrow \Omega$ maps the two scrolls X_i onto two planes in $\mathbf{P}^5$ meeting at a point, and thus spanning a unique hyperplane in $\mathbf{P}^5$. This concludes the proof of the lemma. □

Proof of Proposition 3.4 continued. Suppose that $y \in \mathbf{P}_+^2 \setminus (C_+ \cup B)$, and thus that $X_{5,y} = \{s_1 \wedge s_2 = 0\} \subset \mathbf{P}^4(x)$, with $s_i \in H^0(\mathcal{F}_{\mathrm{HM}}(3))$ vanishing doubly on elliptic quintic scrolls $X_i \subset \mathbf{P}^4(x)$. The elliptic quintic scrolls X_1 and X_2 meet only in 25 points, namely the $\mathbf{H}_5$ orbit of $y \in \mathbf{P}_+^2$. Moreover, since rank $M_5'(y,y) \leqslant 3$, it follows that all the $\mathbf{H}_5$ translates of y are in the singular locus of the quintic $X_{5,y}$.

By fact (III) above, on each scroll X_i, the linear system $|-2K_{X_i}|$ is a base-point free pencil, with general member a smooth elliptic curve of degree 10. We denote by D_i the unique degree 10 elliptic curve in the pencil $|-2K_{X_i}|$ which passes through the parameter point y. Each curve in the pencil $|-2K_{X_i}|$ is Heisenberg invariant, since at least three curves in the pencil (the doubled elliptic quintic curves) are Heisenberg invariant, and only the identity automorphism on $|-2K_{X_i}| \cong \mathbf{P}^1$ has $\geqslant 3$ fixed points. Thus $D_1 \cap D_2 = X_1 \cap X_2$ is the $\mathbf{H}_5$ orbit of y.

We show now that rank $M_5'(x,y) \leqslant 3$ for all $x \in D_1 \cup D_2$, and thus that $X_{5,y}$ is singular along the union of these two elliptic curves. The space of Heisenberg invariant quintics containing the elliptic curve D_i (the elliptic scroll X_i) is 4-dimensional (respectively, three-dimensional), so D_i is linked on X_i to a $\mathbf{H}_5$-invariant curve G_i of degree 15. The elliptic curve G_i is a section of the scroll X_i, described in [ADHPR2], Proposition 4.10 (iii), and is the unique $\mathbf{H}_5 \rtimes$-invariant curve of class $C_0 + 12l$ on X_i. By [ADHPR2], Proposition 4.12 (ii), for a fixed point $z \in G_i$, not on a Horrocks–Mumford line, the quintic $\{\det M_5'(x,z)=0\} \subset \mathbf{P}^4(x)$ is the trisecant variety of the elliptic scroll X_i. It then follows, as in [ADHPR1], Proposition 24, that rank $M_5'(x,y) \leqslant 3$ for all x on the unique curve of numerical equivalence class $4C_0 - 2l$ passing through z. However, if $z \in D_i$, this curve is precisely D_i. On the other hand the matrices $M_5'(x,y)$ and $M_5'(y,x)$ coincide up to a permutation of their columns, and so do their collection of 4×4 minors. Thus it follows that rank $M_5'(x,y) \leqslant 3$ for all $x \in D_i \cap G_i$. Now on the scroll X_i we have $D_i \cdot G_i = 50$, and since both these curves are $\mathbf{H}_5$-invariant, $D_i \cap G_i$ must consist of at least 25 distinct point. By [ADHPR2], Proposition 4.8, $G_1 \cap G_2 = \varnothing$, and since rank $M_5'(x,y) \leqslant 3$ for all $x \in D_1 \cap D_2$, it follows that each $4{\times}4$ minor of $M_5'(x,y)$ vanishes in at least 50 points along D_i, and thus by Bézout's theorem, vanishes identically on the curve D_i. It follows that rank $M_5'(x,y) \leqslant 3$ for all $x \in D_1 \cup D_2$, which means that the quintic $X_{5,y} \subset \mathbf{P}^4$ is singular along the union of the two elliptic curves D_i.

One checks now in *Macaulay/Macaulay2* that for a general $y \in \mathbf{P}^2_+ \setminus (C_+ \cup B)$ the quintic $X_{5,y}$ is singular only along the union $D_1 \cup D_2$.

To show (4) and (5), note that it is well-known that for general $y \in \mathbf{P}^4$, $\tilde{X}_{5,y} \subseteq \mathbf{P}^4 \times \mathbf{P}^4$ has precisely 50 ordinary double points. (See, for example, [Au] or [Bor1].) Thus, if this is also the case for one $y \in \mathbf{P}^2_+$, it will be the case for general such y. But this can be checked using *Macaulay/Macaulay2*. Now $p_1\colon \tilde{X}_{5,y} \longrightarrow X_{5,y}$ is an isomorphism away from $p_1^{-1}(\mathrm{Sing}(X_{5,y}))$, since rank $M_5'(x,y) \leqslant 3$ only for points of $\mathrm{Sing}(X_{5,y})$. Furthermore, $p_1^{-1}(\mathrm{Sing}(X_{5,y})) \longrightarrow \mathrm{Sing}(X_{5,y})$ is a $\mathbf{P}^1$-bundle, since rank $M_5'(x,y) = 3$ on $\mathrm{Sing}(X_{5,y})$. Since generically p_1 resolves the singularities of $X_{5,y}$, $X_{5,y}$ must generically have cA_1 singularities. Because $\tilde{X}_{5,y}$ has only ordinary double points, it is then clear that $X_{5,y}$ has only cDV singularities, and by deformation theory of du Val singularities, y must be at least cA_3, and at least two ordinary double points of $\tilde{X}_{5,y}$ are sitting over y. By Heisenberg invariance, there are precisely two ordinary double points over each point in the Heisenberg orbit of y, and thus each of these points is a cA_3 point, while away from the Heisenberg orbit of y, points of $\mathrm{Sing}(X_{5,y})$ are cA_1 points.

Finally, to show (6), first note that the equation $M_5'(x,y)z = 0$ of $\tilde{X}_{5,y}$ is equivalent to the equation ${}^tL(z,y)x = 0$, given that $y \in \mathbf{P}^2_+$. Thus the equation of $X_{5,y}' = p_2(\tilde{X}_{5,y})$ is given by $\{\det L(z,y) = 0\}$. Now let $N(x,y)$ be a $4{\times}5$ matrix of linear forms whose rows are general linear combinations of the rows of $M_5'(x,y)$, and let

$$\Delta = \{x \in \mathbf{P}^4 \mid \mathrm{rank} N(x,y) \leqslant 3\}.$$

Then it is well known (see [Au] or [Bor1]) that the linear system $|4H{-}\Delta|$ on $X_{5,y}$ induces a birational map $\varphi\colon X_{5,y} \dashrightarrow X_{5,y}'$ which lifts to the projection $p_2\colon \tilde{X}_{5,y} \longrightarrow X_{5,y}'$. Now $X_{5,y}$ contains the elliptic quintic scrolls X_1 and X_2. Each scroll X_i is embedded via $|C_0 + 2l|$ by fact (I) above, and hence $\varphi|_{X_i}$ is induced by a subsystem of $|4C_0 + 8l - (\Delta \cap X_i)|$. But Δ certainly includes the curve D_i, which is of class $-2K_{X_i} = 4C_0 - 2l$. Furthermore, for a general (1,5)-polarized Abelian surface $A \subseteq X_{5,y}$, $\Delta \cap A$ is degree 20. It then follows from the intersection theory on a resolution of $X_{5,y}$ that $\Delta \cap X_i = D_i \cup l_i^1 \cup \cdots \cup l_i^5$, where l_i^j are lines of the ruling on X_i, and thus $\varphi|_{X_i}$ is induced by a linear system numerically equivalent to $5l$. Thus $\varphi|_{X_i}$ maps X_i to an elliptic normal quintic curve in $X_{5,y}' \subseteq \mathbf{P}^4$, and $\mathrm{Sing}(X_{5,y}')$ contains at least two elliptic normal curves. However, checking for general $y \in \mathbf{P}^2_+ \setminus (C_+ \cup B)$, one finds via *Macaulay/Macaulay2* precisely this singular locus, so this describes the singular locus for $X_{5,y}'$ for general $y \in \mathbf{P}^2_+$. $\square$

Next we study (2,10)-polarized Abelian surfaces in $\mathbf{P}^4$.

PROPOSITION 3.6. *Let A be a general Abelian surface with an ample line bundle $\mathcal{L}=(\mathcal{L}')^{\otimes 2}$ of type* (2,10), *and let $x_0 \in A$. Let $\pi\colon \widetilde{A} \longrightarrow A$ be the blow-up of A at the* 25 *points of the set $x_0+K(\mathcal{L}')$, and suppose there exists elements $f_0,\dots,f_4 \in H^0(\widetilde{A}, \pi^*\mathcal{L}\otimes w_{\widetilde{A}}^{-1})$ inducing a morphism $f=(f_0,\dots,f_4)\colon \widetilde{A} \to \mathbf{P}^4$. Suppose furthermore this map is equivariant with respect to the action of $K(\mathcal{L}')$ on $\tilde{A}$ and the Schrödinger action of $\mathbf{H}_5$ on $\mathbf{P}^4$, and is also equivariant with respect to negation on $\tilde{A}$ and the Heisenberg involution ι on $\mathbf{P}^4$. Then, possibly after changing the origin on A, we can take $x_0=0$, $\mathcal{L}$ a line bundle of characteristic* 0 *with respect to some decomposition. Furthermore, f is an embedding, and for any $y \in f(\widetilde{A})$ the quintic hypersurface $\{\det M_5'(x,y)=0\}$ contains $f(\widetilde{A})$.*

Proof. The fact that negation on A lifts to negation on $\widetilde{A}$ tells us that x_0 is a two-torsion point. Furthermore, $(-1)^*\mathcal{L}\cong\mathcal{L}$, so $\mathcal{L}$ is a symmetric line bundle. Since x_0 is two-torsion and $A_2\subseteq K(\mathcal{L})$, we have $t_{x_0}^*\mathcal{L}\cong\mathcal{L}$, and in particular if we change the origin of A to be x_0, $\mathcal{L}$ will still be a symmetric line bundle. Indeed, if $(-1)'$ denotes negation on A with origin x_0, then $(-1)'^*=t_{-x_0}^*\circ(-1)^*\circ t_{x_0}^*$, so $(-1)'^*\mathcal{L}\cong(-1)^*\mathcal{L}\cong\mathcal{L}$. In addition, the action of $(-1)'$ on sections of $\mathcal{L}$ is the same as that of the negation (-1), so changing the origin to x_0 does not affect the hypotheses of the proposition.

Now let $H\subseteq\mathbf{P}^4$ be a hyperplane containing $\mathbf{P}^2_+$, let $\tilde{D}=f^*H$ be the corresponding divisor on $\widetilde{A}$, and set $D=\pi_*\tilde{D}$. Let

$$A_2^-(D)=\{x\in A_2 \mid \mathrm{mult}_x(D)\equiv 1 \bmod 2\}.$$

By [LB], IV, (7.6), if $\mathcal{L}$ is of characteristic zero with respect to some decomposition, then $\#A_2^-(D)$ is 0 or 16, while if $\mathcal{L}$ is not of characteristic zero, then $\#A_2^-(D)=8$. We claim the latter does not occur. Indeed, since $\tilde{D}^2=15$, we expect $f^{-1}(\mathbf{P}^2_+)$ to consist of 15 points with D vanishing to order 1 on each point, ruling out the latter case. To show that $f^{-1}(\mathbf{P}^2_+)$ in fact consists of 15 distinct points, we proceed as follows. We have $\pi(f^{-1}(\mathbf{P}^2_+))\subseteq A_2$, so $f^{-1}(\mathbf{P}^2_+)$ consists of 2-torsion points and possibly a number of copies of $E_1=\pi^{-1}(0)$. Now if p is an isolated point in $f^{-1}(\mathbf{P}^2_+)$, then ι acts on the tangent plane $T_{A,p}$ to A at p as negation. In particular, $f_*T_{A,p}$ must intersect $\mathbf{P}^2_+$ transversally, so that p occurs with multiplicity one in $f^{-1}(\mathbf{P}^2_+)$. Similarly, if $E_1\subseteq f^{-1}(\mathbf{P}^2_+)$, then E_1 must occur with multiplicity one. Thus either $E_1\not\subseteq f^{-1}(\mathbf{P}^2_+)$, and then $f^{-1}(\mathbf{P}^2_+)$ consists of 15 distinct points, or $E_1\subseteq f^{-1}(\mathbf{P}^2_+)$, and we also have a residual $(\tilde{D}-E_1)^2=12$ points. In either case, we have $\#A_2^-(D)>8$ and, hence, $\#A_2^-(D)=16$ and $\mathcal{L}$ is of characteristic 0.

Now consider the map $\Theta\colon \mathbf{P}^4 \dashrightarrow \Omega\subseteq\mathbf{P}^5$ introduced in fact (VI) of the proof of Proposition 3.4. Recall from [Au] (see also [ADHPR1], Remark 37) that if L_1 is an α-plane in Ω, then $\Theta^{-1}(L_1)$ is a (1,5)-polarized Abelian surface union the base locus of Θ. Moreover, if L_1 is linked to a β-plane L_2 via a hyperplane section of Ω, then $\Theta^{-1}(L_2)$ must be a degree 15 nonminimal (2,10)-polarized Abelian surface (see [ADHPR1], pg. 898). In particular, there is a three-dimensional family of such $\mathbf{H}_5$ and ι-invariant surfaces. These surfaces are embedded in $\mathbf{P}^4$ by a linear system

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of the type given in the hypotheses of this Proposition, and since these surfaces are linearly normal, the embedding is given by a complete linear system $|H^0(\pi^*\mathcal{L} \otimes \omega_{\tilde{A}}^{-1})|$, where $\mathcal{L}$ is of characteristic zero with respect to some decomposition.

Now the moduli space of (2,10)-polarized Abelian surfaces along with a choice of symplectic basis is the Siegel upper half-space $\mathcal{H}_2$. This is three dimensional. There is a universal family $\mathcal{A} \longrightarrow \mathcal{H}_2$ with a zero section, and a line bundle $\mathcal{L}$ on $\mathcal{A}$ such that for any $Z \in \mathcal{H}_2$, $\mathcal{L}|_{\mathcal{A}_Z}$ is the line bundle of type (2,10) and of characteristic zero with respect to the decomposition on $\mathcal{A}_Z$ induced by the choice of symplectic basis (see [LB], Chapter 8, (7.1)). Furthermore, we can blow up the submanifold $\bigcup_{Z\in\mathcal{H}_2} 2K(\mathcal{L}|_{\mathcal{A}_Z}) \subseteq \mathcal{A}$ to obtain a three-dimensional family of non-minimal (2,10)-polarized Abelian surfaces $\widetilde{\mathcal{A}} \longrightarrow \mathcal{H}_2$, along with the line bundle $\pi^*\mathcal{L} \otimes \omega_{\widetilde{\mathcal{A}}/\mathcal{H}_2}^{-1}$. Here $\pi\colon \widetilde{\mathcal{A}} \longrightarrow \mathcal{A}$ denotes the blow-up map. This defines the universal family of polarized non-minimal Abelian surfaces of precisely the sort we are interested, and every such surface appears in this family. Now the point is that this is a three-dimensional family, and further for each $Z \in \mathcal{H}_2$, there are only a finite number of bases of $H^0(\pi^*\mathcal{L} \otimes \omega_{\widetilde{\mathcal{A}}/\mathcal{H}_2}^{-1}|_{\mathcal{A}_Z})$ for which the induced map $f\colon \widetilde{\mathcal{A}}_Z \dashrightarrow \mathbf{P}^4$ is equivariant with respect to translation by elements of $2K(\mathcal{L}|_{\mathcal{A}_Z})$ on $\widetilde{\mathcal{A}}_Z$ and the Schrödinger representation of $\mathbf{H}_5$ on $\mathbf{P}^4$. Indeed, all such bases are related by an element of $\mathrm{SL}_2(\mathbf{Z}_5)$ acting on $H^0(\pi^*\mathcal{L} \otimes \omega_{\widetilde{\mathcal{A}}/\mathcal{H}_2}^{-1}|_{\mathcal{A}_Z})$. Since we already have a three-dimensional family of such Abelian surfaces embedded in $\mathbf{P}^4$, these two families must coincide. Thus for a general $Z \in \mathcal{H}_2$, $|H^0(\pi^*\mathcal{L} \otimes \omega_{\widetilde{\mathcal{A}}/\mathcal{H}_2}^{-1}|_{\mathcal{A}_Z})|$ is very ample, while the embedded surface $f(\widetilde{\mathcal{A}}_Z)$ is of the form $\Theta^{-1}(L_2)$ for some β-plane $L_2 \in \Omega \subset \mathbf{P}^5$. Now if $y \in f(\widetilde{\mathcal{A}}_Z)$, and H is a hyperplane in $\mathbf{P}^5$ tangent to Ω at $\Theta(y)$, then $\Theta^{-1}(H)$ is the Horrocks–Mumford quintic $\{\det M_5'(x, y) = 0\}$, and clearly $f(\widetilde{\mathcal{A}}_Z) \subseteq \Theta^{-1}(H)$, since $L_2 \subseteq H$. □

4. Moduli of (1,6)-Polarized Abelian Surfaces

We will show that the general Abelian surface $A \subset \mathbf{P}^5$ with a (1,6) polarization is determined by the cubics containing it, and this will allow us to define a rational map Θ_6, essentially taking A to the set of cubics containing it. In fact, we will find out in addition that through a general point y of $\mathbf{P}^5$ passes exactly one (1,6)-polarized Abelian surface! The strategy for determining this Abelian surface is to consider the cubics passing through the $\mathbf{H}_6$-orbit of a given $y \in \mathbf{P}^5$. These cubics (or a specific subspace of these cubics) will also contain the unique Abelian surface passing through y.

We first need to discuss the representation theory of $\mathbf{H}_6$ acting on the vector space $H^0(\mathcal{O}_{\mathbf{P}^5}(3))$, the space of cubic forms on $\mathbf{P}^5$. There are no Heisenberg invariant cubic forms on $\mathbf{P}^5$; however if $\mathbf{H}' \subseteq \mathbf{H}_6$ is the subgroup generated by σ^2 and τ^2, then there are $\mathbf{H}'$-invariant cubic forms. We denote the space of such forms by $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$. It is easy to see that $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ has as a basis $f_0, \ldots, f_3$,

$\sigma f_0, \ldots, \sigma f_3$, where

$$f_0 = x_0^3 + x_2^3 + x_4^3, \qquad f_1 = x_1^2 x_4 + x_3^2 x_0 + x_5^2 x_2,$$
$$f_2 = x_1 x_2 x_3 + x_3 x_4 x_5 + x_5 x_0 x_1, \qquad f_3 = x_0 x_2 x_4.$$

Furthermore, $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ is a representation of $\mathbf{H}_6$, which splits up into four isomorphic representations

$$H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'} \cong \bigotimes_{i=0}^{3} \langle f_i, \sigma f_i \rangle.$$

We will identify $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ with $V_0 \otimes W$, where V_0 is a two-dimensional representation of $\mathbf{H}_6$ and W is a four dimensional complex vector space with basis e_0, e_1, e_2, e_3 so that $V_0 \otimes \langle e_i \rangle = \langle f_i, \sigma f_i \rangle$.

The importance of $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ comes from the following key lemma:

LEMMA 4.1. *Let $A \subseteq \mathbf{P}^5$ be a* (1,6)*-polarized Abelian surface which is invariant under the action of $\mathbf{H}_6^e = \mathbf{H}_6 \rtimes \langle \iota \rangle$. Then* dim $H^0(\mathcal{I}_A(3))^{\mathbf{H}'} \geqslant 4$.

Proof. This is a very similar argument to that in [HM], bottom of page 76. We consider the restriction map

$$H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'} \longrightarrow H^0(\mathcal{L}^3)^{\mathbf{H}'},$$

where $\mathcal{L}$ is the line bundle on A inducing the embedding. Let $A' = A/2K(\mathcal{L})$, and π: $A \longrightarrow A'$ the quotient map. Then $2K(\mathcal{L})$ acts on $\mathcal{L}^3$, and there exists a bundle $\mathcal{M}$ on A' such that $\mathcal{L}^3 = \pi^*(\mathcal{M})$. Furthermore, $H^0(\mathcal{L}^3)^{\mathbf{H}'} = H^0(\mathcal{M})$. Now $c_1(\mathcal{M})^2 = c_1(\mathcal{L}^3)^2/\deg(\pi) = 12$, so $\mathcal{M}$ is a line bundle inducing a polarization of type (1,6) on A'. Thus ι acts on $H^0(\mathcal{M}) = \mathbf{C}^6$ in the usual way so that it has two eigenspaces, one of dimension four and one of dimension two. Now $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ is ι-invariant, so that it must map to one of these two eigenspaces. Hence the kernel of the above restriction map is at least four dimensional. $\square$

Remark 4.2. Note that the Riemann-Roch theorem tells us that we should only expect $H^0(\mathcal{I}_A(3))$ to be two dimensional. Thus $A \subset \mathbf{P}^5$ is not cubically normal.

Eventually we will show that for a general Abelian surface $A \subseteq \mathbf{P}^5$, dim $H^0(\mathcal{I}_A(3))^{\mathbf{H}'} = 4$. This will allow us to define a rational map Θ_6: $\mathcal{A}_6^{\mathrm{lev}} \dashrightarrow \mathrm{Gr}(2, W)$ by taking A to the two-dimensional subspace $V \subseteq W$ such that $V_0 \otimes V = H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$. (See Definition 4.6 for the precise construction.)

DEFINITION 4.3. Define a rational map ϕ: $\mathbf{P}^5 \dashrightarrow \mathrm{Gr}(2, W)$ by taking a point $y \in \mathbf{P}^5$ to a subspace $V \subseteq W$ such that $V_0 \otimes V$ is the largest $\mathbf{H}_6$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ vanishing at y.

Equivalently, we first identify $\mathrm{Gr}(2, W)$ with $\mathrm{Gr}(2, W^\vee)$ by identifying a subspace $V \subseteq W$ with its annihilator $V^0 \subseteq W^\vee$, and then we can define the map ϕ:

$\mathbf{P}^5 \dashrightarrow \mathrm{Gr}(2, W^\vee)$ in coordinates, using the dual basis to $e_0, \ldots, e_3$, as follows: a point $y \in \mathbf{P}^5$ is mapped to the subspace spanned by $(f_0(y), \ldots, f_3(y))$ and $((\sigma f_0)(y), \ldots, (\sigma f_3)(y)) \in W^\vee$.

We note also that $V_0 \otimes V$ is the subspace of $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ vanishing on the $\mathbf{H}_6$-orbit of $y \in \mathbf{P}^5$.

We will use Plücker coordinates for $\mathrm{Gr}(2, W^\vee)$: the coordinates for a 2-plane spanned by $(x_0, \ldots, x_3)$ and $(y_0, \ldots, y_3)$ will be $p_{ij} = x_i y_j - x_j y_i$. These satisfy the Plücker relation $p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23} = 0$, which gives the equation of the Plücker embedding $\mathrm{Gr}(2, W^\vee) \subseteq \mathbf{P}^5$. Thus, more explicitly, ϕ is defined in Plücker coordinates by

$$y \in \mathbf{P}^5 \mapsto (f_0(y)(\sigma f_1)(y) - f_1(y)(\sigma f_0)(y), \ldots, f_2(y)(\sigma f_3)(y) - f_3(y)(\sigma f_2)(y)) \in \mathrm{Gr}(2, W^\vee).$$

Remark 4.4. Another way to obtain the $\mathbf{H}_6$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ vanishing at a general point $y \in \mathbf{P}^5$ is to take the four cubics $\det(M_3(x, y))$, $\det(M_3(\sigma^3(x), y))$, $\det(M_3(\tau^3(x), y))$, and $\det(M_3(\sigma^3\tau^3(x), y))$; see Section 1 and [GP1] for their explicit form. As observed in [GP1], Remark 2.13, these four cubics do not always span a four-dimensional space, e.g. when y is contained in an elliptic normal curve.

LEMMA 4.5. *Let $A \subseteq \mathbf{P}^5$ be a general $\mathbf{H}_6$-invariant Abelian surface. Then the map ϕ in Definition 4.3 is defined at the general point of A,* $\dim H^0(\mathcal{I}_A(3))^{\mathbf{H}'} = 4$, *and the cubics in $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ cut out a scheme of dimension $\leqslant 2$.*

Proof. First, let $X(\Gamma_6) \subseteq \mathbf{P}^5$ be the hexagon of [GP1], § 3. Then ϕ is defined on an open subset of $\mathrm{Sec}(X(\Gamma_6)) = X(\partial C(6, 4))$, with notation as in [GP1]. Indeed, ϕ is defined for instance at the point $(1{:}\,1{:}\,2{:}\,1{:}\,0{:}\,0) \in \mathrm{Sec}(X(\Gamma_6))$, as one checks by explicit evaluation. Since $X(\Gamma_6) \subset \mathbf{P}^5$ is a degeneration of a general $\mathbf{H}_6$-invariant elliptic normal curve, ϕ must be defined on a non-empty open subset of the secant variety of a general such curve. Thus ϕ is defined generically on a general translation scroll, and thus also on the general (1,6)-polarized Abelian surface by [GP1], Theorem 3.1.

Let $A \subseteq \mathbf{P}^5$ be such an Abelian surface. Let $y \in A$ be a general point of the Abelian surface, let $V \subseteq W^\vee$ be the subspace corresponding to $\phi(y)$, and let V^0 be its annihilator. Then $V_0 \otimes V^0$ is a four-dimensional space of cubics whose zero-locus contains the orbit of y under $\mathbf{H}_6$, and no other cubic of $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ vanishes along this orbit. Hence, $V_0 \otimes V^0 \supseteq H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$. On the other hand, $\dim H^0(\mathcal{I}_A(3))^{\mathbf{H}'} \geqslant 4$, by Lemma 4.1, so equality holds.

Finally, suppose that $A \subset \mathbf{P}^5$ is a general translation scroll, so that $A \subseteq \mathrm{Sec}(E)$ for some nonsingular elliptic normal curve $E \subset \mathbf{P}^5$. As before, $\dim H^0(\mathcal{I}_A(3))^{\mathbf{H}'} = 4$, and by [GP1], Theorem 5.2, two of these cubics cut out $\mathrm{Sec}(E)$, which is irreducible. It follows that the cubics of $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ must cut out a surface, and the same holds for the general (1,6)-polarized Abelian surface, again by the degeneration argument. □

DEFINITION 4.6. Let $U \subseteq \mathbf{P}^5$ be the largest open set on which ϕ is defined. We define a rational map $\Theta_6 : \mathcal{A}_6^{\mathrm{lev}} \dashrightarrow \mathrm{Gr}(2, W^\vee)$ by sending the general Heisenberg invariant (1,6)-polarized Abelian surface $A \subseteq \mathbf{P}^5$ to $\phi(A \cap U)$. We note that $\phi(A \cap U)$ is the single point in $\mathrm{Gr}(2, W^\vee)$ whose annihilator corresponds to $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$.

Now in the diagram

$$\begin{tikzcd} & \mathcal{X} \subseteq \mathbf{P}^5 \times \mathrm{Gr}(2, W^\vee) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbf{P}^5 & & \mathrm{Gr}(2, W^\vee) \end{tikzcd}$$

let $\mathcal{X}$ be the closure of the graph of ϕ, so that if π_1 and π_2 are the projections to $\mathbf{P}^5$ and $\mathrm{Gr}(2, W^\vee)$, respectively, then $\pi_1 : \mathcal{X} \longrightarrow \mathbf{P}^5$ is birational and $\pi_2 = \phi \circ \pi_1$ on the open set of $\mathcal{X}$ where $\phi \circ \pi_1$ is defined.

We now define another subvariety $\mathcal{C} \subseteq \mathbf{P}^5 \times \mathrm{Gr}(2, W^\vee)$ as follows: If $V \subseteq W^\vee$ is a two-dimensional subspace, and $V^0 \subseteq W$ is its annihilator, then $V_0 \otimes V^0$ is a subrepresentation of $V_0 \otimes W$, which, by the above discussion, may be identified with a four-dimensional space of cubics. Let $\mathcal{C} \subseteq \mathbf{P}^5 \times \mathrm{Gr}(2, W^\vee)$ be the universal family defined by these cubics: i.e., if $V \in \mathrm{Gr}(2, W^\vee)$, then $\mathcal{C}_V$ is the scheme of zeros in $\mathbf{P}^5$ of the ideal generated by the cubics in $V_0 \otimes V^0$.

Finally, let $Q \subseteq \mathrm{Gr}(2, W^\vee)$ be the quadric defined by the hyperplane section $\{p_{03} + p_{12} = 0\}$ in the Plücker embedding.

Note first that $\mathcal{X} \subseteq \mathcal{C}$. Indeed, if $y \in \mathbf{P}^5$ is a point where ϕ is defined, and $\phi(y) = V \subset W^\vee$, then the cubics in $V_0 \otimes V^0$ necessarily vanish at y.

We will now show the following

THEOREM 4.7. (1) $\pi_2(\mathcal{X}) = Q$, *the nonsingular hyperplane section of the Plücker embedding of* $\mathrm{Gr}(2, W^\vee)$ *given by* $\{p_{03} + p_{12} = 0\}$.

(2) $\Theta_6 : \mathcal{A}_6^{\mathrm{lev}} \dashrightarrow Q$ *is a birational map, and* $\pi_2 : \mathcal{X} \longrightarrow Q$ *is birational to a twist of the universal family over* $\mathcal{A}_6^{\mathrm{lev}}$. *In particular,* $\mathcal{A}_6^{\mathrm{lev}}$ *is a rational threefold.*

Proof. (1) First note, by direct calculation, that

$$f_0(\sigma f_3) - f_3(\sigma f_0) + f_1(\sigma f_2) - f_2(\sigma f_1) = 0,$$

which means exactly $\pi_2(\mathcal{X}) \subseteq Q$. To show that $\pi_2(\mathcal{X}) = Q$, we will find a fibre of π_2: $\mathcal{X} \longrightarrow Q$ which is non-empty and has dimension $\leqslant 2$. Then since $\dim(\mathcal{X}) = 5$ and $\dim(Q) = 3$, π_2 must be surjective, and have generic fibre dimension 2.

To show that $\pi_{2|\mathcal{X}}$ has nonempty fibres of dimension 2, let $A \subseteq \mathbf{P}^5$ be a general Heisenberg invariant (1,6)-polarized Abelian surface. As observed in Definition 4.6, $\phi(A \cap U)$ consists of exactly one point, say, $V \in \mathrm{Gr}(2, W^\vee)$. If $\tilde{A} \subseteq \mathcal{X}$ is the proper transform of A in $\mathcal{X}$, then $\pi_2(\tilde{A}) = V$. Furthermore, $\mathcal{X}_V \subseteq \mathcal{C}_V$ and by Lemma 4.5,

$V_0 \otimes V^0 = H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ and $\mathcal{C}_V$ is a scheme of dimension $\leqslant 2$. Hence $\mathcal{X}_V$ is nonempty of dimension 2.

(2) To prove the first part, it is enough to show that for general $V \in Q$, the fibre $\mathcal{C}_V$ of $\pi_2\colon \mathcal{C} \longrightarrow Q$ is a scheme whose two-dimensional component is of degree 12. Then there cannot be two distinct Abelian surfaces $A, A' \subseteq \mathcal{C}_V$, and so Θ_6 must be generically 1 to 1.

To check this claim about $\mathcal{C}_V$, we choose one specific $V \in Q$ spanned by (1,0,0,0) and (0,–1,1,0) in $W^\vee$, so that $V^0 \subseteq W$ is spanned by (0,1,1,0) and (0,0,0,1). Then $\mathcal{C}_V$ is the subscheme of $\mathbf{P}^5$ defined by the equations

$$\{f_1 + f_2 = f_3 = \sigma f_1 + \sigma f_2 = \sigma f_3 = 0\},$$

or explicitly

$$\begin{aligned} x_0x_2x_4 = x_1x_3x_5 &= x_1^2x_4 + x_3^2x_0 + x_5^2x_2 + x_1x_2x_3 + x_3x_4x_5 + x_5x_0x_1 \\ &= x_1x_4^2 + x_3x_0^2 + x_5x_2^2 + x_2x_3x_4 + x_4x_5x_0 + x_0x_1x_2 = 0. \end{aligned}$$

The first two equations yield the degree 9 (Stanley–Reisner) threefold $X(\partial C(6,4))$ (see [GP1], Proposition 4.1) consisting of the union of linear subspaces L_{ij}, $i \in \{0,2,4\}$, $j \in \{1,3,5\}$, with $L_{ij} = \{x_i = x_j = 0\}$ If $i \not\equiv j+3 \mod 6$, then $L_{ij} \cap \mathcal{C}_V$ is given by the equations

$$\begin{aligned} \{x_i = x_j &= x_k^2x_{k+3} + x_{k-2}x_{k-1}x_k + x_kx_{k+1}x_{k+2} \\ &= x_kx_{k+3}^2 + x_{k+1}x_{k+2}x_{k+3} + x_{k-3}x_{k-2}x_{k-1} = 0\}, \end{aligned}$$

where $2k-3 = i+j \mod 6$, and $k \in \{1,3,5\}$. This is a quadric surface

$$\{x_i = x_j = x_kx_{k+3} + x_{k-2}x_{k-1} = 0\}$$

or

$$\{x_i = x_j = x_kx_{k+3} + x_kx_{k+1} = 0\},$$

together with a line, depending on the value of i and j. If $i \equiv j+3 \mod 6$, then $L_{ij} \cap \mathcal{C}_V$ is given by the equations

$$x_i = x_j = \sum_{\substack{k\in\{0,2,4\}\\ k\neq i}} x_k^2x_{k+3} = \sum_{\substack{k\in\{0,2,4\}\\ k\neq i}} x_kx_{k+3}^2 = 0,$$

which is easily seen to be a curve. Thus $\mathcal{C}_V$ is a union of 6 quadric surfaces and a number of (possibly embedded) curves. This verifies the claim, and thus by the above discussion the general fibre of $\pi_2\colon \mathcal{X} \longrightarrow Q$ is a (1,6)-polarized Heisenberg invariant Abelian surface. If $V = \Theta_6(A)$, then $\mathcal{X}_V = A$. This shows the last part of (2) in the theorem. □

Remark 4.8. (1) A more careful analysis of the specific set of cubics studied in the proof of Theorem 4.7, (2) shows that in fact the scheme $\mathcal{C}_V$ they define

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consists of the union of the six quadric surfaces $S, \sigma(S), \ldots, \sigma^5(S)$, where

$$S = \{x_0 = x_1 = x_2x_5 + x_3x_4 = 0\} \subset \mathbf{P}^5,$$

and the $\mathbf{H}_6$-orbit of the involution $\iota(-1)$-eigenspace $\mathbf{P}^1_-$. This orbit consists of exactly nine lines. Conversely, if A is a $\mathbf{H}_6^e$-invariant (1,6) Abelian surface, $\mathbf{P}^1_- \cap A$ consists of four points, namely the odd two-torsion points of A, and so $\mathbf{P}^1_-$ (and all its Heisenberg translates) are contained in any cubic hypersurface containing $A \subset \mathbf{P}^5$. This shows that for a general (1,6)-polarized Abelian surface $A \subset \mathbf{P}^5$, the cubics containing it cut out the union of A with the $\mathbf{H}_6$-orbit of $\mathbf{P}^1_-$.

(2) Furthermore, one may directly calculate the ideal of $\bigcup_{i=0}^5 \sigma^i(S)$, and one finds that the ideal is generated by the four cubics $f_1 + f_2$, $\sigma f_1 + \sigma f_2$, f_3, σf_3, and six additional quartics. However, we have seen that for an arbitrary (1,6)-polarized Abelian surface A, $\dim H^0(\mathcal{I}_A(3)) \geqslant 4$. Thus we see that for the general such A, $\dim H^0(\mathcal{I}_A(3)) = 4$, and that $\mathcal{I}_A$ is generated by quartics.

We next study the complete intersections of type (3,3) arising in this construction. They turn out to be partial smoothings of the degenerate 'Calabi–Yau' threefolds described in Proposition 1.3 and Proposition 1.4.

DEFINITION 4.9. For a point $p \in \mathbf{P}(W)$ corresponding to a one-dimensional subspace $T \subseteq W$, let $V_{6,p} \subseteq \mathbf{P}^5$ be the complete intersection of type (3,3) determined by the cubic hypersurfaces in $V_0 \otimes T \subset H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$. Notice that, by construction, $V_{6,p} \subseteq \mathbf{P}^5$ is $\mathbf{H}_6$-invariant.

THEOREM 4.10. *For general $p \in \mathbf{P}(W)$ corresponding to $T \subseteq W$ we have*

(1) *$V_{6,p}$ contains a pencil of (1,6)-polarized Abelian surfaces, parametrized by a pencil of two-dimensional subspaces contained in the annihilator of T.*
(2) *$V_{6,p}$ is an irreducible threefold whose singular locus consists of 72 ordinary double points. These 72 ordinary double points are the base locus of the pencil in (1).*
(3) *There is a small resolution $V^1_{6,p} \longrightarrow V_{6,p}$ of the ordinary double points, with $V^1_{6,p}$ a Calabi–Yau threefold, and such that there is a map $\pi_1: V^1_{6,p} \longrightarrow \mathbf{P}^1$ whose fibres form the pencil of Abelian surfaces of part (1).*
(4) *$\chi(V^1_{6,p}) = 0$ and $h^{1,1}(V^1_{6,p}) = h^{1,2}(V^1_{6,p}) = 6$.*

Proof. (1) The choice of $V_{6,p}$ is given by $p \in \mathbf{P}(W)$, with $p = \mathbf{P}(T)$, while the set of (1,6)-polarized Abelian surfaces contained in $V_{6,p}$ is parametrized by $V \in Q \subseteq \mathrm{Gr}(2, W^\vee)$ such that $T \subseteq V^0$. Now as is well known, the lines in a nonsingular hyperplane section of $\mathrm{Gr}(2, W^\vee)$ correspond to pencils of lines in $\mathbf{P}(W)$, and each point of $\mathbf{P}(W)$ is the center of exactly one of these pencils. Hence, for $p \in \mathbf{P}(W)$, with $p = \mathbf{P}(T)$, the set $\{V \in Q \mid V^0 \supseteq T\}$ is a line in Q. Each such V gives rise to an Abelian surface in $V_{6,p}$, and hence we obtain a pencil of Abelian surfaces in $V_{6,p}$.

(2) That $V_{6,p}$ is irreducible for general $p \in \mathbf{P}(W)$ follows, for example, from the fact that for a Heisenberg invariant elliptic normal curve $E \subseteq \mathbf{P}^5$, $\mathrm{Sec}(E)$ is of the form $V_{6,p}$ for some p. This is part of [GP1], Remark 2.13 and Theorem 5.2.

To prove that in general $V_{6,p}$ has only 72 ordinary double points seems difficult without resorting to computational means. First note that if f_1 and f_2 are two cubic hypersurfaces in $\mathbf{P}^5$ containing a (1,6)-polarized Abelian surface A, and defining a threefold $X = \{f_1 = f_2 = 0\}$, then f_1 and f_2 induce sections of $(\mathcal{I}_A/\mathcal{I}_A^2)(3)$, which are linearly dependent precisely on the set $\mathrm{Sing}(X) \cap A$. Now $c_2((\mathcal{I}_A/\mathcal{I}_A^2)(3)) = 72$, so we would expect to have exactly 72 singularities of X on A, counted with multiplicities. On the other hand, 72 ordinary double points can be easily accounted for. Namely, in the pencil $V_0 \otimes T \subset H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ defining $V_{6,p}$, there are exactly 4 cubic hypersurfaces which are determinants of 3×3-matrices $M_3(x, y)$, for certain values of the parameter y. Such a cubic hypersurface is singular along the locus where its 2×2-minors vanish, which is an elliptic normal curve in $\mathbf{P}^5$. In particular, $V_{6,p}$ has 18 ordinary double points at the points of intersection of such an elliptic curve with (any) other cubic in the pencil $V_0 \otimes T$. Using *Macaulay*/*Macaulay2*, one can easily find examples of threefolds $V_{6,p}$ which do have precisely 72 ordinary double points. Thus $V_{6,p}$ for a general $p \in \mathbf{P}(W)$ possesses exactly 72 ordinary double points.

Now, blowing up a smooth Abelian surface $A \subseteq V_{6,p}$ in the pencil, we obtain a small resolution $V^2_{6,p} \longrightarrow V_{6,p}$. Flopping simultaneously the 72 exceptional $\mathbf{P}^1$'s gives a small resolution $V^1_{6,p} \longrightarrow V_{6,p}$ which is a Calabi–Yau threefold containing a pencil of minimal Abelian surfaces. Thus by Lemma 1.2, $V^1_{6,p}$ has an Abelian surface fibration. In particular, the base points of the pencil of Abelian surfaces on $V_{6,p}$ consist precisely of the 72 nodes. This completes the proof of (2) and (3).

For (4), the Euler characteristic computation follows immediately from the fact that the Euler characteristic of a nonsingular (3, 3) complete intersection in $\mathbf{P}^5$ is -144. See Remark 4.11 below for the Hodge numbers. □

Remark 4.11. We sketch here a method of computing the Hodge numbers of $V^1_{6,p}$. An effective method of computing the Hodge numbers of nodal quintics in $\mathbf{P}^4$ is well known (see [Scho] and [We]); we essentially generalize this to nodal type (3,3) and (2,2,2,2) complete intersections in $\mathbf{P}^5$ and $\mathbf{P}^7$, respectively.

Let X be a complete intersection Calabi–Yau threefold in $\mathbf{P}^{n+3}$ with only ordinary double points as singularities, and assume that X is given by equations $f_1 = \cdots = f_n = 0$, with all equations being of the same degree d. We assume also that there exists a projective small resolution $\tilde{X} \longrightarrow X$, for instance obtained by blowing up a smooth Weil divisor passing through all the nodes, which is the case for $V_{6,p}$ in Theorem 4.10.

Let S be the homogeneous coordinate ring of X, that is $S \cong \mathbf{C}[x_0, \dots, x_{n+3}]/(f_1, \dots, f_n)$. Let $T^1 \cong \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ be the tangent space to the deformation space of X. It is easy to see that there is a natural isomorphism

$$T^1 \cong \left(\frac{S^n}{\{(\partial f_1/\partial x_i, \ldots, \partial f_n/\partial x_i) \mid 0 \leqslant i \leqslant n+3\}}\right)_d,$$

where the subscript d refers to the degree d homogeneous part. Let T^1_{loc} be the tangent space of the deformation space of the germ of the singular locus of X. There is a natural map $T^1 \longrightarrow T^1_{\mathrm{loc}}$ and we wish to compute the dimension of the kernel of this map, which is the tangent space of the deformation space of $\tilde{X}$. Thus, we need to know when $(g_1, \ldots, g_n) \in S^n$ representing an element of T^1 yields a trivial deformation on the ordinary double points. This is precisely when the matrix

$$\begin{pmatrix} g_1 & \partial f_1/\partial x_0 & & \partial f_1/\partial x_{n+3} \\ \vdots & \vdots & \cdots & \vdots \\ g_n & \partial f_n/\partial x_0 & & \partial f_n/\partial x_{n+3} \end{pmatrix}$$

has rank $< n$ at the singular points of X. Thus the $n \times n$ minors of this matrix must be contained in the ideal of the singular locus of X. This is a calculation which can be easily performed using *Macaulay*/*Macaulay2*. One thus determines the dimension of the kernel of $T^1 \longrightarrow T^1_{\mathrm{loc}}$, and this dimension is precisely $h^{1,2}(\tilde{X})$. From the knowledge of the topological Euler characteristic of $\tilde{X}$, we can then compute the Picard number of $\tilde{X}$.

We also note that even if $X \subseteq \mathbf{P}^m$ is not a complete intersection, but the homogeneous ideal of X is generated by elements $f_1, \ldots, f_n$ all of the same degree d, then T^1 can be computed as a subspace of

$$\left(\frac{S^n}{\{(\partial f_1/\partial x_i, \ldots, \partial f_n/\partial x_i) \mid 0 \leqslant i \leqslant m\}}\right)_d.$$

See, for example, [Tei]. This makes it possible to compute Hodge numbers even in this case. We omit the details, but will make use of this on occasion to compute Hodge numbers of other Calabi–Yau threefolds.

Remark 4.12. Because the Picard number of $V^1_{6,p}$ is fairly large, it would be difficult to understand all details of the geometry of $V_{6,p}$! However, it is already interesting to identify several different minimal models, by focusing only on the subgroup of $\mathrm{Pic}(V^1_{6,p})$ generated by H, the pull-back of a hyperplane in $\mathbf{P}^5$, and A, the class of a (1,6)-polarized Abelian surface, that is a fibre of the Abelian surface fibration $\pi_1\colon V^1_{6,p} \longrightarrow \mathbf{P}^1$. There are several curves of interest in $V^1_{6,p}$ also. Let $[e]$ be the class of one of the 72 exceptional lines. (Caution: these lines are not all in the same homology class, but their homology classes cannot be distinguished using intersection numbers with only H and A.) By Remark 4.8 we have $\mathbf{P}^1_- \subset V_{6,p}$; let $[l]$ be the class of $\mathbf{P}^1_- \subset V_{6,p}$. It is not difficult to check that, for general p, $\mathbf{P}^1_-$ is disjoint from the singular locus of $V_{6,p}$, and hence the proper transform of $\mathbf{P}^1_-$ is disjoint from all of the exceptional curves. With these preparations, in $V^1_{6,p}$, we find the following intersection numbers:

$$H^3 = 9, \qquad H^2A = 12, \qquad A^2 = 0,$$
$$H \cdot e = 0, \qquad H \cdot l = 1, \qquad A \cdot e = 1, \qquad A \cdot l = 4.$$

Here $A \cdot l = 4$ since, by Remark 4.8, $\mathbf{P}^1_-$ intersects $A \subseteq \mathbf{P}^5$ in the four odd two-torsion points of A.

As in the proof of Theorem 4.10, by flopping simultaneously the 72 exceptional curves we obtain the model $V^2_{6,p}$. There is still a map $V^2_{6,p} \longrightarrow V_{6,p}$, which is obtained by blowing up a general $A \subseteq V_{6,p}$. In $V^2_{6,p}$, we now have the intersection numbers

$$H^3 = 9, \qquad H^2A = 12, \qquad HA^2 = 0, \qquad A^3 = -72,$$
$$H \cdot e = 0, \qquad H \cdot l = 1, \qquad A \cdot e = -1, \qquad A \cdot l = 4.$$

Now, since the ideal of a general Abelian surface A is generated by cubics and quartics, the linear system $|4H - A|$ is base-point free on $V^2_{6,p}$. However, $(4H - A) \cdot l = 0$, and hence $|4H - A|$ induces a contraction $V^2_{6,p} \longrightarrow \overline{V^2_{6,p}}$, contracting, by Remark 4.8, (2), only l and its Heisenberg translates, thus a total of 9 lines. Furthermore, the normal bundle of l can be easily computed to be $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. These 9 lines can be simultaneously flopped to obtain a model $V^3_{6,p}$. In this model,

$$H^3 = 0, \qquad H^2A = -24, \qquad HA^2 = -144, \qquad A^3 = -648,$$
$$H \cdot e = 0, \qquad H \cdot l = -1, \qquad A \cdot e = -1, \qquad A \cdot l = -4.$$

Now in $V^2_{6,p}$, the linear system $|3H - A|$ had the union of l and its Heisenberg translates as base locus, by Remark 4.8. Thus in $V^3_{6,p}$, $|3H - A|$ is base-point free, and $\dim |3H - A| \geqslant 1$. Hence two distinct elements of $|3H - A|$ are in fact disjoint, and thus $|3H - A|$ gives rise to a fibration $\pi_2 \colon V^3_{6,p} \longrightarrow \mathbf{P}^1$ whose (general) fibres are then nonsingular surfaces. A calculation shows that $(3H - A) \cdot c_2(V^3_{6,p}) = 0$ and, hence, π_2 is an Abelian surface fibration. Let A' be the proper transform in $V_{6,p} \subseteq \mathbf{P}^5$ of a nonsingular fibre of π_2. Then $A' \subseteq \mathbf{P}^5$ is a nonminimal Abelian surface, containing nine exceptional lines, precisely $\mathbf{P}^1_-$ and its Heisenberg translates. In addition, we compute from the above tables that $\deg A' = 15$. Thus the linear system embedding A' in $\mathbf{P}^5$ is $|L - \sum_{i=1}^9 E_i|$, where $E_1 \ldots, E_9$ are the exceptional curves and L is a polarization on the minimal model $A'_{\min}$ either of type (1,12) or (2,6).

To see that this polarization is in fact of type (2,6), first note by inspection that all cubics in $H^0(\mathcal{O}_{\mathbf{P}^5}(3))^{\mathbf{H}'}$ are in fact invariant under the Heisenberg involution ι. Thus ι acts on the Abelian surface A'. The fixed points of this action are exactly $A' \cap \mathbf{P}^3_+$ and $A' \cap \mathbf{P}^1_- = \mathbf{P}^1_-$. Intersection theory shows that $\# A' \cap \mathbf{P}^3_+ = 15$. On the other hand ι descends to the involution $x \mapsto -x$ on the minimal model $A'_{\min}$ of A'. In particular, if H is a hyperplane in $\mathbf{P}^5$ containing $\mathbf{P}^3_+$, then $H \cap A'$ descends to a symmetric divisor D on $A'_{\min}$ vanishing with multiplicity 1 at all 16 two-torsion points of $A'_{\min}$. Therefore by [LB], Proposition 4.7.5, L must be of type (2,6) rather than (1,12).

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Remark 4.13. The general cubic hypersurface containing a (1,6)-polarized Abelian surface is smooth. An interesting question is whether it is rational. See also [Has] for a detailed discussion of such rationality issues.

5. Moduli of (1,7)-Polarized Abelian Surfaces

The rationality of the moduli space $\mathcal{A}_7^{\mathrm{lev}}$ of (1,7)-polarized Abelian surfaces was proved by Manolache and Schreyer [MS], who show that this moduli space is birational to a special Fano threefold of type V_{22} of index 1 and genus 12. Their approach is based on a detailed description of the minimal free resolution of the ideal sheaf of a (1,7)-polarized Abelian surface in $\mathbf{P}^6$.

We will give in the following a somewhat different proof of their result, by using the fact that the general Abelian surface A with a (1,7) polarization is determined by the Pfaffian cubics containing it. This will focus attention on certain Pfaffian Calabi–Yau threefold containing a pencil of (1,7)-polarized Abelian surfaces which were first discovered by Aure and Ranestad (unpublished). We will also show in [GP3] the existence on such Pfaffian Calabi–Yau threefolds of a second pencil of (1,14)-polarized Abelian surfaces.

Let $N(\mathbf{H}_7)$ denote the normalizer of the Heisenberg group $\mathbf{H}_7$ inside $\mathrm{SL}(V)$, where as in Section 1 the inclusion $\mathbf{H}_7 \hookrightarrow \mathrm{SL}(V)$ is via the Schrödinger representation. We have a sequence of inclusions

$$\langle I\rangle \subseteq Z(\mathbf{H}_7) = \mu_7 \subseteq \mathbf{H}_7 \subseteq N(\mathbf{H}_7),$$

and as is well-known, $N(\mathbf{H}_7)/\mathbf{H}_7 \cong \mathrm{SL}_2(\mathbf{Z}_7)$, and in fact $N(\mathbf{H}_7)$ is a semi-direct product $\mathbf{H}_7 \rtimes \mathrm{SL}_2(\mathbf{Z}_7)$ (see [HM], §1 for an identical discussion for the group $\mathbf{H}_5$). Therefore the Schrödinger representation of $\mathbf{H}_7$ induces a seven-dimensional representation ρ_7: $\mathrm{SL}_2(\mathbf{Z}_7) \longrightarrow \mathrm{SL}(V)$. In terms of generators and relations (see for example [BeMe]), one has

$$\mathrm{PSL}_2(\mathbf{Z}_7) = \langle S, T \mid S^7 = 1, (ST)^3 = T^2 = 1, (S^2TS^4T)^3 = 1\rangle,$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

while the representation ρ_7 is given projectively by

$$\rho_7(S) = (\xi^{\frac{-ij}{2}}\delta_{ij})_{i,j\in\mathbf{Z}_7}, \qquad \rho_7(T) = \frac{1}{\sqrt{-7}}(\xi^{ij})_{i,j\in\mathbf{Z}_7},$$

see [Ta] and [Si] for details. Here ξ is a fixed primitive 7th root of unity.

The center of $\mathrm{SL}_2(\mathbf{Z}_7)$ is generated by T^2, and $\rho_7(T^2) = -\iota$. Thus the representation ρ_7 is reducible. In fact, if V_+ and V_- denote the positive and negative eigenspaces, respectively, of the Heisenberg involution ι acting on V, then V_+

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and V_- are both invariant under ρ_7, and ρ_7 splits as $\rho_+ \oplus \rho_-$, where $\rho_\pm$ is the representation of $\mathrm{SL}_2(\mathbf{Z}_7)$ acting on $V_\pm$. Note that ρ_- is trivial on the center of $\mathrm{SL}_2(\mathbf{Z}_7)$, so in fact it descends to give an irreducible representation ρ_-: $\mathrm{PSL}_2(\mathbf{Z}_7) \longrightarrow \mathrm{GL}(V_-)$.

For the reader's convenience, we reproduce from the *Atlas of Finite Groups* [CNPW] the character table for $\mathrm{PSL}_2(\mathbf{Z}_7)$:

Size of conjugacy class representative	1 I	21 S	42 $T^2ST^{-2}ST^2$	56 ST	24 T	24 T^{-1}
Characters						
χ_1	1	1	1	1	1	1
χ_2	3	-1	1	0	β	$\bar{\beta}$
χ_3	3	-1	1	0	$\bar{\beta}$	β
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	-1	1	0	0
χ_6	8	0	0	-1	1	1

where $\beta = \frac{1}{2}(-1+\sqrt{-7})$. We will denote in the sequel the corresponding representations by their characters. The representation ρ_- is irreducible and has character χ_2. The polynomial invariants of this representation are classical and they have first been determined by F. Klein, see [Kl1, Kl2]. It turns out that there are no invariants of degree <4, and the quartic

$$f_4 = x_1^3 x_2 - x_2^3 x_3 - x_3^3 x_1,$$

is the unique invariant in degree 4. The smooth quartic curve defined by this invariant $\overline{X(7)} = \{f_4 = 0\} \subset \mathbf{P}^2_-$ is in fact an isomorphic image of the modular curve of level 7, and has $\mathrm{PSL}_2(\mathbf{Z}_7)$ as its full automorphism group. (See for instance [GP1], [Kl1], and [Ve] for details.) The other primary invariants of this representation are a sextic f_6, which is the determinant of the Hessian matrix of f_4, and f_{14} a polynomial of degree fourteen which is obtained as the determinant of a bordered Hessian of f_4.

We will also need a number basic facts concerning apolarity and polars of hypersurfaces, and we recall them briefly here.

Let V be an n-dimensional vector space, and fix a basis for it. For a point a with coordinates $a = (a_1, \ldots, a_n) \in V$, and a homogeneous polynomial $F \in S^d(V^\vee)$ of degree d, one defines

$$P_a(F) := \frac{1}{d} \sum_{i=1}^{n} a_i \frac{\partial F}{\partial x_i},$$

where $x_1, \ldots, x_n \in V^\vee$ form the dual of the chosen basis. It is easy to see that the previous definition is independent of the choice of basis, and so if we further set

$$P_{a_1 \ldots a_k}(F) := P_{a_1}(P_{a_2}(\ldots P_{a_k}(F))\ldots),$$

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and then extend by linearity $P_\Phi(F)$ to all $\Phi \in S^k(V)$, we have defined a pairing between $S(V^\vee)$ and $S(V)$, called the *apolarity* pairing (see [DK] for a modern account and for details). The resulting form $P_\Phi(F)$ is called the *polar of F with respect to* Φ. Two homogeneous forms Φ and F are called *apolar* if $P_\Phi(F) = 0 = P_F(\Phi)$ (cf. [Sal] who says that the term was coined by Reye). One says that $\Phi \in S^{d-k}(V)$ is a k^{th} *antipolar* of a hyperplane H with respect to F if $P_\Phi(F) = H^k$.

Finally, if $l_1, \dots, l_s \in V^\vee$ are linear forms such that $F = l_1^d + \cdots + l_s^d$, then $P_\Phi(F) = 0$ for all $\Phi \in I_\Gamma \subset S(V)$, where I_Γ is the homogeneous ideal of $\Gamma = \{H_1, H_2, \dots, H_s\} \subset \mathbf{P}(V^\vee)$, the collection of hyperplanes $H = \{l_i = 0\}$. Conversely, if $P_\Phi(F) = 0$ for all $\Phi \in I_\Gamma$, with $\Gamma = \{H_1, H_2, \dots, H_s\}$ a collection of points in the dual space, then F is a sum of powers $F = l_1^d + \cdots + l_s^d$, for suitably rescaled linear forms. One says that $l_1, \dots, l_s$ (or more precisely that the corresponding points $H_1, H_2, \dots, H_s$ in $\mathbf{P}(V^\vee)$) form a *polar s-polyhedron* to F if $F = \lambda_1 l_1^d + \cdots + \lambda_s l_s^d$, for suitable scalars λ_i (see [DK] and [RS] for modern accounts of apolarity).

We start by looking at the equations of an Abelian surface in $\mathbf{P}^6$:

PROPOSITION 5.1. *Let* $A \subseteq \mathbf{P}^6$ *be a* (1,7)-*polarized Abelian surface. Then* $h^0(\mathcal{I}_A(2)) = 0$, $h^0(\mathcal{I}_A(3)) = 21$, *and the ideal of A is generated by these* 21 *cubics.*

Proof. The first claim is proved by a direct argument in [MS], Lemma 2.3. Alternatively, [La] or [MS], Lemma 2.4, show that a (1,7)-polarized Abelian surface $A \subseteq \mathbf{P}^6$ is projectively normal, and hence Riemann–Roch gives $h^0(\mathcal{I}_A(2)) = 0$ and $h^0(\mathcal{I}_A(3)) = 21$.

Projective normality, together with Kodaira vanishing, imply also that $\mathcal{I}_A$ is 4-regular in the Castelnuovo–Mumford sense. The obstruction to being 3-regular is that $h^3(\mathcal{I}_A) = h^2(\mathcal{O}_A) = 1$. However, $h^i(\mathcal{I}_A(3-i)) = 0$ for $i \neq 3$, and the co-multiplication map

$$H^3(\mathcal{I}_A(-1)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^6}(1))^* \otimes H^3(\mathcal{I}_A),$$

is dual to the natural multiplication $H^0(\mathcal{O}_{\mathbf{P}^6}(1)) \otimes H^0(\mathcal{O}_A) \longrightarrow H^0(\mathcal{O}_A(1))$ and thus is an isomorphism. Therefore we may apply [EPW], Lemma 8.8, to see that $\mathcal{I}_A$ is generated by 21 cubics, and in fact to determine all Betti numbers in the minimal resolution of this ideal sheaf. An alternative however partly incomplete argument may be found in [MS], Corollary 2.4. $\square$

Recall from [GP1], Corollary 2.8 that the 7×7-matrix

$$M_7'(x, y) = \left(x_{\frac{(i+j)}{2}}\, y_{\frac{(i-j)}{2}}\right)_{i,j \in \mathbf{Z}_7}$$

has rank at most 4 on an embedded $\mathbf{H}_7$ invariant (1,7)-polarized Abelian surface in $\mathbf{P}^6$. On the other hand, for any parameter point $y = (0: y_1: y_2: y_3: -y_3: -y_2: -y_1) \in \mathbf{P}^2_-$, the matrix M_7' is alternating. We will denote in the sequel by $I_3(y) \subset \mathbf{C}[x_0, \dots, x_6]$ the homogeneous ideal generated by the 6×6-Pfaffians of the alternating matrix $M_7'(x, y)$, and by $V_{7,y} \subset \mathbf{P}^6$ the closed subscheme defined by this ideal.

PROPOSITION 5.2.

(1) *For $y \in \overline{X(7)} = \{f_4 = 0\} \subset \mathbf{P}^2_-$, the scheme $V_{7,y}$ is the secant variety of an elliptic normal curve in $\mathbf{P}^6$ (the level 7 structure elliptic curve corresponding to the point y on the modular curve $\overline{X(7)}$).*

(2) *For a general $y \in \mathbf{P}^2_-$, the scheme $V_{7,y}$ is a projectively Gorenstein irreducible threefold of degree 14 and sectional genus 15.*

Proof. If $V_{7,y}$ is of the expected codimension three, then it will be of the degree and genus stated, and will be projectively Gorenstein, as any such Pfaffian subscheme has these properties. On the other hand, [GP1], Theorem 5.4, shows that for $y \in \mathbf{P}^2_-$ the origin of a Heisenberg invariant elliptic normal curve $E \subset \mathbf{P}^6$, we have $V_{7,y} = \mathrm{Sec}(E)$. Thus for general $y \in \mathbf{P}^2_-$, $V_{7,y}$ is irreducible and of the expected codimension. □

PROPOSITION 5.3

(1) *For all $y \in \mathbf{P}^2_- \setminus \overline{X(7)}$, the scheme $V_{7,y}$ meets $\mathbf{P}^2_-$ along a conic C_y and the point y.*

(2) *The conic C_y is defined by the second polar $P_{y^2}(f_4)$ with respect to y of the Klein modular quartic curve $\overline{X(7)} = \{f_4 = 0\} \subset \mathbf{P}^2_-$. Furthermore, $C_y = C_{y'}$ if and only if $y = y' \in \mathbf{P}^2_-$, and C_y is a singular conic if and only if $y \in \mathrm{Hess}(\overline{X(7)}) = \{f_6 = 0\} \subset \mathbf{P}^2_-$.*

(3) *The point y lies on C_y if and only if $y \in \overline{X(7)}$. Moreover, if $y \in \overline{X(7)}$, then the conic C_y touches $\overline{X(7)}$ at the point y (with multiplicity 2, if y is not a flex of $\overline{X(7)}$).*

Proof. Let I' be the bihomogeneous ideal in $\mathbf{C}[x_1, x_2, x_3] \otimes \mathbf{C}[y_1, y_2, y_3]$ generated by the 6×6-Pfaffians of the alternating matrix $M_7'(x, y)$, where this time both sets of coordinates $y = (0\colon y_1\colon y_2\colon y_3\colon -y_3\colon -y_2\colon -y_1)$ and $x = (0\colon x_1\colon x_2\colon x_3\colon -x_3\colon -x_2\colon -x_1)$ are chosen in $\mathbf{P}^2_-$.

We wish to show that the ideal I' takes the form $J \cdot P_{y^2}(f_4)$, where J is the bihomogeneous ideal of the diagonal Δ in $\mathbf{P}^2_- \times \mathbf{P}^2_-$, namely

$$J\colon = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2),$$

and $P_{y^2}(f_4)$ is the second polar of the Klein quartic curve $\overline{X(7)} \subseteq \mathbf{P}^2_-$ with respect to y, namely

$$P_{y^2}(f_4) = \tfrac{1}{2}(y_1y_2x_1^2 - y_2y_3x_2^2 - y_1y_3x_3^2 + y_1^2x_1x_2 - y_2^2x_2x_3 - y_3^2x_1x_3).$$

While this can be shown by direct computation, it is easier to do so via representation theory.

One may check by inspection that the bihomogeneous ideal I' is equivariant with respect to the diagonal action of $\mathrm{PSL}_2(\mathbf{Z}_7)$ on $\mathbf{P}^2_- \times \mathbf{P}^2_-$ given by the representation ρ_- introduced at the beginning of this section. In addition, the zero locus in

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$\mathbf{P}^2_- \times \mathbf{P}^2_-$ of I' contains the diagonal Δ, since the matrix $M_7'(y, y)$ has rank at most 4 for every $y \in \mathbf{P}^2_-$. Thus $I' \subseteq J$, and so the bihomogeneous (3,3)-part $(I')_{(3,3)} \subseteq J \cdot H^0(\mathcal{O}_{\mathbf{P}^2_- \times \mathbf{P}^2_-}(2, 2))$. Now

$$\begin{aligned}
&H^0(\mathcal{O}_{\mathbf{P}^2_- \times \mathbf{P}^2_-}(1, 1)) \cong \chi_3 \oplus \chi_4,\\
&H^0(\mathcal{O}_{\mathbf{P}^2_- \times \mathbf{P}^2_-}(2, 2)) \cong \chi_1 \oplus 2\chi_4, \oplus \chi_5 \oplus 2\chi_6,\\
&H^0(\mathcal{O}_{\mathbf{P}^2_- \times \mathbf{P}^2_-}(3, 3)) \cong \chi_1 \oplus 2\chi_2 \oplus \chi_3 \oplus 5\chi_4 \oplus 4\chi_5 \oplus 4\chi_6,
\end{aligned}$$

as can be easily calculated from the given character table of $\mathrm{PSL}_2(\mathbf{Z}_7)$. In particular, as $\mathrm{PSL}_2(\mathbf{Z}_7)$-representations, $J_{(1,1)} \cong \chi_3$, and since $\dim (I')_{(3,3)} \leqslant 7$, because I' is generated by 7 Pfaffians there is then no choice but for $(I')_{(3,3)} \cong \chi_3 \otimes \chi_1 = \chi_3$. This means that $(I')_{(3,3)} = J_{(1,1)} \cdot \langle f \rangle$, where f is the unique $\mathrm{PSL}_2(\mathbf{Z}_7)$-invariant of bidegree (2,2). This invariant is in fact $P_{y^2}(f_4)$. Thus for a fixed $y \in \mathbf{P}^2_-$, the scheme $V_{7,y}$ meets $\mathbf{P}^2_-$ in the point y and the conic $C_y = \{P_{y^2}(f_4) = 0\}$. This proves part (1).

To finish the proof of (2) and (3) note that

$$P_{y^2}(f_4)(y, y) = y_1^3 y_2 - y_2^3 y_3 - y_3^3 y_1$$

which vanishes if and only if $y \in \overline{X(7)} \subset \mathbf{P}^2_-$. And in case $y \in \overline{X(7)} \subset \mathbf{P}^2_-$, the conic C_y is tangent to $\overline{X(7)}$ at the point y since

$$\begin{aligned}
T_y(C_y) &= T_y(\overline{X(7)})\\
&= \{x \in \mathbf{P}^2_- \mid (3y_1^2 y_2 - y_3^3)x_1 + (y_1^3 - 3y_2^2 y_3)x_2 - (3y_1 y_3^2 + y_2^3)x_3 = 0\}.
\end{aligned}$$

Miele shows in his thesis [Mie] that the intersection multiplicity of C_y and $\overline{X(7)}$ at y is two when $y \in \overline{X(7)}$ is not one of the flexes, and that in this case all of the other intersection points of $\overline{X(7)}$ and C_y are in fact simple.

Finally, the conic C_y is singular if and only if

$$\operatorname{rank}\left(\frac{\partial^2 f_4(y)}{\partial y_i \partial y_j}\right) < 3,$$

that is $y \in \operatorname{Hess}(\overline{X(7)}) = \{f_6 = 0\}$ by the definition of the Hessian locus. □

PROPOSITION 5.4. *Let $A \subset \mathbf{P}^6$ be a general Heisenberg invariant* (1,7)*-polarized Abelian surface, and let $A \cap \mathbf{P}^2_- = \{p_1, p_2, \ldots, p_6\}$ be the odd* 2*-torsion points of A. Then*:

(1) *The points p_i form a polar hexagon to the Klein quartic curve $\overline{X(7)}$ in the dual space $(\mathbf{P}^2_-)^*$.*

(2) *The surface A is contained in V_{7,p_i}, for all $i = 1, \ldots, 6$. Moreover,* 21 *cubic Pfaffians defining three of the six V_{7,p_i}'s generate the homogeneous ideal I_A of A.*

Proof. It follows from [GP1], Corollary 2.8, that for $y = p_i$ one of the odd 2-torsion points of A, the matrix $M_7'(x, y)$ is alternating and of rank at most 4 along the surface A, hence A is contained in the Pfaffian scheme V_{7,p_i}, for all $i = 1, \ldots, 6$.

None of the points $p_i \in A \cap \mathbf{P}^2_-$ lie on the quartic $\overline{X(7)} \subset \mathbf{P}^2_-$ since otherwise, by Proposition 5.3 (3), the corresponding scheme V_{7,p_i} would be the secant variety of an elliptic normal curve in $\mathbf{P}^6$, which in turn does not contain smooth Abelian surfaces. Thus, from Proposition 5.3, we deduce that $p_i \notin C_{p_i}$, for all $i = 1, \dots, 6$. In particular, for all $i \neq j$, the point p_j lies necessarily on the conic C_{p_i}, and the point p_i lies on the conic C_{p_j}. The four remaining odd 2-torsion points of A, namely $\{p_k | k \neq i, j\}$, are then well determined as the four points of intersection of the two (smooth) conics C_{p_i} and C_{p_j}. Now, [DK], Theorem 6.14.2, gives a criterion for when 6 points in the plane form a polar hexagon for a plane quartic curve, and the above description of $\{p_1, \dots, p_6\}$ fits that criterion precisely. This proves part (1).

We next show that 21 cubics given by the submaximal Pfaffians of three of the skew-symmetric matrices $M_7'(x, p_i)$, $i = 1, \dots, 6$, are linearly independent, and thus generate the ideal I_A. Indeed, obviously any 14 such Pfaffians coming from two different matrices $M_7'(x, p_i)$ and $M_7'(x, p_j)$, are linearly independent. To show that a third set of submaximal Pfaffians coming from $M_7'(x, p_k)$, for some $k \neq i,j$, would be linearly dependent on the first two it is enough to check this for a degenerate Abelian surface $A \subset \mathbf{P}^7$. This may be checked directly for the Stanley–Reisner degeneration in [GP1], Proposition 4.4. By Proposition 5.1, this proves (2). □

Remark 5.5. One may check that for a general (1,7)-polarized Abelian surface A, the 21 cubic Pfaffians defining any three of the six associated V_{7,p_i}'s generate the homogeneous ideal I_A.

We have now obtained the same description of the moduli space as in [MS], Theorem 4.9:

COROLLARY 5.6. *The moduli space of* (1,7)*-polarized Abelian surfaces with canonical level structure is birational to the space* $\mathrm{VSP}(\overline{X(7)}, 6)$ *of polar hexagons to the Klein quartic curve* $\overline{X(7)} \subset (\mathbf{P}^2_-)^*$*. This is a smooth special Fano threefold of type* V_{22} *of index* 1 *and genus* 12*. In particular,* $\mathcal{A}_7^{\mathrm{lev}}$ *is rational.*

Proof. By Proposition 5.4 (2), the polar hexagon of odd two-torsion points of a general (1,7)-polarized Abelian surface A with level structure uniquely determines A. Mukai [Muk1, Muk2] has shown that the space of polar hexagons to a general plane quartic curve is a smooth Fano threefold of index 1 and genus 12. On the other hand, the analysis of [MS], Theorem 4.4 and (4.5), shows that the variety $\mathrm{VSP}(\overline{X(7)}, 6)$ of polar hexagons to the Klein quartic $\overline{X(7)} \subset (\mathbf{P}^2_-)^*$ is general in Mukai's sense and thus also a smooth Fano threefold. In particular, being of the same dimension as the moduli space $\mathcal{A}_7^{\mathrm{lev}}$, the general polar hexagon to $\overline{X(7)} \subset (\mathbf{P}^2_-)^*$ is the set of odd two-torsion points of some (1,7)-polarized Abelian surface with canonical level structure.

A discussion of the properties (smoothness, type of Fano threefold) of $\mathrm{VSP}(\overline{X(7)}, 6)$ can be found in [MS], see also [Schr] and [Muk2]. The rationality of such a Fano threefold appears to have been known to Mukai [Muk1], and Ivskovskih [Isk]. See also [MS], Theorem 4.10, for the sketch of a proof. □

Remark 5.7. For a general $y \in \mathbf{P}^2_-$, the smooth conic C_y parametrizes a pencil of (1,7)-polarized Abelian surfaces contained in $V_{7,y}$. The conic C_y, is in fact also a conic in the anticanonical embedding of the special Fano threefold $\mathrm{VSP}(\overline{X(7)}, 6)$. See [Schr] for a description of the conics on a Fano threefold of index 1 and genus 12.

PROPOSITION 5.8. *Let $y \in \mathbf{P}^2_-$, be a general point. Then*

(1) *The threefold $V_{7,y}$ has as singularities* 49 *ordinary double points, which occur at the $\mathbf{H}_7$-orbit of the point y.*
(2) *There is a small resolution $V^1_{7,p} \longrightarrow V_{7,p}$ of the ordinary double points such that $V^1_{7,p}$ is a Calabi–Yau threefold, and such that there is a map $\pi_1\colon V^1_{7,p} \longrightarrow \mathbf{P}^1$ whose fibres form the pencil of* (1,7)*-polarized Abelian surfaces in Remark 5.7.*
(3) $\chi(V^1_{7,y}) = 0$ *and* $h^{1,1}(V^1_{7,y}) = h^{1,2}(V^1_{7,y}) = 2$.

Proof. For (1), we know of no better proof than that given in Rodland's thesis [Rod]. One calculates the tangent cone of $V_{7,y}$ at y and finds in general an ordinary double point; thus the $\mathbf{H}_7$-orbit of the point y accounts for 49 ordinary double points. On the other hand, a *Macaulay/Macaulay2* calculation shows that one can find $y \in \mathbf{P}^2_-$, for which $V_{7,y}$ has only 49 singular points. Thus for $y \in \mathbf{P}^2_-$ general, $V_{7,y}$ has precisely 49 ordinary double points.

The small resolution in (2) is obtained, as in Theorem 4.10, by blowing up a smooth (1,7)-polarized Abelian surface contained in $V_{7,y}$ to obtain a small resolution $V^2_{7,y} \longrightarrow V_{7,y}$, and then by flopping the 49 resulting exceptional curves.

Part (3) follows from the fact that the general nonsingular Pfaffian Calabi–Yau has Euler characteristic $\chi = -98$, while the calculation of the Hodge numbers $h^{1,1}$ and $h^{1,2}$ is done via the techniques of Remark 4.11. □

Remark 5.9. We now discuss the Kähler cone of various models of $V_{7,y}$. First note that $H_2(V^1_{7,y}, \mathbf{Z})$ contains two classes of interest: e, the class of an exceptional curve in the small resolution, and c the class of the conic C_y contained in $V_{7,y} \cap \mathbf{P}^2_-$. It is then clear that in $V^1_{7,y}$, we have

$$\begin{array}{llll} H^3 = 14, & H^2A = 14, & A^2 = 0, & \\ H \cdot e = 0, & H \cdot c = 2, & A \cdot e = 1, & A \cdot c = 5. \end{array}$$

This in fact shows that $\mathrm{Pic}(V^1_{7,y})/\mathrm{Torsion}$ is generated by H and A. Indeed, if not, first note since $A \cdot e = 1$, A must be primitive in $\mathrm{Pic}(V^1_{7,y})/\mathrm{Torsion}$, so A and $aA+bH$ form a basis for $\mathrm{Pic}(V^1_{7,y})/\mathrm{Torsion}$, where $a,b \in \mathbf{Q}$. But since $(aA+bH) \cdot e = a$, $a \in \mathbf{Z}$, and since $H^3 = 14$ is not divisible by a cube, we must have $b \in \mathbf{Z}$.

We next consider $V^2_{7,y}$, the model obtained from $V^1_{7,y}$ by flopping the above 49 exceptional curves. One sees easily that

$$\begin{array}{llll} H^3 = 14, & H^2A = 14, & HA^2 = 0, & A^3 = -49, \\ H\cdot e = 0, & H\cdot c = 2, & A\cdot e = -1, & A\cdot c = 5. \end{array}$$

We shall show in [GP3] that the Kähler cone of $V^2_{7,y}$ is spanned by H and $5H{-}2A$, and that $|5H{-}2A|$ contracts the $\mathbf{H}_7$-orbit of the conic C_y. Flopping these curves will then yield a third model $V^3_{7,y}$ with intersection table

$$\begin{array}{llll} H^3 = -378, & H^2A = -966, & HA^2 = -2450, & A^3 = -6174, \\ H\cdot e = 0, & H\cdot c = -2, & A\cdot e = -1, & A\cdot c = -5. \end{array}$$

The Kähler cone of $V^3_{7,y}$ will be seen to be spanned by $5H{-}2A$ and $7H{-}3A$. Finally, in this model $|7H{-}3A|$ is a base-point free pencil of Abelian surfaces with a polarization of type (1,14).

Finally we will see in [GP3] that the linear system $|5H{-}2A|$ maps the threefold $V^2_{7,y}$ into $\mathbf{P}^{13}$ as a codimension 7 linear section of the Plücker embedding of the Grasmmanian Gr(2,7). This should potentially explain some of the numerical similarities in the mirror symmetry computations in [Rod] and [Tjo].

6. Moduli of (1,8)-Polarized Abelian Surfaces

We start by determining the quadratic equations of a (1,8)-polarized Abelian surface in $\mathbf{P}^7$.

Much as in the (1,6) case, let $\mathbf{H}'$ be the subgroup of the Heisenberg group $\mathbf{H}_8$ generated by σ^4 and τ^4. We can easily compute the space of $\mathbf{H}'$-invariant quadrics, $H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'}$, and we see that it splits up into three four-dimensional isomorphic $\mathbf{H}_8$-representations

$$H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'} \cong \bigoplus_{i=0}^{2} \langle f_i, \sigma f_i, \sigma^2 f_i, \sigma^3 f_i\rangle,$$

where

$$f_0 = x_0^2 + x_4^2, \qquad f_1 = x_1x_7 + x_3x_5, \quad \text{and} \quad f_2 = x_2x_6.$$

Remark 6.1. For a point $y \in \mathbf{P}^2_-$, we can, as in the (1,6) case, ask what is the largest $\mathbf{H}_8$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'}$ vanishing at y, or equivalently, what is the subspace of $H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'}$ vanishing on the $\mathbf{H}_8$-orbit of y. To determine this, for $y = (0{:}\,y_1{:}\,y_2{:}\,y_3{:}\,0{:}\,-y_3{:}\,-y_2{:}\,-y_1) \in \mathbf{P}^2_-$, consider the $4{\times}3$-matrix

$$(\sigma^i(f_j(y)))_{\substack{0\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 2}} = \begin{pmatrix} 0 & -y_1^2 - y_3^2 & -y_2^2 \\ y_1^2 + y_3^2 & 0 & -y_1y_3 \\ 2y_2^2 & 2y_3y_1 & 0 \\ y_1^2 + y_3^2 & 0 & -y_1y_3 \end{pmatrix}.$$

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Note that for any $y \in \mathbf{P}^2_-$, this matrix has rank 2, and its kernel is spanned by $(y_1y_3, -y_2^2, y_1^2 + y_3^2)$. Thus the representation of $\mathbf{H}_8$ spanned by

$$f = y_1y_3f_0 - y_2^2f_1 + (y_1^2 + y_3^2)f_2, \quad \sigma(f), \quad \sigma^2(f), \quad \text{and} \quad \sigma^3(f)$$

is the subspace of $H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'}$ vanishing along the $\mathbf{H}_8$ orbit of y. We note that this representation is also spanned by the 4×4 Pfaffians of the matrices $M_4(x, y)$, $M_4(\sigma^4(x), y)$, $M_4(\tau^4(x), y)$ and $M_4(\sigma^4\tau^4(x), y)$, though we will not need this fact in what follows.

LEMMA 6.2. *If $A \subseteq \mathbf{P}^7$ is an Abelian surface invariant under the Schrödinger representation of $\mathbf{H}_8$, then $\dim H^0(\mathcal{I}_A(2))^{\mathbf{H}'} = 4$.*

Proof. This is similar to Lemma 4.1. First consider the restriction map

$$H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'} \longrightarrow H^0(\mathcal{L}^{\otimes 2})^{\mathbf{H}'},$$

where $\mathcal{L}$ is the line bundle inducing the embedding of A in $\mathbf{P}^7$. Let $A' = A/4K(\mathcal{L})$, and let $\pi\colon A \longrightarrow A'$ be the quotient map. Then $4K(\mathcal{L})$ acts on $\mathcal{L}^{\otimes 2}$, and there exists a bundle $\mathcal{M}$ on A' such that $\mathcal{L}^{\otimes 2} = \pi^*\mathcal{M}$, and $H^0(\mathcal{L}^{\otimes 2})^{\mathbf{H}'} \cong H^0(\mathcal{M})$. Now $c_1(\mathcal{M})^2 = c_1(\mathcal{L}^{\otimes 2})^2/\deg(\pi) = 16$, so $\dim H^0(\mathcal{M}) = 8$. Thus, the dimension of the kernel of the restriction map is at least 4, so $\dim H^0(\mathcal{I}_A(2))^{\mathbf{H}'} \geqslant 4$. Equality then follows from Remark 6.1, since $A \cap \mathbf{P}^2_-$ is nonempty. □

Note that $\mathbf{H}'$ acts on $\mathbf{P}^2_-$, and the quotient $\mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ is easily seen to be isomorphic to $\mathbf{P}^2$. From [GP1], §6, we have a morphism

$$\Theta_8\colon \mathcal{A}^{\mathrm{lev}}_{(1,8)} \longrightarrow \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2 \cong \mathbf{P}^2,$$

that essentially associates to an (1,8)-polarized Abelian surface with canonical level structure the class of its odd 2-torsion points. Note that this map is defined on all of $\mathcal{A}^{\mathrm{lev}}_{(1,8)}$, even for Abelian surfaces where the (1,8)-polarization is not very ample, since one still has a concept of where the odd 2-torsion points are mapped.

THEOREM 6.3. *The map $\Theta_8\colon \mathcal{A}^{\mathrm{lev}}_{(1,8)} \longrightarrow \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ is dominant.*

Proof. Let $Z \subseteq \mathbf{P}^2_-$ be the inverse image of $\mathrm{im}(\Theta_8)$ under the projection $\mathbf{P}^2_- \longrightarrow \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$; we wish to show that $\bar{Z} = \mathbf{P}^2_-$. Let $\mathcal{A} \subseteq \mathbf{P}^7_\Delta$ be a Heisenberg invariant degeneration of a (1,8)-polarized Abelian surface with canonical level structure to a translation scroll $S_{E,\tau}$ of an $\mathbf{H}_8$ invariant elliptic normal curve $E \subset \mathbf{P}^7$, as given by [GP1], Theorem 3.1; see also the discussion after Proposition 1.3. Then $S_{E,\tau} \cap \mathbf{P}^2_- \subseteq \overline{Z}$, and so $\mathrm{Sec}(E) \cap \mathbf{P}^2_- \subseteq \overline{Z}$ for a general Heisenberg invariant elliptic normal curve E in $\mathbf{P}^7$.

What is $\mathrm{Sec}(E) \cap \mathbf{P}^2_-$? By the proof of [GP1], Lemma 6.1 (b), a secant line $\langle x,y \rangle$ of $E \subset \mathbf{P}^7$ intersects $\mathbf{P}^2_-$ if and only if $x = -y$ on E. Thus $\mathrm{Sec}(E) \cap \mathbf{P}^2_-$ is the projection of E from $\mathbf{P}^4_+$ to $\mathbf{P}^2_-$. Since $E \cap \mathbf{P}^4_+$ consists of the four two-torsion points of E (see [LB], Corollary 4.7.6), this projection is a two-to-one cover of a conic in $\mathbf{P}^2_-$.

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Thus to prove that $\bar{Z} = \mathbf{P}^2_-$, it is enough to show that as E varies, $\mathrm{Sec}(E) \cap \mathbf{P}^2_-$ sweeps out $\mathbf{P}^2_-$. To do this, it is enough to find two (possibly degenerate) Heisenberg invariant elliptic normal curves E_1 and E_2 such that $\mathrm{Sec}(E_1) \cap \mathbf{P}^2_-$ and $\mathrm{Sec}(E_2) \cap \mathbf{P}^2_-$ do not coincide. We can take E_1 to be the octagon $X(\Gamma_8)$, see [GP1], Theorem 3.2, and E_2 to be any nonsingular Heisenberg invariant elliptic normal curve in $\mathbf{P}^7$. Then using the equations of [GP1], Proposition 4.1 and the fact proved there that $\mathrm{Sec}(E_1) = X(\partial C(8,4))$, one finds that, with coordinates $(x_1{:}\,x_2{:}\,x_3)$ on $\mathbf{P}^2_-$, $\mathrm{Sec}(E_1) \cap \mathbf{P}^2_- = \{x_1x_3 = 0\}$, a reducible conic. On the other hand, $\mathrm{Sec}(E_2) \cap \mathbf{P}^2_-$ is an irreducible conic, so these two curves cannot coincide. This shows that Θ_8 is dominant, and concludes the proof. $\square$

DEFINITION 6.4. For a fixed point $y \in \mathbf{P}^2_-$, let $V_{8,y}$ denote the scheme in $\mathbf{P}^7$ defined by the quadrics in $H^0(\mathcal{O}_{\mathbf{P}^7}(2))^{\mathbf{H}'}$, vanishing on the $\mathbf{H}_8$-orbit of y.

THEOREM 6.5. *For general* $y \in \mathbf{P}^2_-$, $V_{8,y} \subset \mathbf{P}^7$ *is a* (2,2,2,2)-*complete intersection, which is singular precisely at the* 64 *points which form the* $\mathbf{H}_8$-*orbit of* y*, and each of these singular points is an ordinary double point.*

Proof. This can be checked computationally in much the same way as in Theorem 4.10, (2). However, in this case the calculation can be carried out by hand, and we do this here.

It is easy to see that for general y, $V_{8,y} \subset \mathbf{P}^7$ is a threefold. Being a complete intersection, $V_{8,y}$ is singular at $x \in V_{8,y}$ if and only if there is a quadric $Q \in \mathbf{P}(H^0(\mathcal{I}_{V_{8,y}}(2)))$ which is singular at x. Thus, we need to identify all singular quadrics in the web $\mathbf{P}(H^0(\mathcal{I}_{V_{8,y}}(2))) = \mathbf{P}^3$. Now $H^0(\mathcal{I}_{V_{8,y}}(2))$ is an $\mathbf{H}_8$-representation, but since the subgroup $\mathbf{H}'$ acts trivially, it is in fact an $\mathbf{H}_8/\mathbf{H}' \cong \mathbf{H}_4$ representation. Using coordinates $z = (z_0{:}\,z_1{:}\,z_2{:}\,z_3)$ on $\mathbf{P}^3 = \mathbf{P}(H^0(\mathcal{I}_{V_{8,y}}(2)))$ so that z corresponds to the quadric $\sum_{i=0}^3 z_i\sigma^i(f)$, the action of $\mathbf{H}_4$ on $\mathbf{P}^3$ becomes the standard Schrödinger representation of $\mathbf{H}_4$. We will continue to write σ and τ for the generators of $\mathbf{H}_4$; this should not create any confusion.

Put

$$w = (w_0{:}\,w_1{:}\,w_2{:}\,w_3) = (2y_1y_3{:}\,-y_2^2{:}\,y_1^2 + y_3^2{:}\,-y_2^2),$$

so, with notation as in Remark 6.1, $f = \frac{1}{2}w_0f_0 + w_1f_1 + w_2f_2$. The point w is considered fixed.

To find the singular quadrics in the web, we compute the Hessian of the quadric $\sum_{i=0}^3 z_i\sigma^i(f)$ with respect to the variables $x_0, \ldots, x_7$. It is convenient to order these variables in two blocks as x_0, x_2, x_4, x_6, and x_1, x_3, x_5, x_7, for then the Hessian is a block matrix $R(z, w) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with

$$A = (z_{i+j}w_{i-j})_{i,j\in\mathbf{Z}_4} \quad \text{and} \quad B = \sigma(A) = (z_{i+j-1}w_{i-j})_{i,j\in\mathbf{Z}_4},$$

where σ acts only on the variables $z_0, \ldots, z_3$. Keeping in mind that $w_3 = w_1$, one can

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further write $A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$, with

$$A_1 = \begin{pmatrix} z_0 w_0 & z_1 w_1 \\ z_1 w_1 & z_2 w_0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} z_2 w_2 & z_3 w_1 \\ z_3 w_1 & z_0 w_2 \end{pmatrix}.$$

Similarly, $B = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}$ with $B_i = \sigma(A_i)$. Then

$$\det R(z, w) = \det(A_1 + A_2) \cdot \det(\mathrm{A}_1 - \mathrm{A}_2) \cdot \det(\mathrm{B}_1 + \mathrm{B}_2) \cdot \det(\mathrm{B}_1 - \mathrm{B}_2).$$

If Q is the quadric in $\mathbf{P}^3(z_0, z_1, z_2, z_3)$ given by the equation $\det(A_1 + A_2) = 0$, then the locus $\{\det R(z, w) = 0\} \subseteq \mathbf{P}^3(z_0, z_1, z_2, z_3)$ is given by $Q \cup \sigma(Q) \cup \tau(Q) \cup \sigma\tau(Q)$. In other words, this is the locus of singular quadrics in the web $|H^0(\mathcal{I}_{V_{8,y}}(2))|$.

We note the following facts about the geometry of this discriminant locus, which hold for general $y \in \mathbf{P}^2_-$:

LEMMA 6.6. *For general* $y \in \mathbf{P}^2_-$,

(1) *Q is a quadric cone, singular at the point $p = (0\colon 1\colon 0\colon -1) \in \mathbf{P}^3$, which is precisely where $A_1 + A_2 = 0$.*
(2) $p \notin \sigma(Q) \cup \tau(Q) \cup \sigma\tau(Q)$.
(3) rank $R(z,w) = 8$ *if* $z \notin Q \cup \sigma(Q) \cup \tau(Q) \cup \sigma\tau(Q)$.
(4) rank $R(z,w) = 7$ *if z is contained in precisely one of Q, $\sigma(Q)$, $\tau(Q)$ and $\sigma\tau(Q)$, and* $z \in \{p,\sigma(p),\tau(p),\sigma\tau(p)\}$.
(5) rank $R(z,w) \geqslant 6$ *if $z \in \{p,\sigma(p),\tau(p),\sigma\tau(p)\}$, or z is contained in precisely two of the quadrics Q, $\sigma(Q),\tau(Q),\sigma\tau(Q)$.*
(6) rank $R(z,w) \geqslant 5$ *if z is contained in precisely three of $Q,\sigma(Q),\tau(Q),\sigma\tau(Q)$.*
(7) rank $R(z,w) \geqslant 5$ *for all* $z \in \mathbf{P}^3$.
(8) *There are precisely 16 points z in $\mathbf{P}^3$ such that* rank $R(z,w) = 5$, *and they form the $\mathbf{H}_4$-orbit of the point* $(w_3\colon w_2\colon w_1\colon w_0)$.

Proof. (1) and (2) are easily checked. For (3)–(6), one notes that

$$\operatorname{rank} R(z, w) \geqslant \operatorname{rank}(A_1 + A_2) + \operatorname{rank}(A_1 - A_2) + \operatorname{rank}(B_1 + B_2) + \operatorname{rank}(B_1 - B_2),$$

from which (3)–(6) follow, using (2) to note that if at least two of these four matrices drop rank, none have rank 0. For (7), one needs to check that $Q \cap \sigma(Q) \cap \tau(Q) \cap \sigma\tau(Q) = \varnothing$ for general $y \in \mathbf{P}^2_-$. It is sufficient to check this at one y, say $(y_1\colon y_2\colon y_3) = (0\colon 1\colon 0)$, in which case $w = (0\colon -1\colon 0\colon -1)$ and $Q = \{(z_1 + z_3)^2 = 0\}$, in which case $Q \cap \sigma(Q) \cap \tau(Q) \cap \sigma\tau(Q) = \varnothing$.

For (8), one can check by hand that

$$\begin{aligned}(4w_1^4 - 2w_0^3 w_2 - 2w_0 w_2^3)(z_0 + z_2)^2 + (w_0 + w_2)^2 \det(A_1 + A_2) + \\ + 4w_1^2 \sigma(\det(A_1 + A_2)) + (w_0 - w_2)^2 \tau(\det(A_1 + A_2)) = 0.\end{aligned} \tag{6.1}$$

Thus the net of quadrics spanned by Q, $\sigma(Q)$ and $\tau(Q)$ contains the doubled plane $(z_0 + z_2)^2 = 0$, and hence $Q \cap \sigma(Q) \cap \tau(Q)$ consists of at most 4 distinct points,

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counted doubly. The same then holds true by symmetry for any other intersection of three of the four quadrics, and hence there are at most 16 points contained in the intersection of three of the four quadrics. On the other hand, the point $(w_3 : w_2 : w_1 : w_0)$ is seen to be in $Q \cap \tau(Q) \cap \sigma\tau(Q)$, and the $\mathbf{H}_4$-orbit of $(w_3 : w_2 : w_1 : w_0)$ consists of 16 points (for general y), all contained in the intersection of 3 of the four quadrics. Thus this accounts for all such points. One checks that rank $R(z,w) = 5$ for $z = (w_3 : w_2 : w_1 : w_0)$. □

Proof of Theorem 6.5 continued. Having understood the discriminant locus of the web, we now have to understand the loci of vertices of singular quadrics in the web. For a given point $z \in \mathbf{P}^3$, the corresponding quadric $\sum_{i=0}^{3} z_i \sigma^i(f)$ has vertex $\mathbf{P}(\ker R(z, w)) \subseteq \mathbf{P}^7$.

Suppose $z \in Q$, and $z \notin \sigma(Q) \cup \sigma\tau(Q)$. Then rank $(B) = 4$, and thus

$$\mathbf{P}(\ker R(z, w)) \subseteq \{x_1 = x_3 = x_5 = x_7 = 0\}.$$

Thus

$$\begin{aligned} V_{8,y} \cap \mathbf{P}(\ker R(z, w)) \subseteq \{x_1 = x_3 = x_5 = x_7 &= \tfrac{1}{2} w_0(x_0^2 + x_4^2) + w_2 x_2 x_6 \\ &= w_1(x_0 x_6 + x_2 x_4) \\ &= \tfrac{1}{2} w_0(x_6^2 + x_2^2) + w_2 x_0 x_4 \\ &= w_1(x_6 x_4 + x_0 x_2) = 0\} \end{aligned}$$

the latter of which is easily seen to be empty for general y.

If $z \in Q$, and $z \notin \tau(Q) \cup \sigma\tau(Q)$, then $\mathrm{rank}(A_1 - A_2) = 2$, $\mathrm{rank}(B_1 - B_2) = 2$, and it follows that

$$\mathbf{P}(\ker R(z, w)) \subseteq \{x_0 - x_4 = x_1 - x_5 = x_2 - x_6 = x_3 - x_7 = 0\}.$$

Thus

$$\begin{aligned} V_{8,y} \cap \mathbf{P}(\ker R(z, w)) \subseteq \{x_0 - x_4 = x_1 - x_5 = x_2 - x_6 = x_3 - x_7 & \\ = w_0 x_0^2 + 2 w_1 x_3 x_1 + w_2 x_2^2 & \\ = w_0 x_3^2 + 2 w_1 x_2 x_0 + w_2 x_1^2 & \\ = w_0 x_2^2 + 2 w_1 x_1 x_3 + w_2 x_0^2 & \\ = w_0 x_1^2 + 2 w_1 x_0 x_2 + w_2 x_3^2 &= 0\} \end{aligned}$$

which again is seen to be empty for general w.

Finally, if $z \in Q$, and $z \notin \sigma(Q) \cup \tau(Q)$, then $\mathrm{rank}(B_1 + B_2) = 2$, $\mathrm{rank}(A_1 - A_2) = 2$, and

$$\mathbf{P}(\ker R(z, w)) \subseteq \{x_0 - x_4 = x_1 + x_5 = x_2 - x_6 = x_3 + x_7 = 0\},$$

and

$$\begin{aligned} V_{8,y} \cap \mathbf{P}(\ker R(z,w)) \subseteq \{x_0 - x_4 = x_1 + x_5 = x_2 - x_6 = x_3 + x_7 \\ = w_0x_0^2 - 2w_1x_1x_3 + w_2x_2^2 \\ = w_0x_3^2 + 2w_1x_2x_0 - w_2x_1^2 \\ = w_0x_2^2 + 2w_1x_1x_3 + w_2x_0^2 \\ = w_0x_1^2 + 2w_1x_2x_0 - w_2x_3^2 = 0\} \end{aligned}$$

which is again empty.

By the $\mathbf{H}_4$ symmetry, it follows that for general y and any point $z \in \mathbf{P}^3$ such that rank $R(z,w) \geqslant 6$, the singular locus of the quadric $\{\sum z_i\sigma^i(f) = 0\}$ is disjoint from $V_{8,y}$. Thus the only contribution to the singularities of $V_{8,y}$ comes from the quadric $\sum_{i=0}^{3} w_{3-i}\sigma^i(f) = 0$ and its Heisenberg translates. Now $\sigma(\mathbf{P}(\ker R((w_3{:}\,w_2{:}\,w_1{:}\,w_0), w)))$ is easily seen to be $\mathbf{P}^2_-$, and $V_{8,y} \cap \mathbf{P}^2_- = \{y, \sigma^4(y), \tau^4(y), \sigma^4\tau^4(y)\}$ for general y. Thus we see the singular locus of $V_{8,y}$, for general $y \in \mathbf{P}^2_-$, is precisely the $\mathbf{H}_8$-orbit of y.

To figure out the nature of the singularities of $V_{8,y}$, we only need now to observe that for general $y \in \mathbf{P}^2_-$, Lemma 6.2 implies there is an Abelian surface $A \subseteq V_{8,y}$, and then f, $\sigma(f)$, $\sigma^2(f)$, $\sigma^4(f)$ yield four sections of $(\mathcal{I}_A/\mathcal{I}_A^2)(2)$ which are linearly dependent precisely on $\mathrm{Sing}(V_{8,y}) \cap A$. Now a Chern class calculation as in Theorem 4.10 shows that one expects 64 such points. However, any such singular point which is not an ordinary double point counts with some nontrivial multiplicity. Since we have identified precisely 64 distinct singular points, these must all be ordinary double points. $\square$

THEOREM 6.7. *The fibre of Θ_8 over a general point $y \in \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ corresponds to a pencil of Abelian surfaces contained in the singular Calabi–Yau complete intersection $V_{8,y} \subset \mathbf{P}^7$. In particular Θ_8 gives $\mathcal{A}^{\mathrm{lev}}_{(1,8)}$ birationally the structure of a $\mathbf{P}^1$-bundle over an open set of $\mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2 \cong \mathbf{P}^2$.*

Proof. By Theorem 6.5, there is an open set $U \subseteq \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2 \cong \mathbf{P}^2$ such that $V_{8,y}$ is a singular Calabi–Yau threefold with 64 ordinary double points for $y \in U$. For $y \in U \cap \mathrm{im}(\Theta_8)$ (which is non-empty by Theorem 6.3,) there is a (1,8)-polarized Heisenberg invariant Abelian surface A for which y is the class of an odd two-torsion point. In particular, $A \subseteq V_{8,y}$, and A contains $\mathrm{Sing}(V_{8,y})$, which is the $\mathbf{H}_8$-orbit of y. Thus we obtain a projective small resolution $V^2_{8,y} \longrightarrow V_{8,y}$ which is a Calabi–Yau threefold, by blowing-up A. After flopping the 64 exceptional curves, we obtain a small resolution $V^1_{8,y} \longrightarrow V_{8,y}$ which, by Lemma 1.2, contains a base-point free pencil of Abelian surfaces. On the other hand the fibre of Θ_8 is one-dimensional, so we obtain in this way a one-parameter family of Abelian surfaces in $V^1_{8,y}$. If this one parameter family of Abelian surfaces is connected, then it must coincide with the pencil we have already constructed. However, if it were not connected, then $V_{8,y}$ would contain at least two distinct pencils of Abelian surfaces. Let A, $A' \subseteq V_{8,y}$ be Abelian surfaces in these two pencils. Then $A \cap A'$ necessarily contains

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a curve. Indeed, otherwise their proper transforms in $V^1_{8,y}$ meet only at points, hence have an empty intersection as $V^1_{8,y}$ is non-singular. But this is only possible if A and A' belong to the same pencil.

Let C be the one-dimensional component of $A \cap A'$. If A, A' are general, then their Neron-Severi groups are generated by H, so C is numerically equivalent to nH for some n. Furthermore, since A and A' are Heisenberg invariant, so is C. Now we must have $n \leqslant 3$. Indeed, by Riemann–Roch, dim $H^0(\mathcal{I}_A(3))/H^0(\mathcal{I}_{V_{8,y}}(3)) \geqslant 16$. If $f \in H^0(\mathcal{I}_A(3))$ is a cubic not vanishing on $V_{8,y}$, then certainly for general choice of A' in the second pencil, f does not vanish on A', so $A \cap A'$ is contained in a divisor of type $3H$.

Finally, to rule out this possibility, we note there are no Heisenberg invariant curves on A numerically equivalent to nH for $n \leqslant 3$; this follows from [LB], Ex. (4), p. 179. Hence, there is only one component of the fibre of Θ_8, and it is contained in a $\mathbf{P}^1$, which concludes the proof. □

The above theorem allows us to conclude that $\mathcal{A}^{\text{lev}}_{(1,8)}$ is uniruled, but it does not show it is rational or unirational. This is because the $\mathbf{P}^1$-bundle may not be the projectivization of a rank 2 vector bundle. To determine rationality, one needs to know the open set of $\mathbf{P}^2$ over which the $\mathbf{P}^1$-bundle structure is defined. This can be done through a careful analysis of the discriminant locus of the family of (2,2,2,2)-complete intersection Calabi–Yau threefolds of type $V_{8,y}$. This is quite a tedious exercise, so we will only sketch the results below.

THEOREM 6.8. *$\mathcal{A}^{\text{lev}}_{(1,8)}$ is birational to a conic bundle over $\mathbf{P}^2$ with discriminant locus contained in the plane quartic $D = \{2w_1^4 - w_0^3w_2 - w_0w_2^3 = 0\}$. In particular $\mathcal{A}^{\text{lev}}_{(1,8)}$ is rational.*

Proof. The basic idea is that the $\mathbf{P}^1$-bundle structure can only break down over those points $y \in \mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ for which $V_{8,y}$ is degenerate, i.e. has worse than 64 ordinary double points. However, some of these degenerations may also contain pencils of Abelian surfaces, so a more careful analysis of the discriminant locus is required. To determine the discriminant locus of the family of Calabi–Yau threefolds, one looks in detail at the proof of Theorem 6.5 and determines precisely where each step breaks down.

Examining facts (1)–(8) in Lemma 6.6 above, one sees that (1) breaks down on the locus $L = \{w_1(w_0 + w_2)(w_0 - w_2) = 0\}$, which is a union of three lines. Outside of these three lines, (2)–(6) continue to hold. However, for (7) to hold, we need to know that $Q \cap \sigma(Q) \cap \tau(Q) \cap \sigma\tau(Q) = \emptyset$. Via a straightforward calculation, one can see that this occurs precisely on the curve $D = \{2w_1^4 - w_0^3w_2 - w_0w_2^3 = 0\}$. In fact, by equation (6.1) in the proof of Lemma 6.6, one even sees that on this curve the four quadrics Q, $\sigma(Q)$, $\tau(Q)$ and $\sigma\tau(Q)$ only span a pencil and, hence, intersect along an elliptic curve. Now (8) holds off of C and L.

Finally, in the last part of the proof of Theorem 6.5, one finds that

$$V_{8,y} \cap \{x_1 = x_3 = x_5 = x_7 = 0\} \neq \emptyset$$

if and only if $w_0 w_1 w_2 (w_0^2 + w_2^2) = 0$, while

$$V_{8,y} \cap \{x_0 - x_4 = x_1 - x_5 = x_2 - x_6 = x_3 - x_7 = 0\} \neq \emptyset$$

if and only if

$$(w_0 - w_2)((w_0 + w_2)^4 - (2w_1)^4) = 0,$$

while

$$V_{8,y} \cap \{x_0 - x_4 = x_1 + x_5 = x_2 - x_6 = x_3 + x_7 = 0\} \neq \emptyset$$

if and only if

$$(w_0 + w_2)((w_0 - w_2)^4 + (2w_1)^4) = 0.$$

Putting this all together, one finds that $V_{8,y}$ might have worse singularities than 64 ordinary double points only over the locus

$$\begin{aligned}\Delta := D \cup \{&w_0 w_1 w_2 (w_0^2 + w_2^2)(w_0^2 - w_2^2)((w_0 + w_2)^4 - \\ &- (2w_1)^4)((w_0 - w_2)^4 + (2w_1)^4) = 0\},\end{aligned}$$

which is a union of the smooth quartic curve D and 15 lines.

To aid the further analysis, we need to bring in the additional $\mathrm{SL}_2(\mathbf{Z}_8)$ symmetry present. Recall there is an exact sequence (see, for example, [LB], Exercise 6.14)

$$0 \longrightarrow \mathcal{H}(8) \longrightarrow N(\mathcal{H}(8)) \longrightarrow \mathrm{SL}_2(\mathbf{Z}_8) \longrightarrow 0$$

where $N(\mathcal{H}(8))$ is the normalizer of the Heisenberg group $\mathcal{H}(8) \subseteq \mathrm{GL}(H^0(\mathcal{O}_A(1)))$ via the Schrödinger representation. Letting $\xi = e^{2\pi i/16}$ be a fixed primitive 16th root of unity, let S and T be the 8×8 matrices

$$(S_{ij}) = (\xi^{-(i-j)^2})_{0 \leqslant i,j \leqslant 7}, \qquad (T_{ij}) = (\xi^{-2ij})_{0 \leqslant i,j \leqslant 7}.$$

It is easy to check that $S, T \in N(\mathcal{H}(8))$, and give via conjugation an action on $\mathcal{H}(8)/\mathbf{C}^* = \mathbf{Z}_8 \times \mathbf{Z}_8$ defined by the matrices $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, respectively. In particular, S and T along with $\mathcal{H}(8)$ generate the normalizer $N(\mathcal{H}(8))$, though we do not need this fact. What is important for us is that for $\alpha \in N(\mathcal{H}(8))$, we have $V_{8,\alpha(y)} = \alpha(V_{8,y})$. Furthermore, one sees easily that S and T act on the components of the discriminant locus Δ and there are three orbits of this action: the first being D, which is an $\mathrm{SL}_2(\mathbf{Z}_8)$ invariant, the second being $L = \{w_1(w_0^2 - w_2^2) = 0\}$, and the third being the remaining 12 lines in Δ. Thus to understand the generic degeneration of $V_{8,y}$ along each component, it is enough to study the components D, $\{w_1 = 0\}$, and $\{w_0 = 0\}$.

Looking at the equations for $V_{8,y}$ when $w_1 = 0$, it is immediate that $V_{8,y}$ is the join $\mathrm{Join}(E_1, E_2)$ of two elliptic normal quartic curves as in Proposition 1.4, and these

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threefolds still have a natural pencil of Abelian surfaces by Proposition 1.5. Over $w_0 = 0$, one finds a Calabi–Yau with 72 ordinary double points, and these too still possess a pencil of Abelian surfaces. Thus it is only over the quartic D that the $\mathbf{P}^1$-bundle structure may be lost. It then follows from [Bea] that $\mathcal{A}_8^{\mathrm{lev}}$ is rational. □

The following summarizes information about the threefold $V_{8,y}$:

THEOREM 6.9. *For general* $y \in \mathbf{P}^2_-$, *let* $V^2_{8,y} \longrightarrow V_{8,y}$ *be the small resolution obtained by blowing up a smooth* (1,8)*-polarized Abelian surface* $A \subseteq V_{8,y}$, *and let* $V^1_{8,y}$ *be the small resolution of* $V_{8,y}$ *obtained by flopping the 64 exceptional curves on* $V^2_{8,y}$. *Then*

(1) *There exists an Abelian surface fibration* $\pi_1 \colon V^1_{8,y} \longrightarrow \mathbf{P}^1$;
(2) $\chi(V^1_{8,y}) = 0$ *and* $h^{1,1}(V^1_{8,y}) = h^{1,2}(V^1_{8,y}) = 2$.

Proof. (1) has already been proved in Theorem 6.7. (2) follows from calculations similar to those in Remark 4.11, so in particular we may see $\mathbf{P}^2_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ as a compactification of the moduli space of $V_{8,y}$. □

Remark 6.10. The structure of the birational models of $V_{8,y}$ is quite interesting and can be completely determined. Let H denote the pullback of a hyperplane section of $V_{8,y}$ to $V^1_{8,y}$. Let A be the class of a (1,8)-polarized Abelian surface in $V^1_{8,y}$. Classes of curves in $V^1_{8,y}$ include

- $[l]$, the class of a line in $V_{8,y}$ disjoint from the singular locus and contained in a translation scroll fibre of the fibration $V^1_{8,y} \longrightarrow \mathbf{P}^1$, and
- $[e]$, the class of an exceptional curve of the small resolution of $V_{8,y}$.

In general $V^1_{8,y} \longrightarrow \mathbf{P}^1$ has a translation scroll fibre because, as in the proof of Theorem 6.3, there is a translation scroll containing a general point $y \in \mathbf{P}^2_-$. Then in the model $V^1_{8,y}$,

$$\begin{array}{llll} H^3 = 16, & H^2A = 16, & A^2 = 0, & \\ H \cdot e = 0, & H \cdot l = 1, & A \cdot e = 1, & A \cdot l = 0. \end{array}$$

Thus, in particular, $\mathrm{Pic}(V^1_{8,y})/\mathrm{Torsion}$ is generated by H and A. In the model $V^2_{8,y}$,

$$\begin{array}{llll} H^3 = 16, & H^2A = 16, & HA^2 = 0, & A^3 = -64, \\ H \cdot e = 0, & H \cdot l = 1, & A \cdot e = -1, & A \cdot l = 0. \end{array}$$

The Kähler cone of $V^1_{8,y}$ is spanned by H and A. We will see shortly that in $V^2_{8,y}$ the linear system $|4H-2A|$ is a base-point free pencil of Abelian surfaces, by Proposition 6.14 below, so the Kähler cone of $V^2_{8,y}$ is spanned by H and $2H-A$. Furthermore, it is then clear that $V^1_{8,y}$ and $V^2_{8,y}$ are the only minimal models of $V_{8,y}$. Computational evidence leads us to speculate that the family of Calabi–Yau threefolds $V_{8,y}$ could possibly be self-mirror!

To determine the Kähler cone of the model $V^2_{8,y}$ in Remark 6.10 we first need to determine the generators of the ideal of a (1,8)-polarized Abelian surface.

DEFINITION 6.11. If $z \in \mathbf{P}^7$, let $W_{8,z}$ be the vanishing locus of the three by three minors of the matrix $M_4(x, z)$; see Section 1 for notation.

Recall now the degenerations X_8^λ, $\lambda \in \mathbf{C}^*$, of (1,8)-polarized Abelian surfaces with canonical level structure defined in [GP1], §4. For fixed $\lambda \in \mathbf{C}^*$, the surface X_8^λ is the union of eight quadric surfaces $X_8^\lambda := \cup_{i \in \mathbf{Z}_8} Q_i^\lambda$, with

$$Q_i^\lambda := \{x_i x_{i+2} + \lambda x_{i-1} x_{i+3} = x_j = 0, \quad \text{for} \quad j \in \mathbf{Z}_8 \setminus \{i, i+2, i-1, i+3\}\}.$$

THEOREM 6.12

(1) *The homogeneous ideal* $I(X_8^\lambda)$ *is generated by the quadrics* $\{\sigma^i(f_1 + \lambda f_2) \mid 0 \leqslant i \leqslant 3\}$ *and the cubics*

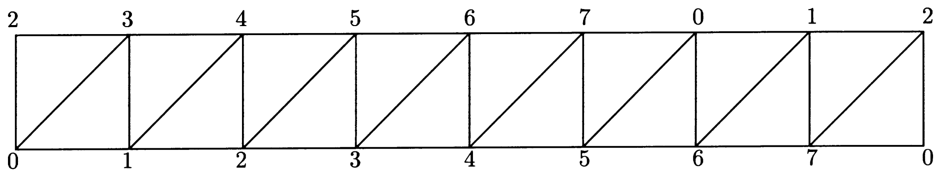
$$\{x_{i-2} x_i x_{i+2} \mid i \in \mathbf{Z}_8\} \cup \{x_{i-1} x_i x_{i+1} \mid i \in \mathbf{Z}_8\}.$$

(2) *For all* $i \in \mathbf{Z}_8$, $x_i x_{i+2} x_{i+5} \in I(X_8^\lambda)$.

(3) *The family* $\{X_8^\lambda | \lambda \in \mathbf{C}^*\} \subseteq \mathbf{C}^* \times \mathbf{P}^7$ *extends uniquely to a flat family*

$$\{X_8^\lambda \mid \lambda \in \mathbf{P}^1 \setminus \{0\}\} \subseteq (\mathbf{P}^1 \setminus \{0\}) \times \mathbf{P}^{n-1},$$

and X_8^∞ *is the face variety of the triangulation of the torus*



In particular, X_8^λ *has the same Hilbert polynomial as a smooth* (1,8)*-polarized Abelian surface.*

Proof. This uses very similar combinatorics to the proof of [GP1], Theorem 4.6, where we neglected to mention in that theorem that part (a) held only for $n \geqslant 10$. We omit the details. □

THEOREM 6.13. *Let* A *be a general* (1,8)*-polarized Heisenberg invariant Abelian surface in* $\mathbf{P}^7$. *Then the embedding is projectively normal and the homogeneous ideal of* A *is generated by the quadrics of* $H^0(\mathcal{I}_A(2))^{\mathbf{H}'}$, *and the 3×3 minors of a matrix* $M_4(x, y)$, *for* $y \in A$ *a general point.*

Proof. This is a standard degeneration argument. Define $S_8 \subseteq \mathbf{P}^2_- \times \mathbf{P}^7$, with coordinates y_1, y_2, y_3 on the first factor and $z_0, \ldots, z_7$ on the second as the locus cut out by the equations

$$S_8 = \{\sigma^i f(y_1, y_2, y_3, z_0, \dots, z_7) = 0 \mid i = 0, \dots, 3\} \subset \mathbf{P}^2_- \times \mathbf{P}^7,$$

where $f := y_1 y_3 f_0 - y_2^2 f_1 + (y_1^2 + y_3^2) f_2$. Though one can show that this scheme is irreducible by using the analysis of Theorem 6.8, we will avoid this, so if necessary, replace S_8 with the irreducible component of S_8 which dominates $\mathbf{P}^2_-$. Note that if $\dim V_{8,y} = 3$, then $\{y\} \times V_{8,y} \subseteq S_8$. Next consider the scheme $\mathcal{S} \subseteq S_8 \times \mathbf{P}^7$ defined in $S_8 \times \mathbf{P}^7$ by the equations $\sigma^i f(y_1, y_2, y_3, x_0, \dots, x_7) = 0$, $i = 0, \dots, 3$, and the 3×3 minors of a matrix $M_4(x, z)$. A general fibre of the projection $\mathcal{S} \longrightarrow S_8$ over a point $(y, z) \in S_8$ is obviously contained in the Calabi–Yau threefold $V_{8,y}$ and, moreover, by Theorem 6.7, there is a unique (1,8)-polarized Heisenberg invariant Abelian surface $A \subseteq V_{8,y}$ containing the point $z \in V_{8,y}$. By [GP1], Corollary 2.7, the Abelian surface A is contained in the fibre $\mathcal{S}_{(y,z)}$. We show that $A = \mathcal{S}_{(y,z)}$ for the general (y,z), by checking equality for one special choice of (y,z), namely $y = (0{:}1{:}\sqrt{-\lambda})$, with $\lambda \neq 0$, and $z = (1{:}1{:}0{:}0{:}0{:}0{:}0{:}0) \in V_{8,y}$. For this choice $\dim V_{8,y} = 3$, and hence $(y, z) \in S_8$. Moreover, for this choice, the ideal of the fiber $\mathcal{S}_{(y,z)}$ is generated by the quadrics $\sigma^i(f_1 + \lambda f_2)$, and the 3×3 minors of the matrix

$$\begin{pmatrix} x_0 & 0 & 0 & x_7 \\ x_1 & x_2 & 0 & 0 \\ 0 & x_3 & x_4 & 0 \\ 0 & 0 & x_5 & x_6 \end{pmatrix}.$$

Therefore, by Theorem 6.12, we deduce that $\mathcal{S}_{(y,z)} = X_8^\lambda$, and so $\mathcal{S}_{(y,z)}$ has the same Hilbert function as a (1,8)-polarized Abelian surface and is projectively normal. It follows that for general (y,z), we have $A = \mathcal{S}_{(y,z)}$, which concludes the proof. $\square$

PROPOSITION 6.14. *For general $y \in \mathbf{P}^2_-$, $|4H-2A|$ is a base-point free pencil on $V^2_{8,y}$, inducing a second Abelian surface fibration π_2: $V^2_{8,y} \longrightarrow \mathbf{P}^1$. A fibre A' of π_2 is a* (2,8) *or* (4,4)*-polarized Abelian surface, mapped to $\mathbf{P}^7$ as a surface of degree* 32.

Proof. By [GP1], Corollary 2.7, $A \subset W_{8,z}$ for all $z \in A \subset \mathbf{P}^7$, so in particular the quartic hypersurface $Q_z = \{\det M_4(x, z) = 0\}$ vanishes doubly along the Abelian surface A and, hence, defines an element of $|4H-2A|$ on $V^2_{8,y}$, for all $z \in A$. One checks for a special value of y, and therefore also for the general y, that the divisors Q_z's in $|4H-2A|$ span at least a pencil. For example, one may take $y = (0{:}1{:}\sqrt{-\lambda}) \in \mathbf{P}^2_-$ for some $\lambda \in \mathbf{C}^*$ and $A = X_8^\lambda$, so that $V_{8,y}$ is given by the equations $\{\sigma^i(f_1 + \lambda f_2) = 0\}$, and then take $z_1 = (1{:}0{:}\cdots{:}0)$, $z_2 = (0{:}1{:}0:\cdots:0)$. Then $Q_{z_1} = \{x_0x_2x_4x_6 = 0\}$, while $Q_{z_2} = \{x_1x_3x_5x_7 = 0\}$, and one can check by hand that $V_{8,y} \cap Q_{z_1} \cap Q_{z_2} = X_8^\lambda$, at least set-theoretically. Thus Q_{z_1} and Q_{z_2} in particular define two independent elements of $|4H-2A|$, and hence this linear system is at least a pencil. In addition, this shows that $|4H-2A|$ has base locus contained in A. Since A could have been chosen to be any member of $|A|$, the base locus of $|4H-2A|$ must be supported on the base locus of $|A|$. In the model $V^2_{8,y}$, this base locus is the union of the 49 exceptional $\mathbf{P}^1$'s of numerical class $|e|$. On the other hand, from the

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intersection tables in Remark 6.10, we compute on $V_{8,y}^2$ that $(4H-2A)\cdot e=2$, so $4H-2A$ is nef.

Since $(4H-2A)^3=(4H-2A)^2H=(4H-2A)^2A=0$ it follows by [Og] that $|4H-2A|$ is base-point free and induces a map to $\mathbf{P}^1$, with fibres either isomorphic to a K3 or an Abelian surface. Since, moreover, $|2H-A|$ is empty as the quadrics containing A are precisely those containing $V_{8,y}$, we see that an element $A'\in|4H-2A|$ is irreducible. It is in fact an Abelian surface: by [Og], it is enough to note that $(4H-2A)\cdot c_2(V_{8,y}^2)=0$, and this can be verified easily.

Fix now a general point $z\in A\subset\mathbf{P}^7$ and define $\psi_z\colon \mathbf{P}^7 \dashrightarrow \mathbf{P}^{15}$ to be the rational map induced by the 3×3-minors of the symmetric matrix $M_4(x,z)$. We regard here ψ_z as the restriction to $\mathbf{P}^7$ of the Cremona involution $\Phi\colon \mathbf{P}^{15}\dashrightarrow\mathbf{P}^{15}$ which associates to each 4×4 matrix its adjoint matrix. Φ is not defined on the locus X_2 of rank $\leqslant 2$ matrices (the secant variety to the Segre embedding of $\mathbf{P}^3\times\mathbf{P}^3$ into $\mathbf{P}^{15}$), it contracts the locus X_3 of rank $\leqslant 3$ matrices (a quartic hypersurface defined by the determinant) to the locus X_1 of rank $\leqslant 1$ matrices (the Segre embedding of $\mathbf{P}^3\times\mathbf{P}^3$ into $\mathbf{P}^{15}$), and it is one-to-one outside of the locus X_3.

In particular ψ_z is birational onto its image and an isomorphism outside the quartic hyper-surface Q_z. Since $\mathcal{O}_{A'}(3H-A)=\mathcal{O}_{A'}(H+D)$, where D is a 2-torsion element, possibly zero, we also deduce that ψ_z maps any Abelian surface A' in $|4H-2A|$ via a linear system induced by the restriction of $|H+D|$. Let A'_z be the surface in $|4H-2A|$ corresponding to the quartic Q_z. Then ψ_z maps A'_z into the intersection of the Segre embedding of $\mathbf{P}^3\times\mathbf{P}^3\subset\mathbf{P}^{15}$ and $\psi_z(V_{8,y})$; in other words the restriction of $H+D$ to A'_z decomposes as the sum L_1+L_2, where each L_i induces the map on one of the two $\mathbf{P}^3$ components. Interchanging L_1 and L_2 amounts to transposing $M_4(x,z)$, and since $M_4(x,z)^t=M_4(x,\iota(z))$, we deduce that L_1 and L_2 are numerically of the same type, and so on A', we have $H\equiv 2L$, where L induces a polarization of type (1,4) or (2,2). Hence H is of type (2,8) or (4,4) on A'. $\square$

Remark 6.15. A straightforward *Macaulay/Macaulay2* computation shows that $|L|$ induces a 2:1 map of A'_z onto a quartic surfaces in $\mathbf{P}^3$ which is singular along two skew lines. Thus, by [BLvS], L must induce a polarization of type (1,4), and thus the Abelian surfaces in the pencil $|4H-2A|$ are in fact (2,8)-polarized.

Remark 6.16. By [GP1], Corollary 2.7, if $A\subseteq\mathbf{P}^7$ is a Heisenberg invariant (1,8)-polarized Abelian surface and $z\in A$, then $A\subseteq W_{8,z}$. The expected codimension of the variety determined by the 3×3 minors of a 4×4 matrix of linear forms is 4. On the other hand, by [GP1], Theorem 5.2, for special values of z, $W_{8,z}$ is the secant scroll of an elliptic normal curve in $\mathbf{P}^7$, so in general $W_{8,z}\subset\mathbf{P}^7$ is a threefold of degree 20, which is a partial smoothing of a degenerate 'Calabi–Yau' threefold, by Proposition 1.3. It turns out that $W_{8,z}$ is in fact a singular model of Calabi-Yau threefold.

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We mention without proof interesting facts about $W_{8,z}$:

$W_{8,z} \subset \mathbf{P}^7$ is invariant under the subgroup of $\mathbf{H}_8$ generated by σ and τ^2 and has, in general, 32 ordinary double points. The (1,8)-polarized Abelian surface $A \subset W_{8,z}$ moves in a pencil whose base locus contains the singular locus of the threefold.

By blowing up $W_{8,y}$ along an Abelian surface A we obtain a small resolution $W^2_{8,y}$ which is a Calabi–Yau threefold. A Calabi–Yau threefold in $\mathbf{P}^7$ defined by the 3×3 minors of a general 4×4 matrix of linear forms, has Hodge numbers $h^{1,1} = 2$, $h^{1,2} = 34$, while for $W^2_{8,y}$ one has $h^{1,1} = 4$, $h^{1,2} = 4$. Let H denote the pullback of a hyperplane section of $W_{8,y}$ to $W^2_{8,y}$. Classes of curves in $W^2_{8,y}$ includes $[c]$, the class of the conic in which $W_{8,y}$ meets $\mathbf{P}^2_-$ (see the proof of Theorem 6.3 for the fact that $W_{8,y}$ intersects $\mathbf{P}^2_-$ in a conic), and $[e]$ the class of an exceptional $\mathbf{P}^1$ of the small resolution of $W_{8,y}$. Then on the model $W^2_{8,y}$,

$$\begin{array}{llll} H^3 = 20, & H^2A = 16, & HA^2 = 0, & A^3 = -32, \\ H\cdot e = 0, & H\cdot c = 2, & A\cdot e = -1, & A\cdot c = 4. \end{array}$$

The linear system $|2H-A|$ is at least three-dimensional, having a subsystem defined by the quadrics containing A. This gives a morphism Ψ: $W^2_{8,y} \longrightarrow \mathbf{P}^3$, whose image is in fact a smooth quadric surface and whose fibres are elliptic curves. Ψ maps the exceptional lines of class e onto the rulings of the quadric Q, while the conics of class c are contracted. Each ruling of Q induces on $W^2_{8,y}$ a K3-surface fibration. A detailed analysis of the geometry of $W_{8,y}$ seems difficult due to the fairly large rank of the Picard group, which is generated over $\mathbf{Q}$ by H, A and the fibres of the above two K3-fibrations.

Remark 6.17. It is interesting to compare the structures described above with those of a (2,4)-polarized Abelian surface $A \subseteq \mathbf{P}^7$, as studied in [Ba]. While numerically such a surface A looks similar to a (1,8)-polarized Abelian surface, in fact it is cut out by six quadrics. Any four of these quadrics defined a complete intersection Calabi–Yau X in $\mathbf{P}^7$ containing A, again in general with 64 ordinary double points. A Calabi–Yau small resolution $\tilde{X} \longrightarrow X$ exists, but now $h^{1,1}(\tilde{X}) = h^{1,2}(\tilde{X}) = 10$. We will say nothing more about this threefold.

7. Moduli of (1,10)-Polarized Abelian Surfaces

We consider first the representation theory of $\mathbf{H}_{10}$ on the space of quadrics $H^0(\mathcal{O}_{\mathbf{P}^9}(2))$. There are four types of five-dimensional irreducible representations appearing in $H^0(\mathcal{O}_{\mathbf{P}^9}(2))$. Indeed, we can write $H^0(\mathcal{O}_{\mathbf{P}^9}(2)) \cong 3V_1 \oplus 3V_2 \oplus 3V_3 \oplus 2V_4$;. The three representations of type V_1 are given by the span of f, $\sigma(f), \dots, \sigma^4(f)$, where $f = x_0^2 + x_5^2$, $x_1x_9 + x_6x_4$, or $x_2x_8 + x_7x_3$. Three representations of type V_2 are given similarly with $f = x_0x_5$, $x_9x_6 + x_4x_1$, or $x_8x_7 + x_3x_2$. Three representations of type V_3 are given by $f = x_0^2 - x_5^2$, $x_1x_9 - x_6x_4$ or

$x_2x_8 - x_7x_3$, and finally the two representations of type V_4 are given by $f = x_9x_6 - x_4x_1$ or $x_8x_7 - x_3x_2$.

Note that for $y \in \mathbf{P}^3_-$, the matrix $M_5(x, y)$ is skew-symmetric, while for $y \in \mathbf{P}^5_+$, $M_5(x, y)$ is symmetric. For a general parameter point $y \in \mathbf{P}^3_-$, the 4×4 Pfaffians of $M_5(x, y)$ cut out a variety $G_y \subset \mathbf{P}^9$ which can be identified with a Plücker embedding of Gr(2,5) in $\mathbf{P}^9$. The corresponding varieties defined by the Pfaffians of $M_5(\sigma^5(x), y)$, $M_5(\sigma^5\tau^5(x), y)$ and $M_5(\tau^5(x), y)$ are then $\sigma^5(G_y)$, $\sigma^5\tau^5(G_y)$ and $\tau^5(G_y)$, respectively.

Note that the subgroup $\langle\sigma^5, \tau^5\rangle$ of $\mathbf{H}_{10}$ acts on $\mathbf{P}^3_-$, and we denote the quotient of this action by $\mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$. By [GP1], Theorem 6.2, we have a map

$$\Theta_{10}\colon \mathcal{A}^{\mathrm{lev}}_{(1,10)} \longrightarrow \mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$$

which essentially maps an Abelian surface to the orbit of its odd 2-torsion points. The map Θ_{10} is birational onto its image. Since both spaces are three dimensional, we obtain the following result:

THEOREM 7.1. *$\mathcal{A}^{\mathrm{lev}}_{10}$ is birationally equivalent to $\mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$.* □

Remark 7.2. The quotient $\mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$, being a quotient of $\mathbf{P}^3$ by a finite Abelian group, is rational by [Miya]. One can also compute explicitly the ring of invariants of this action, and this ring is generated by quadrics and quartics. There are 11 independent invariants of degree 4, and these can be used to map $\mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$ into $\mathbf{P}^{10}$, where one finds the image to be a singular Fano threefold of index 1 and genus 9.

We also have

THEOREM 7.3. *Let $y \in \mathbf{P}^3_-$ be a general point. Then*

(1) *The 20 quadrics defining the four Plücker embedded Grassmannians G_y, $\sigma^5(G_y)$, $\sigma^5\tau^5(G_y)$, $\tau^5(G_y)$ span only a 15-dimensional space of quadrics $R_y \subset H^0(\mathcal{O}_{\mathbf{P}^9}(2))$, which as a representation of $\mathbf{H}_{10}$ is of type $V_1 \oplus V_2 \oplus V_3$;*
(2) *The subspace R_y generates the homogeneous ideal of the (1,10)-polarized Abelian surface corresponding to the image via Θ_{10} of y in $\mathbf{P}^3_-/\mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. By [GP1], Corollary 2.7, if A is a $\mathbf{H}_{10}$ invariant Abelian surface in $\mathbf{P}^9$ and $y \in A \cap \mathbf{P}^3_-$ is an odd 2-torsion point, then $A \subseteq G_y \cap \sigma^5(G_y) \cap \tau^5(G_y) \cap \sigma^5\tau^5(G_y)$, so all quadrics in the subspace R_y vanish on A. In fact, Theorem 6.2 of [GP1] states that $A = G_y \cap \sigma^5(G_y) \cap \tau^5(G_y)$, and, moreover, the corresponding space of 15 quadrics generates the homogeneous ideal of A in $\mathbf{P}^9$. Thus we must have dim $R_y = 15$. To see how R_y decomposes as an $\mathbf{H}_{10}$ representation, it is sufficient to check the isomorphism $R_y \cong V_1 \oplus V_2 \oplus V_3$ for a special value of y. One may use the point $y = (0\colon 1\colon 1\colon 0\colon 0\colon 0\colon 0\colon 0\colon -1\colon -1) \in \mathbf{P}^3_-$, which in the notation of [GP1], §4, yields the degenerate surface X^1_{10}, whose homogeneous ideal, by [GP1] Theorem 3.6, is generated by the quadrics

$$\{x_i x_{i+2} + x_{i-1} x_{i+3} \mid i \in \mathbf{Z}_{10}\} \cup \{x_i x_{i+5} \mid i \in \mathbf{Z}_{10}\}.$$

This clearly decompose as $V_1 \oplus V_2 \oplus V_3$. $\square$

THEOREM 7.4. (1) *For general $y \in \mathbf{P}^3_-$, $V_{10,y} = G_y \cap \tau^5(G_y) \subset \mathbf{P}^9$ is a Calabi–Yau threefold with* 50 *ordinary double points, which is invariant under the subgroup generated by σ^2 and τ. The singular locus of $V_{10,y}$ is the $\langle\sigma^2, \tau\rangle$-orbit of $\{\sigma^5(y)\}$. If $A \subseteq V_{10,y}$ is the $\mathbf{H}_{10}$-invariant* (1,10)*-polarized Abelian surface which has y as an odd 2-torsion point, then the linear system* $|2H{-}A|$ induces a $2{:}1$ *cover from $V_{10,y}$ to the symmetric HM-quintic threefold $X_{5,y'} \subset \mathbf{P}^4$, where*

$$y' = (y_1y_2 + y_3y_4 : -y_2y_3 : y_1y_4 : y_1y_4 : -y_2y_3) \in \mathbf{P}^2_+ \subseteq \mathbf{P}^4$$

(see Proposition 3.4).

(2) *If $y \in A \cap \mathbf{P}^5_+$, let $W_{10,y} \subset \mathbf{P}^9$ be the variety defined by the* $3{\times}3$ *minors of the matrix $M_5(x, y)$. For general y, this is a Calabi–Yau threefold of degree* 35 *with* 25 *ordinary double points. Furthermore, $W_{10,y}$ is not simply connected, but has an unbranched double cover birational to $X_{5,y''} \subset \mathbf{P}^4$, with*

$$y'' = (2(y_3y_4 - y_1y_2) : y_0y_1 - y_4y_5 : y_2y_5 - y_0y_3 : y_2y_5 - y_0y_3 : y_0y_1 - y_4y_5) \\ \in \mathbf{P}^2_+ \subseteq \mathbf{P}^4.$$

Proof: (1) To understand the singularity structure of $V_{10,y}$, we will first understand the image of the map induced by the linear system $|2H{-}A|$ in $\mathbf{P}^4$, and show that this map expresses $V_{10,y}$ as a partial resolution of a double cover of $X_{5,y'} \subset \mathbf{P}^4$ branched over its singular locus.

First we note that for general $y \in \mathbf{P}^3_-$, $V_{10,y}$ is an irreducible threefold of degree 25. To see this, we show it is true for special choice of y. Choose $y \in \mathbf{P}^3_-$ so that $y_i = y_{5+i}$. Then $M_5(x, y) = (x'_{i+j} y_{i-j})$ where $x'_{i+j} = x_{i+j} + x_{i+j+5}$. Thus G_y is just a cone over an elliptic normal curve E in the $\mathbf{P}^4$ given by $\{x_i - x_{i+5} = 0\}$, with vertex the $\mathbf{P}^4$ given by $\{x_i + x_{i+5} = 0\}$. $G_{\tau^5(y)}$ is a similar cone over $\tau^5(E)$, and then $V_{10,y} = G_y \cap G_{\tau^5(y)}$ is just the linear join of E and $\tau^5(E)$, which is an irreducible threefold of degree 25, by Proposition 1.4. For general y, G_y and $G_{\tau^5(y)}$ have dimension 6 and degree 5 in $\mathbf{P}^9$, so $G_y \cap G_{\tau^5(y)}$ is expected to be of dimension 3 and degree 25. Since this holds for special y, it holds for general y.

Next we show that $V_{10,y}$ contains a pencil of Abelian surfaces. Let l be the line joining y and $\tau^5(y)$, so that $\sigma^5(l)$ is the line joining $\sigma^5(y)$ and $\tau^5\sigma^5(y)$. Let $z \in \sigma^5(l)$, such that $z \notin \{\sigma^5(y) + \sigma^5\tau^5(y), \sigma^5(y) - \sigma^5\tau^5(y)\}$. Then there is a linear transformation $T \in \mathrm{PGL}(\mathbf{P}^9)$ of the form $T = \mathrm{diag}(a, b, \ldots, a, b)$ such that $T^2(z) = \sigma^5(y)$. Note that $\tau^5 T = T\tau^5$, $\sigma^5 T = T^{-1}\sigma^5$ in $\mathrm{PGL}(\mathbf{P}^9)$, and that for any $w \in \mathbf{P}^3_-$, we have $T(G_w) = G_{T^{-1}(w)}$. (The significance of these communtation relations will be explained in more detail in § 1 of [GP3].) Denote by $A_{T(z)} \subset \mathbf{P}^9$ the Abelian surface determined by $T(z) \in \mathbf{P}^3_-$, i.e.

$$A_{T(z)} = G_{T(z)} \cap G_{\sigma^5 T(z)} \cap G_{\tau^5 T(z)} \cap G_{\sigma^5\tau^5 T(z)}.$$

Then

$$\begin{aligned} T(A_{T(z)}) &= G_z \cap G_{\sigma^5 T^2(z)} \cap G_{\tau^5(z)} \cap G_{\sigma^5\tau^5 T^2(z)} \\ &= G_z \cap G_{\tau^5(z)} \cap (G_y \cap G_{\tau^5(y)}) \\ &\subseteq V_{10,y}. \end{aligned}$$

As z varies on $\sigma^5(l)$, the point $T(z)$ also varies on this line, and thus we obtain a pencil of distinct (1,10)-polarized Abelian surfaces in $V_{10,y}$, filling up $V_{10,y}$ by irreducibility. Note that in particular, $\sigma^5(z) = T(\sigma^5(T(z))) \in T(A_{T(z)})$ moves along the line l. Thus $l \subseteq V_{10,y}$, and each Abelian surface in the pencil intersects the line l.

Now we will study the linear system $|2H - A|$ on $V_{10,y}$. To do so, we must first understand the equations of $V_{10,y}$ more deeply. The Pfaffian of the matrix obtained by deleting the first row and column of $M_5(x, y)$ is

$$\begin{aligned} p = &- x_0x_5(y_1y_2 + y_3y_4) + (x_1x_4 + x_6x_9)y_2y_3 - (x_2x_3 + x_7x_8)y_1y_4 + \\ &+ (x_3x_7y_1^2 + x_2x_8y_4^2) - (x_4x_6y_2^2 + x_1x_9y_3^2) + (x_5^2y_1y_3 + x_0^2y_2y_4). \end{aligned}$$

The ideal of $V_{10,y}$ is then generated by

$$\{\sigma^{2i}(p) \mid 0 \leqslant i \leqslant 4\} \cup \{\tau^5\sigma^{2i}(p) \mid 0 \leqslant i \leqslant 4\}.$$

On the other hand, the ideal of A is generated by these ten quadrics plus the additional quadrics $\{\sigma^{2i+5}(p) \mid 0 \leqslant i \leqslant 4\}$. Thus in particular, if we set

$$f_i := \tfrac{1}{2}\sigma^{2i}(\sigma^5(p + \tau^5(p)) + p + \tau^5(p))$$

we see that

$$\begin{aligned} f_i = &(x_{3+2i}x_{7+2i} + x_{2+2i}x_{8+2i})(y_1^2 + y_4^2) + \\ &+ (x_{5+2i}^2 + x_{2i}^2)(y_1y_3 + y_2y_4) - (x_{4+2i}x_{6+2i}x_{1+2i}x_{9+2i})(y_2^2 + y_3^2), \end{aligned}$$

and that $f_0, \ldots, f_4$ are linearly independent quadrics which are still linearly independent when restricted to $V_{10,y}$. Furthermore, each f_i vanishes on A, and in fact $f_0, \ldots, f_4$ cut out A on $V_{10,y}$. Thus $f_0, \ldots, f_4$ define a four-dimensional linear subsystem of $|H^0(\mathcal{O}_{V_{10,y}}(2))|$ whose base locus is precisely the Abelian surface A.

We now use $f_0, \ldots, f_4$ to define a rational map $f\colon V_{10,y} \dashrightarrow \mathbf{P}^4$. We describe next its image. The map f is induced by the linear system $|2H - A|$ on $V^2_{10,y}$, where the model $V^2_{10,y} \longrightarrow V_{10,y}$ is obtained by blowing up $V_{10,y}$ along A. We also denote by f the induced morphism $V^2_{10,y} \longrightarrow \mathbf{P}^4$.

Let z be a general point on the line $\sigma^5(l)$ joining $\sigma^5(y)$ and $\sigma^5\tau^5(y)$, and T as before. Let us consider the restriction of f to the surface $A' = T(A_{T(z)})$. There are two alternatives. First if $A \cap A'$ is an isolated set of points, then the number of these points is divisible by 50, since A and A' are invariant under the action of τ and

σ^2. Since $\sigma^5(y) \in A \cap A'$, we have at least 50 such points, and the degree of $f\colon A' \dashrightarrow \mathbf{P}^4$ is $4 \cdot 20 - a \cdot 50$, for some $a \geqslant 1$, which is nonnegative only if $a = 1$. Thus $A \cap A'$ can only consist of one $\langle\sigma^2, \tau\rangle$-orbit of $\sigma^5(y)$, and A must intersect A' transversally. Thus the proper transform of A' in $V^2_{10,y}$, which we denote by $\widetilde{A}'$, is A' blown up in 50 points.

The second alternative is that $A \cap A'$ contains a curve C. In this case, since A is general, the curve C can only be numerically equivalent to nH on A', for some n. Since A is cut out by quadrics, $n \leqslant 2$. On the other hand, since A and A' are invariant with respect to σ^2 and τ, so is C. However, by [LB], Ex. (4), p. 179, this is impossible. Thus this case does not occur!

Next observe that for $x \in V_{10,y}, f(x) = f(\tau^5(x))$, since each f_i is τ^5-invariant. Thus f factors via $V_{10,y} \longrightarrow V_{10,y}/\langle\tau^5\rangle \dashrightarrow \mathbf{P}^4$. Now τ^5 acts on A'. The (1,10) polarization $\mathcal{L}$ on A' descends to a (1,5) polarization $\mathcal{L}'$ on A'/τ^5, and $f_0, \ldots, f_4$ descend to sections of $\mathcal{L}'^{\otimes 2}$. The map $f\colon A'/\tau^5 \dashrightarrow \mathbf{P}^4$ lifts then to a morphism $\tilde{f}\colon \tilde{A}'/\tau^5 \longrightarrow \mathbf{P}^4$. It is easy to see that $\tilde{f}$ satisfies all the hypotheses of Proposition 3.6. Thus f maps $T(A_{T(z)})$, for z general, to a Heisenberg invariant Abelian surface $A'' \subseteq \mathbf{P}^4$ of degree 15, and A'' is contained in a quintic $X_{5,y'} \subset \mathbf{P}^4$ for any $y' \in A''$. If we find a point y' which is contained in $f(T(A_{T(z)}))$ for all $z \in \sigma^5(l)$, then $f(V_{10,y}) \subseteq X_{5,y'}$. A simple calculation shows that

$$f(l) = (y_1y_2 + y_3y_4 : -y_2y_3 : y_1y_4 : y_1y_4 : -y_2y_3) =: y'.$$

Since $T(A_{T(z)})$ intersects l for each z, y' is the desired point, and so $f(V_{10,y}) \subseteq X_{5,y'}$. To show that f maps two-to-one onto $X_{5,y'}$, we observe that on $V^2_{10,y}$, H and A are Cartier divisors, with $H^3 = 25$, $H^2A = 20$, $HA^2 = 0$ and $A^3 = -50$, (the latter holds since $(2H - A)^2 \cdot A = 30$, as f maps A two-to-one to a degree 15 surface in $\mathbf{P}^4$). From this we see that $(2H - A)^3 = 10$. Thus $f(V_{10,y}) = X_{5,y'}$ and f maps two-to-one generically.

To compute the branch locus of the double cover $f\colon V^2_{10,y} \longrightarrow X_{5,y'}$, we need to identify the fixed locus of the involution τ^5 acting on $V_{10,y}$. This is easily done: the fixed locus of τ^5 acting on $\mathbf{P}^9$ consists of

$$L_0 = \{x_0 = x_2 = x_4 = x_6 = x_8 = 0\}$$

and

$$L_1 = \{x_1 = x_3 = x_5 = x_7 = x_9 = 0\}.$$

Note we can write

$$M_5(x, y) = (x_{6(i+j)}y_{6(i-j)} + x_{6(i+j)+5}y_{6(i-j)+5})_{0 \leqslant i,j \leqslant 4},$$

making it clear that

$$M_5(x, y)|_{L_0} = -M_5(\tau^5(x), y)|_{L_0} = (x_{6(i+j)+5}y_{6(i-j)+5})_{0 \leqslant i,j \leqslant 4}$$

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and

$$M_5(x,y)|_{L_1} = M_5(\tau^5(x),y)|_{L_1} = (x_{6(i+j)}y_{6(i-j)})_{0 \leqslant i,j \leqslant 4}.$$

Each of these are skew-symmetric Moore matrices on $\mathbf{P}^4$. Thus by [Hu1] and [ADHPR2], Proposition 4.2, each of these matrices drops rank on an elliptic normal curve. Thus $(L_0 \cup L_1) \cap V_{10,y} = E_0 \cup E_1$, the disjoint union of two elliptic normal curves. This is the fixed locus of τ^5 acting on $V_{10,y}$. It is now clear that f must map E_0 and E_1 to the two singular curves of $X_{5,y'}$ of Proposition 3.4, (3).

We can now understand from this the singularity structure of $V^2_{10,y}$. First note that the exceptional locus of π_2: $V^2_{10,y} \longrightarrow V_{10,y}$ consists of 50 $\mathbf{P}^1$'s. A is a Cartier divisor on $V^2_{10,y}$, so in particular $V^2_{10,y}$ is nonsingular along any nonsingular element of the linear system $|A|$. But the exceptional locus of π_2 is contained in the base locus of $|A|$, hence $V^2_{10,y}$ is nonsingular along the exceptional locus. In particular, since π_2 is the small resolution of 50 ordinary double points and $\omega_{V_{10,y}}$ is trivial, so is $\omega_{V^2_{10,y}}$.

Let $V^2_{10,y} \xrightarrow{s_1} \hat{X}_{5,y'} \xrightarrow{s_2} X_{5,y'}$, be the Stein factorization of f: $V^2_{10,y} \longrightarrow X_{5,y'}$. Then s_2 is branched precisely over $\mathrm{Sing}(X_{5,y'})$, and is a double cover. From the description of the singularities of $X_{5,y'}$ of Proposition 3.4, (5), one sees that $s_2(\mathrm{Sing}(\hat{X}_{5,y'}))$ is the Heisenberg orbit of y', and shows that $\hat{X}_{5,y'}$ has one ordinary double point sitting over each point of the orbit of y'.

Next consider s_1: $V^2_{10,y} \longrightarrow \hat{X}_{5,y'}$. We have already seen that $f(l) = y'$; thus s_1 must contract l and its $\langle \sigma^2, \tau \rangle$-orbit (25 lines) to the 25 ordinary double points of $\hat{X}_{5,y'}$. Since $\omega_{V^2_{10,y}}$ and $\omega_{\hat{X}_{5,y'}}$ are trivial, s_1 must be crepant, and the only possibility then is that s_1 only contracts these 25 lines, while $V^2_{10,y}$ is nonsingular. Thus $V_{10,y}$ itself has 50 ordinary double points obtained by contracting the exceptional locus of π_2.

(2) Fix any point $y \in \mathbf{P}^5_+$. Consider first $\mathbf{P}(\mathrm{Sym}^2(\mathbf{C}^5)) = \mathrm{Proj}\,\mathbf{C}[\{x_{ij} | 0 \leqslant i,j \leqslant 4, x_{ij} = x_{ji}\}]$. The generic symmetric matrix $M = (x_{ij})$ has rank 2 precisely on $\mathrm{Sym}^2(\mathbf{P}^4) \subseteq \mathbf{P}(\mathrm{Sym}^2(\mathbf{C}^5))$. One computes that $\mathrm{Sym}^2(\mathbf{P}^4)$ has degree 35 in this embedding and thus $W_{10,y}$ can be described as $i(\mathbf{P}^9) \cap \mathrm{Sym}^2(\mathbf{P}^4)$, where i: $\mathbf{P}^9 \longrightarrow \mathbf{P}(\mathrm{Sym}^2(\mathbf{C}^5))$ is the linear map given by $x_{ij} = x_{i+j}y_{i-j} + x_{i+j+5}y_{i-j+5}$. Thus, if the intersection is transversal, $W_{10,y}$ has dimension 3 and degree 35.

To understand $W_{10,y}$ further, consider it embedded in $\mathrm{Sym}^2(\mathbf{P}^4)$ via the map i. Let π: $\mathbf{P}^4 \times \mathbf{P}^4 \longrightarrow \mathrm{Sym}^2(\mathbf{P}^4)$ be the quotient map, and let $z_0, \ldots, z_4, w_0, \ldots, w_4$ be coordinates on the first and second $\mathbf{P}^4$'s, respectively. The map π is given by $x_{ij} = z_i w_j + z_j w_i$. Let us determine the equations of $\pi^{-1}(W_{10,y})$, which is a double cover of $W_{10,y}$. One checks easily that the equations of $i(\mathbf{P}^9)$ are

$$\{(y_3y_4 - y_1y_2)x_{i,i} + (y_0y_1 - y_4y_5)x_{i-1,i+1} + (y_2y_5 - y_0y_3)x_{i-2,i+2} = 0 \mid 0 \leqslant i \leqslant 4\}.$$

Hence $\pi^{-1}(W_{10,y})$ is given by the five bilinear equations

$$\{(y_3y_4 - y_1y_2)(2z_iw_i) + (y_0y_1 - y_4y_5)(z_{i-1}w_{i+1} + z_{i+1}w_{i-1}) + \\ + (y_2y_5 - y_0y_3)(z_{i-2}w_{i+2} + z_{i+2}w_{i-2}) = 0\}.$$

If we write

$$y''_0 = 2(y_3y_4 - y_1y_2), \qquad y''_1 = y''_4 = y_0y_1 - y_4y_5, \qquad y''_2 = y''_3 = y_2y_5 - y_0y_3,$$

then we can rewrite the above set of equations as $L(z, y'')w = 0$, where $L(z, y'')$ is the 5×5 matrix $L(z, y'') = (z_{2i-j}y''_{i-j})$. If p_1, p_2: $\mathbf{P}^4 \times \mathbf{P}^4 \longrightarrow \mathbf{P}^4$ are the first and second projections, then $X'_{5,y''} = p_1(\pi^{-1}(W_{10,y}))$ is given by the equation $\{\det L(z, y'') = 0\}$. Thus it is clear that $W_{10,y}$ is of dimension three and hence of degree 35. Note that this argument shows that a generic intersection of $\mathrm{Sym}^2(\mathbf{P}^4)$ and $\mathbf{P}^9$ is a nonsingular threefold (since the singular locus of $\mathrm{Sym}^2(\mathbf{P}^4)$ has codimension 4) and has an unbranched covering which is a Calabi–Yau threefold. We conclude that the generic section of $\mathrm{Sym}^2(\mathbf{P}^4)$ is a nonsingular Calabi–Yau threefold, and hence also $\omega_{W_{10,y}} = \mathcal{O}_{W_{10,y}}$.

Now note that

$${}^tL(z, y'')w = M'_5(w, y'')z$$

given that $y'' \in \mathbf{P}^2_+$. Thus, if $\tilde{X}_{5,y''} \subseteq \mathbf{P}^4(z) \times \mathbf{P}^4(w)$ is given by the equation ${}^tL(z, y'')w = 0$, then $p_1: \tilde{X}_{5,y''} \longrightarrow X'_{5,y''}$ is a birational map and $p_2(\tilde{X}_{5,y''}) = X_{5,y''}$, where $X_{5,y''}$ is the (symmetric) Horrocks–Mumford quintic given by y''. Then $p_2: \tilde{X}_{5,y''} \longrightarrow X_{5,y''}$ is also birational. Thus we see that $\pi^{-1}(W_{10,y})$ is birational to $X_{5,y''}$. In addition, by Proposition 3.4, (4), $\tilde{X}_{5,y''}$ has 50 ordinary double points. Now p_1: $\pi^{-1}(W_{10,y}) \longrightarrow X'_{5,y''}$ is also a crepant partial resolution, and fails to be an isomorphism where rank $L(z, y'') \leqslant 3$. However, rank $L(z, y'') = \mathrm{rank}\,{}^tL(z, y'')$, so p_1: $\tilde{X}_{5,y''} \longrightarrow X'_{5,y''}$ and p_1: $\pi^{-1}(W_{10,y}) \longrightarrow X'_{5,y''}$ fail to be isomorphisms on the same set, which by Proposition 3.4, (6), is a union of two degree 5 elliptic curves $E_1 \cup E_2$. Furthermore, $p_1^{-1}(E_i) \subseteq \tilde{X}_{5,y''}$ and $p_1^{-1}(E_i) \subseteq \pi^{-1}(W_{10,y})$ are both $\mathbf{P}^1$-bundles. It follows from this description that while $\tilde{X}_{5,y''}$ and $\pi^{-1}(W_{10,y})$ are not necessarily isomorphic over $X'_{5,y''}$, they must have the same singularity structure, both being partial crepant resolutions of curves of cA_1 singularities with 50 cA_2 points. In particular, $\pi^{-1}(W_{10,y})$ has 50 ordinary double points.

Finally, it is easy to see that for a general choice of y, $\pi^{-1}(W_{10,y})$ is disjoint from the diagonal in $\mathbf{P}^4 \times \mathbf{P}^4$, so that $\pi^{-1}(W_{10,y}) \longrightarrow W_{10,y}$ is an unbranched covering. To see this, one can check for one point, e.g. a point of the form $y = (y_0, \ldots, y_5, \ldots) = (0, y_1, y_2, y_3, y_4, 0, \ldots)$, in which case $y'' = (2(y_3y_4 - y_1y_2), 0, \ldots, 0)$, and it is particularly easy to see that $\pi^{-1}(W_{10,y})$ is disjoint from the diagonal.

Thus, for general y, $W_{10,y}$ must be a Calabi–Yau threefold with 25 ordinary double points. □

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Remark 7.5. We discuss the structure of the Kähler cones of minimal models of $V_{10,y}$.

First consider $V^1_{10,y}$, obtained by flopping the 50 exceptional curves of $V^2_{10,y} \longrightarrow V_{10,y}$. Then $|A|$ is a base-point free pencil on $V^1_{10,y}$. Let H be the pullback of a hyperplane section. Let e be the class of an exceptional curve of the small resolution, and let l be the proper transform of the line l joining y and $\tau^5(y)$ as in the proof of Theorem 7.4. Then

$$\begin{array}{llll} H^3 = 25, & H^2A = 20, & A^2 = 0, & \\ H \cdot l = 1, & H \cdot e = 0, & A \cdot l = 2, & A \cdot e = 1. \end{array}$$

One can compute that $h^{1,1}(V^1_{10,y}) = h^{1,2}(V^1_{10,y}) = 2$, and then it is clear that $\mathrm{Pic}(V^1_{10,y})/\mathrm{Torsion}$ is generated by H and A, and the Kähler cone of $V^1_{10,y}$ is spanned by H and A.

In $V^2_{10,y}$,

$$\begin{array}{llll} H^3 = 25, & H^2A = 20, & HA^2 = 0, & A^3 = -50, \\ H \cdot l = 1, & H \cdot e = 0, & A \cdot l = 2, & A \cdot e = -1. \end{array}$$

The Kähler cone of $V^2_{10,y}$ is spanned by H and $2H-A$; indeed, it follows from the proof of Theorem 7.4 that $|2H-A|$ on $V^2_{10,y}$ is base-point free and induces the map $V^2_{10,y} \longrightarrow X_{5,y'}$, and the Stein factorization of this map contracts the $\langle\sigma^2, \tau^2\rangle$ orbit of l. Thus these 25 lines can be flopped, to obtain a new model $V^3_{10,y}$. In this model,

$$\begin{array}{llll} H^3 = 0, & H^2A = -30, & HA^2 = -100, & A^3 = -250 \\ H \cdot l = -1, & H \cdot e = 0, & A \cdot l = -2, & A \cdot e = -1. \end{array}$$

Now $|2H-A|$ induces the morphism f: $V^3_{10,y} \longrightarrow X_{5,y'}$, and $X_{5,y'}$ contains a pencil $|A'|$ of minimal (1,5)-polarized Abelian surfaces. Furthermore, if $X^1_{5,y'} \subseteq \mathbf{P}^4 \times |A'|$ defined by

$$X^1_{5,y'} = \{(x, S) \mid x \in S \in |A'|\}$$

is a partial resolution of $X^1_{5,y'}$ in which the pencil $|A'|$ is base-point free, then f factors through $\tilde{f}$: $V^3_{10,y} \longrightarrow X^1_{5,y'}$. It is then easy to see that if S is a general element of this pencil, then $\tilde{f}^{-1}(S)$ is of class $10H-6A$ on $V^3_{10,y}$. Also, $\tilde{f}^{-1}(S) \longrightarrow S$ is a two-to-one unbranched cover. Thus either $\tilde{f}^{-1}(S)$ is irreducible, or $\tilde{f}^{-1}(S)$ splits as a union of two (1,5) Abelian surfaces. However, it is not difficult to see that the divisor $5H-3A$ is effective. For example, choosing a general point $z \in A$, $\det M_5(x, z)$ is a quintic vanishing triply along A. Since a pencil of Abelian surfaces on a Calabi–Yau threefold cannot have a multiple fibre, we conclude that $|5H-3A|$ must be a pencil of Abelian surfaces, and $\tilde{f}^{-1}(S)$ splits. Hence $|5H-3A|$ yields a base-point free pencil of (1,5)-polarized Abelian surface, and the Kähler cone of $V^3_{10,y}$ is spanned by $2H-A$ and $5H-3A$.

We will not address the Kähler cone structure of $W_{10,y}$ here. Because $h^{1,1} \geqslant 3$ for a small resolution of this Calabi–Yau threefold, we in fact expect $W_{10,y}$ to inherit a rather complicated Kähler cone structure from its double cover, a Horrocks–Mumford quintic. See [Fry] for an analysis of the moving cone of the general Horrocks–Mumford quintic.

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References

[Ad] Adler, A.: On the automorphism group of a certain cubic threefold, *Amer. J. Math.* **100**(6) (1978), 1275–1280.

[Au] Aure, A.: Surfaces on quintic threefolds associated to the Horrocks–Mumford bundle, In: *Lecture Notes in Math.* 1399, Springer, New York, 1989, pp. 1–9.

[ADHPR1] Aure, A. B., Decker, W., Hulek, K., Popescu, S. and Ranestad, K.: The geometry of bielliptic surfaces in $\mathbf{P}^4$, *Internat. J. Math.* **4** (1993), 873–902.

[ADHPR2] Aure, A. B., Decker, W., Hulek, K., Popescu, S. and Ranestad, K.: Syzygies of Abelian and bielliptic surfaces in $\mathbf{P}^4$, *Internat. J. Math.* **8** (1997), 849–919.

[Ba] Barth, W.: Abelian surfaces with (1,2) polarization, In: *Algebraic Geometry* (*Sendai 1985*), Adv. Stud. Pure Math. 10, Kinokuniya, Tokyo, 1985, pp. 41–84.

[BM] Barth, W. and Moore R.: Geometry in the space of Horrocks–Mumford surfaces, *Topology* **28**(2) (1989), 231–245.

[BHM] Barth, W., Hulek, K., and Moore, R.: Degenerations of Horrocks–Mumford surfaces, *Math. Ann.* **277**(4) (1987), 735–755.

[BS] Bayer, D. and Stillman, M.: Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from ftp://math.columbia.edu via anonymous ftp.

[Bea] Beauville, A.: Variétés de Prym et jacobiennes intermédiaires, *Ann. Sci. École Norm. Sup.* (*4*) **10**(3) (1977), 309–391.

[BeMe] Behr, H. and Mennicke, J.: A presentation of the groups PSL(2,*p*), *Canad. J. Math.* **20** (1968), 1432–1438.

[BL] Birkenhake, Ch. and Lange, H.: Moduli spaces of Abelian surfaces with isogeny, In: *Geometry and analysis* (*Bombay, 1992*), Tata Inst. Fund. Res., Bombay, 1995, pp. 225–243.

[BLvS] Birkenhake, Ch., Lange, H., and van Straten, D.: Abelian surfaces of type (1,4), *Math. Ann.* **285** (1989), 625–646.

[Bor1] Borcea, C.: On desingularized Horrocks–Mumford quintics, *J. Reine, Angew. Math.* **421** (1991), 23–41.

[Bor2] Borcea, C.: Nodal quintic threefolds and nodal octic surfaces, *Proc. Amer. Math. Soc.* **109**(3) (1990), 627–635.

[Bori] Borisov, L.: A finiteness theorem for Sp(4,$\mathbb{Z}$), Algebraic geometry, 9, *J. Math. Sci.* (*New York*) **94**(1), (1999), 1073–1099; math.AG/9510002.

[Cle] Clemens, C. H.: Double solids, *Adv. in Math.* **47**(3) (1983), 107–230.

[CNPW] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A.: *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*, with computational assistance from J. G. Thackray, Oxford Univ. Press, Oxford, 1985.

[DHS] Debarre, O., Hulek, K., and Spandaw, J.: Very ample linear systems on abelian varieties, *Math. Ann.* **300** (1994), 181–202.

[DK] Dolgachev, I. and Kanev, V.: Polar covariants of plane cubics and quartics, *Adv. Math.* **98**(2) (1993), 216–301.

[DO] Dolgachev, I. and Ortland, D.: Points sets in projective spaces and theta functions, *Astérisque* **165** (1988).

[Do] Donagi, R.: The unirationality of $\mathcal{A}_5$, *Ann. of Math.* (*2*) **119**(2) (1984), 269–307.

[EPW] Eisenbud, D., Popescu, S. and Walter, Ch.: Enriques surfaces and other non-Pfaffian subcanonical subschemes of codimension 3, *Comm. Algebra* **28**(12) (2000), special volume for Robin Hartshorne's 60th birthday, to appear, math.AG/9906171.

[Fry] Fryers, M.: The moveable cones of Calabi–Yau threefolds, Thesis, Univ. Cambridge 1998.

[vG] van Geemen, B.: The moduli space of curves of genus 3 with level 2 structure is rational, Unpublished preprint.

[GS] Grayson, D. and Stillman, M.: Macaulay 2: A computer program designed to support computations in algebraic geometry and computer algebra, Source and object code available from http://www.math.uiuc.edu/Macaulay2/.

[Gri1] Gritsenko, V.: Irrationality of the moduli spaces of polarized Abelian surfaces. With an appendix by the author and K. Hulek, In: *Abelian Varieties* (*Egloffstein, 1993*), de Gruyter, Berlin, 1995, pp. 63–84.

[Gri2] Gritsenko, V.: Irrationality of the moduli spaces of polarized Abelian surfaces, *Internat. Math. Res. Notices* (1994), No. 6, 235 ff., approx. 9 pp. (electronic).

[GP1] Gross, M. and Popescu, S.: Equations of $(1,d)$-polarized Abelian surfaces, *Math. Ann.* **310** (1998), 333–377.

[GP2] Gross, M. and Popescu, S.: The moduli space of (1,11)-polarized Abelian surfaces in unirational, *Compositio Math.* **126** (2001), 1–24.

[GP3] Gross, M. and Popescu, S.: Calabi–Yau threefolds and moduli of Abelian surfaces II, in preparation.

[Ha] Hartshorne, R.: *Algebraic Geometry*, Springer, New York, 1977.

[Has] Hassett, B.: Some rational cubic fourfolds, *J. Algebraic Geom.* **8**(1) (1999), 103–114.

[HM] Horrocks, G. and Mumford, D.: A Rank 2 vector bundle on $\mathbf{P}^4$ with 15,000 symmetries, *Topology*, **12** (1973), 63–81.

[Hu1] Hulek, K.: Projective geometry of elliptic curves, *Astérisque* **137** (1986).

[Hu2] Hulek, K.: Geometry of the Horrocks–Mumford bundle, In: *Algebraic Geometry* (*Bowdoin 1985*), Proc. Symp. Pure Math. 46 (2), Amer. Math. Soc., Providence, p. 69–85.

[HKW] Hulek, K., Kahn, C. and Weintraub, S.: *Moduli Spaces of Abelian Surfaces: Compactification, Degenerations, and Theta Functions*, Walter de Gruyter, Berlin 1993.

[HS1] Hulek, K. and Sankaran, G. K.: The Kodaira dimension of certain moduli spaces of Abelian surfaces, *Compositio Math.* **90**(1) (1994), 1–35.

[HS2] Hulek, K. and Sankaran, G. K.: The geometry of Siegel modular varieties, Preprint math.AG/9810153.

[I] Igusa, J.: Arithmetic variety of moduli for genus two, *Ann. of Math.* (*2*) **72** (1960), 612–649.

[Isk] Iskovskikh, V. A.: Algebraic threefolds (a brief survey), (Russian) In: *Algebra*, Gos. Univ., Moscow, 1982, pp. 46–78.

[Iy] Iyer, J. N.: Projective normality of Abelian surfaces given by primitive line bundles, *Manuscripta Math.* **98**(2) (1999), 139–153.

[Kat] Katsylo, P.: Rationality of the moduli variety of curves of genus 3, *Comment. Math. Helv.* **71**(4) (1996), 507–524.

[Kl1] Klein, F.: Über transformationen siebenter Ordnung der elliptischen Funktionen (1878/79), Abhandlung LXXXIV, in *Gesammelte Werke*, Bd. III, Springer, Berlin, 1924.

[Kl2] Klein, F.: Über die elliptischen Normalkurven der *n*-ten Ordnung (1885), Abhandlung XC, in *Gesammelte Werke*, Bd. III, Springer, Berlin, 1924.

[La] Lazarsfeld, R.: Projectivité normale des surfaces Abeliennes, (written by O. Debarre) Pre-print Europroj 14, Nice.

[LB] Lange, H. and Birkenhake, Ch.: *Complex Abelian Varieties*, Springer-Verlag, New York, 1992.

[MS] Manolache, N. and Schreyer, F.-O.: Moduli of (1,7)-polarized Abelian surfaces via syzygies, *Math. Nachr.* (2001) to appear, preprint math.AG/9812121.

[Mie] Miele, M.: Klassifikation der Durchscnitte Heisenberg-invarianter Systeme von Quadriken in $\mathbf{P}^6$, Thesis, Erlangen, 1995.

[Mi] Miyata, T.: Invariants of certain groups I, *Nagoya Math. J.* **41** (1971), 69–73.

[Moo] Moore, R.: Heisenberg-invariant quintic 3-folds, and sections of the Horrocks–Mumford bundle, Preprint, Canberra 1985.

[MM] Mori, S. and Mukai, S.: The uniruledness of the moduli space of curves of genus 11, In: *Algebraic Geometry* (*Tokyo/Kyoto, 1982*), Lecture Notes in Math. 1016, Springer, New York, 1983, pp. 334–353.

[Muk1] Mukai, S.: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, *Proc. Nat. Acad. Sci. U.S.A.* **86**(9) (1989), 3000–3002.

[Muk2] Mukai, S.: Fano 3-folds, In: *Complex Projective Geometry* (*Trieste, 1989/Bergen, 1989*), London Math. Soc. Lecture Note Ser. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 255.

[Mu1] Mumford, D.: On the equations defining Abelian varieties, *Invent. Math.* **1** (1966), 287–354.

[Mu2] Mumford, D.: *Abelian Varieties*, Oxford Univ. Press, 1974.

[Mu3] Mumford, D.: *Tata Lectures on Theta I*, Progress in Math. 28, Birkhäuser, Basel, 1983.

[O'G] O'Grady, K. G., On the Kodaira dimension of moduli spaces of Abelian surfaces, *Compositio Math.* **72**(2) (1989), 121–163.

[Og] Oguiso, K., On algebraic fiber space structures on a Calabi–Yau 3-fold, With an appendix by Noboru Nakayama. *Internat. J. Math.* **4**(3) (1993), 439–465.

[RS] Ranestad, K. and Schreyer, F.-O.: Varieties of sums of powers, *J. Reine Angew. Math.* **525** (2000), 147–181.

[Rod] Rodland, E.: The Pffafian Calabi–Yau, its mirror, and their link to the Grassmannian $G(2,7)$, *Compositio Math.* **122**(2) (2000), 135–149.

[Sal] Salmon, G.: *Modern Higher Algebra 4*, Hodges, Figgis, and Co., Dublin, 1885.

[Scho] Schoen, C.: Algebraic cycles on certain desingularized nodal hypersurfaces, *Math. Ann.* **270** (1985), 17–27.

[Scho2] Schoen, C.: On the geometry of a special determinantal hypersurface associated to the Horrocks–Mumford vector bundle, *J. Reine Angew. Math.* **364** (1986), 85–111.

[Schr] Schreyer, F.-O.: Geometry and algebra of prime Fano 3-folds of index 1 and genus 12, *Compositio Math.* (2001), to appear, preprint math.AG/9911044.

[SR] Semple, G. and Roth, L.: *Algebraic Geometry*, Chelsea, 1937.

[Si] Silberger, A. J.: An elementary construction of the representations of SL(2,GF(q)), *Osaka J. Math.* **6** (1969), 329–338.

[Ta] Tanaka, S.: Construction and classification of irreducible representations of special linear group of the second order over a finite field, *Osaka J. Math.* **4** (1967), 65–84.

[Tei] Teissier, B.: The hunting of invariants in the geometry of discriminants, In: *Real and Complex Singularities*, (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 565–678.

[Tjo] Tjotta, E.: Quantum cohomology of a Pfaffian Calabi–Yau variety: verifying mirror symmetry predictions, *Compositio Math.* **126** (2001), 79–89.

[Ve] Vélu, J.: Courbes elliptiques munies d'un sous-groupe $\mathbf{Z}/n\mathbf{Z} \times \mu_n$, *Bull. Soc. Math. France*, *Mém.* **57** (1978), 5–152.

[Ver] Verra, A.: A short proof of the unirationality of $\mathcal{A}_5$, *Nederl. Akad. Wetensch. Indag. Math.* **46**(3) (1984), 339–355.

[We] Werner, J.: *Kleine Auflösungen spezieller dreidimensionaler Varietäten*, Bonner Math. Schriften 186, Universität Bonn, Mathematisches Institut, Bonn, 1987.