Preface

This monograph is devoted to geometric inverse problems, with emphasis on the two-dimensional case. *Inverse problems* arise in various fields of science and engineering, frequently in connection with imaging methods where one attempts to produce images of the interior of an unknown object by making indirect measurements outside. A standard example is X-ray computed tomography (CT) in medical imaging. There one sends X-rays through the patient and measures how much the rays are attenuated along the way. From these measurements one would like to determine the attenuation coefficient of the tissues inside. If the X-rays are sent along a two-dimensional cross-section (identified with $\mathbb{R}^2$) of the patient, the X-ray measurements correspond to the *Radon transform* Rf of the unknown attenuation function f in $\mathbb{R}^2$. Here, Rf just encodes the integrals of f along all straight lines in $\mathbb{R}^2$. The easy *direct problem* in X-ray CT would be to determine the Radon transform Rf when f is known. However, in order to produce images, one needs to solve the *inverse problem*: determine f when Rf is known (i.e. invert the Radon transform).

One can divide the mathematical analysis of the Radon transform inverse problem in several parts, including the following:

- (Uniqueness) If $Rf_1 = Rf_2$, does it follow that $f_1 = f_2$?
- (Stability) If Rf_1 and Rf_2 are close, are f_1 and f_2 close in suitable norms? Is there stability with respect to noise or measurement errors?
- (Reconstruction) Is there an efficient algorithm for reconstructing f from the knowledge of Rf?
- (Range characterization) Which functions arise as Rf for some f?
- (Partial data) Can one determine (some information on) f from partial knowledge of Rf?

In this monograph we will study inverse problems in *geometric* settings. For X-ray type problems this will mean that straight lines are replaced by more general curves. A particularly clean setting, which is still relevant for several applications, is given by geodesic curves of a smooth Riemannian metric. We will focus on this setting and formulate our questions on compact Riemannian manifolds (M, g) with smooth boundary. This corresponds to working with compactly supported functions in the Radon transform problem.

We will now briefly describe the main geometric inverse problems studied in this book. Our first question is a direct generalization of the Radon transform problem.

1. Geodesic X-ray transform. Is it possible to determine an unknown function f in (M, g) from the knowledge of its integrals over maximal geodesics?

This is a fundamental inverse problem that is related to several other inverse problems, in particular in seismic imaging applications. A classical related problem is to determine the interior structure of the Earth by measuring travel times of earthquakes. In a mathematical idealization, we may suppose that the Earth is a ball $M \subset \mathbb{R}^3$ and that wave fronts generated by earthquakes follow the geodesics of a Riemannian metric g determined by the sound speed in different substructures. If an earthquake is generated at a point $x \in \partial M$, then the first arrival time of that earthquake to a seismic station at $y \in \partial M$ is the geodesic distance $d_g(x, y)$. The *travel time tomography* problem, originating in geophysics in the early twentieth century, is to determine the metric g (i.e. the sound speed in M) from the geodesic distances between boundary points. The same problem arose much later in pure mathematics and differential geometry. It can be formulated as follows.

2. Boundary rigidity problem. Is it possible to determine the metric in (M, g), up to a boundary fixing isometry, from the knowledge of the boundary distance function $d_g|_{\partial M \times \partial M}$?

The geodesic X-ray transform problem is in fact precisely the linearization of the boundary rigidity problem for metrics in a fixed conformal class. If one removes the restriction to a fixed conformal class, the linearization of the boundary rigidity problem is a *tensor tomography problem*. To describe such a problem, let (M, g) be a compact Riemannian n-manifold with smooth boundary, and let m be a non-negative integer. The geodesic X-ray transform on symmetric m-tensor fields is an operator I_m defined by

$$I_m f(\gamma) = \int_\gamma f_{j_1 \cdots j_m}(\gamma(t)) \dot{\gamma}^{j_1}(t) \cdots \dot{\gamma}^{j_m}(t)\, dt,$$

where γ is a maximal geodesic in M and $f = f_{j_1\cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}$ is a smooth symmetric m-tensor field on M. Here and throughout this monograph we employ the Einstein summation convention where a repeated lower and upper index is summed. In the above case this means that

$$f_{j_1\cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m} = \sum_{j_1,\ldots,j_m=1}^{n} f_{j_1\cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}.$$

If $m \geq 1$ the operator I_m always has a non-trivial kernel: one has $I_m(\sigma\nabla h) = 0$ whenever h is a smooth symmetric $(m-1)$-tensor field with $h|_{\partial M} = 0$, ∇ is the total covariant derivative, and σ denotes the symmetrization of a tensor. Tensors of the form $\sigma\nabla h$ are called *potential tensors*. If $m = 1$, this just means that $I_1(dh) = 0$ whenever $h \in C^\infty(M)$ satisfies $h|_{\partial M} = 0$. Any 1-tensor field f has a solenoidal decomposition $f = f^s + dh$ where f^s is *solenoidal* (i.e. divergence-free) and $h|_{\partial M} = 0$. Thus it is only possible to determine the solenoidal part of a 1-tensor f from $I_1 f$. This decomposition generalizes to tensors of arbitrary order, leading to the following inverse problem.

3. Tensor tomography problem. Is it possible to determine the solenoidal part of an m-tensor field f in (M, g) from the knowledge of $I_m f$?

A variant of the geodesic X-ray transform, arising in applications such as SPECT (single-photon emission computed tomography), includes an attenuation factor. In this case, $f \in C^\infty(M)$ is a source function and $a \in C^\infty(M)$ is an attenuation coefficient, and one can measure integrals such as

$$I_a f(\gamma) = \int_\gamma e^{\int_0^t a(\gamma(s))\, ds} f(\gamma(t))\, dt, \quad \gamma \text{ is a maximal geodesic.}$$

This is the *attenuated geodesic X-ray transform* of f, and a typical inverse problem is to determine f from $I_a f$ when a is assumed to be known. Clearly this reduces to the standard geodesic X-ray transform when $a = 0$. Similar questions appear in mathematical physics, where the attenuation coefficient is replaced by a *connection* or a *Higgs field* on some vector bundle over M. This roughly corresponds to replacing the function $a(x)$ by a matrix-valued function or a 1-form.

4. Attenuated geodesic X-ray transform. Is it possible to determine a function f in (M, g) from its attenuated geodesic X-ray transform, when the attenuation is given by a connection and a Higgs field?

This question also arises as the linearization of the *scattering rigidity problem* (or the *non-Abelian X-ray transform*) for a connection/Higgs field.

One can ask related questions for tensor fields and also for more general weighted X-ray transforms.

Finally, we consider a geometric inverse problem of a somewhat different nature. Consider the Dirichlet problem for the Laplace equation in (M, g),

$$\begin{cases} \Delta_g u &= 0 \text{ in } M, \\ u &= f \text{ on } \partial M. \end{cases}$$

Here Δ_g is the Laplace–Beltrami operator on (M, g), given in local coordinates by

$$\Delta_g u = |g|^{-1/2} \partial_{x_j} \left(|g|^{1/2} g^{jk} \partial_{x_k} u \right),$$

where (g^{jk}) is the inverse matrix of $g = (g_{jk})$, and $|g| = \det(g_{jk})$. This is a uniformly elliptic operator, and there is a unique solution $u \in C^\infty(M)$ for any $f \in C^\infty(\partial M)$. The *Dirichlet-to-Neumann map* Λ_g takes the Dirichlet data of u to Neumann data,

$$\Lambda_g : f \mapsto \partial_\nu u|_{\partial M},$$

where $\partial_\nu u|_{\partial M} = du(\nu)|_{\partial M}$ with ν denoting the inner unit normal to ∂M.

The above problem is related to electrical impedance tomography, where the objective is to determine the electrical properties of a medium by making voltage and current measurements on its boundary. Here the metric g corresponds to the electrical resistivity of the medium, and for a prescribed boundary voltage f one measures the corresponding current flux $\partial_\nu u$ at the boundary. Thus the electrical measurements are encoded by the Dirichlet-to-Neumann map Λ_g. There are natural gauge invariances: the map Λ_g remains unchanged under a boundary fixing isometry of (M, g), and when $\dim M = 2$ there is an additional invariance due to conformal changes of the metric. This leads to the following inverse problem.

5. Calderón problem. Is it possible to determine the metric in (M, g), up to gauge, from the knowledge of the Dirichlet-to-Neumann map Λ_g?

In this monograph we will discuss known results for the above problems, with an emphasis on the case where (M, g) is *two dimensional*. One reason for focusing on the two-dimensional setting is that the available results and methods are somewhat different in three and higher dimensions. This is also suggested by a formal variable count: in the questions above we attempt to determine unknown functions of n variables from data given by a function of $2n - 2$ variables. Thus the inverse problems above are formally determined when $n = 2$ and formally overdetermined when $n \geq 3$. This indicates that there may be less flexibility when solving the two-dimensional problems. On the other hand, the possibility of using methods from complex analysis will give an advantage in two dimensions.

Another reason for focusing on the two-dimensional case is that the two-dimensional theory is at the moment fairly well developed in the context of *simple manifolds*. A compact Riemannian manifold (M, g) with smooth boundary is called simple if

- the boundary ∂M is *strictly convex* (the second fundamental form of ∂M is positive definite),
- M is *non-trapping* (any geodesic reaches the boundary in finite time), and
- M has *no conjugate points*.

Examples of simple manifolds include strictly convex domains in Euclidean space, strictly convex simply connected domains in non-positively curved manifolds, strictly convex subdomains of the hemisphere, and small metric perturbations of these.

In this book we will show that questions 1–4 above have a positive answer on two-dimensional simple manifolds, and question 5 has a positive answer on any two-dimensional manifold. In particular, this gives a positive answer in two dimensions to the boundary rigidity problem posed by Michel (1981/82). The original proof of this result in Pestov and Uhlmann (2005) employs striking connections between the above problems: in fact, it uses the solution of the geodesic X-ray transform problem and the Calderón problem in order to solve the boundary rigidity problem.

We will also see that there are counterexamples to questions 1–4 if one goes outside the class of simple manifolds. However, it is an outstanding open problem whether questions 1–4 have positive answers in the class of strictly convex non-trapping manifolds (i.e. whether the no conjugate points assumption can be removed).

While the emphasis in this monograph is on the two-dimensional case, a large part of the material is valid in any dimension ≥ 2. In Chapters 1–8 the results are either presented in arbitrary dimension, or they are first presented in two dimensions and there is an additional section describing extensions to the higher dimensional case. However, the methods in Chapters 9–14 involve fibrewise holomorphic functions and holomorphic integrating factors, and these are largely specific to the two-dimensional case.

The field of geometric inverse problems is vast, and the present monograph only covers a selection of topics. We have attempted to choose topics that have reached a certain degree of maturity and that lead to a coherent presentation. For the chosen topics, we have tried to give an up-to-date treatment including the most recent results. However, there are several notable omissions such as results specific to three and higher dimensions, the case of closed manifolds, further geometric inverse problems for partial differential equations, inverse spectral problems, and so on. Some of these are briefly discussed in Chapter 15.

As for the references, we have not aimed at a complete historical account of the results presented here. In the main text we have cited a few selected references for each topic, and in Chapter 15 we give a number of further references on related topics. We refer to the bibliographical notes in Sharafutdinov (1994) for an account of results up to 1994. The survey articles Paternain et al. (2014b); Ilmavirta and Monard (2019); Stefanov et al. (2019) contain a wealth of references to further results.

We assume that readers are familiar with basic Riemannian geometry roughly at the level of Lee (1997). We also assume familiarity with elliptic partial differential equations and Sobolev spaces in the setting of Riemannian manifolds, as presented e.g. in Taylor (2011). There are numerous exercises scattered throughout the text and the more challenging ones are marked with a $*$.

Outline

One intent of the present text is to provide a unified approach to the questions 1–4 while exposing the main techniques involved. Having this in mind we have structured the contents as follows.

Chapter 1 considers basic properties of the classical Radon transform in the plane and discusses briefly the Funk transform on the 2-sphere. These homogeneous geometric backgrounds are particularly amenable to the use of standard Fourier analysis and provide a quick introduction to the subject. Chapter 2 studies rotationally symmetric examples and the well-known Herglotz condition that translates into a non-trapping condition for the geodesics.

Chapter 3 discusses at length the necessary geometric background. The starting assumptions on compact Riemannian manifolds is that they have strictly convex boundary and no trapped geodesics. This combination produces an exit time function that is smooth everywhere except at the glancing region, where its behaviour is well understood. This setting is good enough to define all X-ray transforms arising in the book (standard, attenuated, and non-Abelian), and it is also good enough to study regularity results for the transport equation associated with the geodesic vector field as it is done in Chapter 5. As we mentioned above when we add the condition of not having conjugate points we obtain the notion of simple manifold; this is also discussed in Chapter 3.

In Chapter 4 we introduce the geodesic X-ray transform and we establish the important link with the transport equation. This link gives in particular that the geodesic X-ray transform I_0 is injective if and only if a uniqueness result holds for the operator $P = VX$, where X is the geodesic vector field and V the vertical vector field. This brings us to the first core idea in this book. To tackle this uniqueness problem for P we use an energy identity called the *Pestov identity*, which emerges from studying the commutator $[P^*, P]$. The absence of conjugate points gives a way to control the sign of the terms that arise from this commutator. Variations of this identity will be used to study attenuated and non-Abelian X-ray transforms in Chapter 13.

Chapter 6 provides some tools that are specific to two dimensions. Here we follow the approach of Guillemin and Kazhdan (1980a), and we take advantage of the fact that there is a Fourier series expansion in the angular variable (i.e. with respect to the vertical vector field V) and that the geodesic vector field decomposes as $X = \eta_+ + \eta_-$, where $\eta_\pm$ maps Fourier modes of degree k to degree $k \pm 1$. The Fourier expansions make it possible to consider holomorphic functions and Hilbert transforms with respect to the angular variable, and a certain amount of 'vertical' complex analysis becomes available. On the other hand, the operators $\eta_\pm$ are intimately connected with the Cauchy–Riemann operators of the underlying complex structure of the surface determined by the metric. These tools get deployed right away in Chapter 7 where we study solenoidal injectivity and stability for the geodesic X-ray transform under the stronger assumption of having non-positive curvature.

Chapter 8 contains the second core idea in the book. This is based on the central fact that when the manifold (M, g) is simple, the normal operator $I_0^* I_0$ is an elliptic pseudodifferential operator of order -1 in the interior of M. The ellipticity combined with the injectivity of I_0 gives a surjectivity result for the adjoint I_0^*. It is this last solvability result that plays a key role in all subsequent developments, and it may be rephrased as an existence result for first integrals of the geodesic flow with prescribed zero Fourier modes.

Chapter 9 discusses inversion formulas up to a Fredholm error and the range of I_0. The description of the range is possible, thanks to the surjectivity of suitable adjoints following the outline of Chapter 8. Chapter 10 deals with tensor tomography, but also explains how to obtain the important *holomorphic integrating factors* from the surjectivity of I_0^*. Here, the holomorphicity is in the sense of Chapter 6, i.e. in the angular variable.

Chapter 11 is devoted fully to question 2 above on boundary rigidity and its relation to the Calderón problem. Chapter 12 proves injectivity for the attenuated X-ray transform using holomorphic integrating factors and finally Chapters 13 and 14 discuss the non-Abelian X-ray transform and attenuated

X-ray transform for connections and Higgs fields. The book concludes with Chapter 15 including a brief summary of the most relevant open problems and a discussion on selected related topics.

The results presented in this monograph are scattered in research articles, and we have aimed at giving a unified presentation of this theory. Some arguments may appear here for the first time. These include a detailed proof of the equivalence of several definitions of simple manifolds in Section 3.8, a direct proof of a basic regularity result for the transport equation in Section 5.1, a relation between the Pestov–Uhlmann inversion formula and the filtered backprojection formula in Section 9.5, and a proof that the scattering relation determines the Dirichlet-to-Neumann map in Section 11.5 based on boundary values of invariant functions.