ON A COMBINATORIAL PROBLEM OF ERDÖS

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A family $\mathfrak{F}$ of sets is said to possess property $\mathfrak{B}$ if there exists a set $B \subset \cup\mathfrak{F}$ such that $B \cap F \neq \phi$ and $B \not\supset F$ for every $F \in \mathfrak{F}$. We consider the following question raised by P. Erdös [1]: let n and N be positive integers, $n \geq 2$ and $N \geq 2n - 1$ and let S be a set of N elements; what is the least integer $m'_N(n)$, (provided such an integer exists), for which there exists a family $\mathfrak{F}$ of $m'_N(n)$ subsets of S satisfying

(a) $|F| = n$ for each $F \in \mathfrak{F}$;

(b) $\cup\mathfrak{F} = S$;

(c) $\mathfrak{F}$ does not have property $\mathfrak{B}$;

(d) if $\mathfrak{F}' \subset \mathfrak{F}$ and $|\cup\mathfrak{F}'| < N$, then $\mathfrak{F}'$ has property $\mathfrak{B}$?

Erdös pointed out that $m'_{2t+1}(2) = 2t + 1$ and that $m'_N(2)$ does not exist if N is even. In this note we shall prove that $m'_N(n)$ exists for all $n \geq 3$, $N \geq 2n - 1$, and obtain some upper bounds. However, instead of studying $m'_N(n)$ we shall consider $m^*_N(n)$ which is defined in almost the same way as $m'_N(n)$ except that (d) is replaced by

(e) if $\mathfrak{F}'$ is a proper subfamily of $\mathfrak{F}$, then $\mathfrak{F}'$ has property $\mathfrak{B}$.

It is clear that the existence of $m^*_N(n)$ implies the existence of $m'_N(n)$ and in fact we have $m'_N(n) \leq m^*_N(n)$.

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THEOREM 1. If $m^*_N(a)$ exists, then so does $m^*_{N+a+2b-1}(a+b)$ for every positive integer b.

Proof. Let $\mathcal{G}$ be a family of $m^*_N(a)$ sets satisfying (a), (b), (c) and (e). Let T be a set with $a + 2b - 1$ elements. We assume that T is disjoint from every member of $\mathcal{G}$. Let $\mathcal{H}$ be the family of b-subsets of T and let $\mathcal{I}$ be the family of $(a+b)$-subsets of T. Let

$$\mathcal{F} = \{F : F = H \cup G,\ H \in \mathcal{H},\ G \in \mathcal{G} \text{ or } F = L,\ L \in \mathcal{I}\} .$$

It is clear that each member of $\mathcal{F}$ has $a + b$ elements and that $|\cup \mathcal{F}| = N + a + 2b - 1$. We need to show that $\mathcal{F}$ satisfies (c) and (e).

Suppose $\mathcal{F}$ has property $\mathcal{B}$. Then there exists a set $B \subset \cup\mathcal{F}$ such that $0 < |B \cap F| < a + b$ for every $F \in \mathcal{F}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, either there exists a set $G_1 \in \mathcal{G}$ such that $B \cap G_1 = \phi$ or there exists a set $G_2 \in \mathcal{G}$ such that $B \supset G_2$. However $B \cap G_1 = \phi$ implies $B \cap H \neq \phi$ for each $H \in \mathcal{H}$. Hence $|B \cap T| \geq a + b$ and hence $B \supset L$ for some $L \in \mathcal{I}$. This is a contradiction and hence $B \cap G_1 = \phi$ is impossible. Also $B \cap L \neq \phi$ for all $L \in \mathcal{I}$ implies $|B \cap T| \geq b$ and hence $B \supset H_1$ for some $H_1 \in \mathcal{H}$. Thus $B \supset G_2$ is also impossible since it would imply $B \supset H \cup G_2$. Hence $\mathcal{F}$ does not have property $\mathcal{B}$.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$. We need to show that there exists a set B such that $0 < |B \cap F| < a + b$ for all $F \in \mathcal{F}'$. Let $A \in \mathcal{F} - \mathcal{F}'$. Suppose first that $A = G_1 \cup H_1$, $G_1 \in \mathcal{G}$, $H_1 \in \mathcal{H}$. Since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup\mathcal{G}$ such that $0 < |B_1 \cap G| < a$ for all $G \in \mathcal{G} - \{G_1\}$. Set $B = H_1 \cup B_1$. Then clearly $0 < |B \cap F| < a + b$ for all $F \in \mathcal{F}'$. The only other possibility is that $A \in \mathcal{I}$. Then one can take $B = A$. It follows that F' has property $\mathcal{B}$ and the proof of Theorem 1 is complete.

COROLLARY 1. $m^*_N(n)$ exists for all $n \geq 3$ and $N \geq 2n$, N even.

Proof. As was pointed out by Erdös, $m^*_{2t+1}(2)$ exists for all $t \geq 1$. Hence if we take $a = 2$, $N = 2t + 1$ in Theorem 1, we see that $m^*_{2t+2b+2}(b+2)$ exists for all $b \geq 1$, $t \geq 1$. This clearly implies the corollary.

In order to establish the existence of $m^*_N(n)$ for $n \geq 3$, N odd, we must first examine the case $n = 3$.

THEOREM 2. $m^*_{2t+1}(3)$ <u>exists for all</u> $t \geq 2$.

<u>Proof.</u> Let $\mathcal{G}$ be a family of $m^*_{2t-1}(2)$ 2-subsets of a set S of $2t - 1$ elements satisfying (a), (b), (c) and (e). We may assume $S = \{1, 2, \ldots, 2t - 1\}$ and $\mathcal{G} = \{(1, 2). (2, 3), (3, 4), \ldots, (i, i+1), \ldots, (2t - 2, 2t - 1), (1, 2t - 1)\}$, that is, the sets in $\mathcal{G}$ form the edges of an odd circuit. Let $\mathcal{F}$ be the family consisting of the following sets: $G \cup \{a\}$, $G \cup \{b\}$, $\{a, b, 1\}$, $\{a, b, 2\}$, $\{a, b, 3\}$, $\{1, 2, 3\}$, where $G \in \mathcal{G}$ and $a, b \notin S$. To prove the theorem we must show that $\mathcal{F}$ satisfies (c) and (e).

Let $B \subset U\mathcal{F}$ have non-empty intersection with each member of $\mathcal{F}$. If $a \in B$ and $b \in B$ then, since $B \cap \{1, 2, 3\} \neq \phi$, B must contain one of $\{a, b, 1\}$, $\{a, b, 2\}$, $\{a, b, 3\}$. If $a \in B$ and $b \notin B$, then $B \cap G \neq \phi$ for each $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, $B \supset G_1$ for some $G_1 \in \mathcal{G}$. Hence $B \supset G_1 \cup \{a\}$. Finally, if $a \notin B$ and $b \notin B$ then $B \supset \{1, 2, 3\}$. Thus $\mathcal{F}$ does not have property $\mathcal{B}$.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$ and let $A \in \mathcal{F} - \mathcal{F}'$. We need to show that there exists a set B such that $0 < |B \cap F| < 3$ for all $F \in \mathcal{F}$. If $A = G_1 \cup \{a\}$, $G_1 \in \mathcal{G}$, then since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset U\mathcal{G}$ such that $|B_1 \cap G| = 1$ for each $G \in \mathcal{G} - \{G_1\}$ and $B_1 \cap G_1 = \phi$. Since at least one, but not all, of $1, 2, 3$ belong to B_1 we may take $B = B_1 \cup \{b\}$. Then $0 < |B \cap F| < 3$ for all $F \in \mathcal{F}'$. The case $A = G_1 \cup \{b\}$ can be disposed of in the same way. If A is one of $\{a, b, 1\}$, $\{a, b, 2\}$, $\{a, b, 3\}$ we may choose $B = A$. Finally, if $A = \{1, 2, 3\}$ we may take $B = \{a, b\}$. It follows that $\mathcal{F}'$ has property $\mathcal{B}$.

COROLLARY 2. $m^*_N(n)$ <u>exists for</u> $n \geq 3$ <u>and</u> $N \geq 2n - 1$, N <u>odd</u>.

<u>Proof.</u> The case $n = 3$ has been taken care of in Theorem 2. For $n > 3$ take $a = 3$, $N = 2t + 1$ in Theorem 1. This shows that $m^*_{2t+2b+3}(b + 3)$ exists for $b \geq 1$, $t \geq 2$, and hence Corollary 2 holds.

From the above results we get the following upper bounds for $m^*_N(n)$:

$$m_N^*(n) \le \begin{cases} (N - 2n + 3)\binom{2n-3}{n-2} + \binom{2n-3}{n}, & \text{if } N \text{ is even}, \quad n \ge 3 \\ (2N - 4n + 8)\binom{2n-4}{n-3} + \binom{2n-4}{n}, & \text{if } N \text{ is odd}, \quad n \ge 4 . \end{cases}$$

We have been able to prove a number of additional recurrence inequalities for $m_N^*(n)$. These are given in the following theorem.

THEOREM 3. <u>The following inequalities hold</u>:

(1) $m_{KL}^*(k\ell) \le m_K^*(k)\ \{m_L^*(\ell)\}^k$, <u>if</u> $K \ge 2k - 1$, $L \ge 2\ell - 1$;

(2) $m_{N+2n}^*(n) \le n\ m_N^*(n - 2) + 2^{n-1}$, <u>if</u> n <u>is odd</u>;

(3) $m_{N+2n}^*(n) \le n\ m_N^*(n - 2) + 2^{n-1} + 2^{n-2}$, <u>if</u> n <u>is even</u>;

(4) $m_{N+2n-1}^*(n) \le (2n - 1)\ \{m_N^*(n - 2) + 1\}$.

Since the arguments are somewhat long, we shall prove only (2).

<u>Proof of (2)</u>: Let $\mathcal{G}$ be a family of $m_N^*(n - 2)$ sets satisfying (a), (b), (c) and (e). Let $F_i = \{2i - 1, 2i\}$ and let $\mathcal{H} = \{F_i : i = 1, 2, \ldots, n\}$. Let $\mathcal{T}$ be the family consisting of all sets of the form $\{a_1, a_2, \ldots a_n\}$ where $a_i \in F_i$ and $a_i = 2i - 1$ for an even number of values of i. Finally, let

$$\mathcal{F} = \{F : F = G \cup H,\ G \in \mathcal{G},\ H \in \mathcal{H} \text{ or } F = L,\ L \in \mathcal{T}\} .$$

It is clear that the number of sets in $\mathcal{F}$ is $n\ m_N^*(n - 2) + 2^{n-1}$ and that $|\cup \mathcal{F}| = N + 2n$. It remains to be shown that $\mathcal{F}$ satisfies conditions (c) and (e).

Let B be any set such that $B \cap F \ne \phi$ for each $F \in \mathcal{F}$. To show that $\mathcal{F}$ does not have property $\mathcal{B}$ we must show that B contains a member of $\mathcal{F}$.

<u>Case 1. For some</u> i <u>and</u> j, $B \supset F_i$ <u>and</u> $B \cap F_j = \phi$.

$B \cap F_j = \emptyset$ implies that $B \cap G \neq \emptyset$ for each $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, $B \supset G_1$ for some $G_1 \in \mathcal{G}$. Hence $B \supset G_1 \cup F_i$, that is, B contains a member of $\mathfrak{F}$.

<u>Case 2</u>. <u>For some</u> i, $B \supset F_i$, $|B \cap F_j| \geq 1$ <u>for all</u> j.

We may assume without loss of generality that $B \supset F_1, \ldots, F_t$ and $|B \cap F_j| = 1$ for $j = t+1, \ldots, n$. In fact we may assume $B \cap F_j = \{2j-1\}$ for $j = t+1, \ldots, t+r$ and $B \cap F_j = \{2j\}$ if $j = t+r+1, \ldots, n$. If r is even then B contains $\{2, 4, \ldots, 2t, 2t+1, \ldots 2t+2r-1, 2t+2r+2, \ldots 2n\}$ and if r is odd, B contains $\{1, 4, 6, \ldots 2t, 2t+1, \ldots, 2t+2r-1, 2t+2r+2, \ldots, 2n\}$ so that in any case B contains a member of $\mathfrak{F}$.

<u>Case 3</u>. $|B \cap F_i| \leq 1$ <u>for all</u> i, $B \cap F_j = \emptyset$ <u>for some</u> j.

In this case $\overline{B}$ satisfies the conditions of Case 2. Hence $\overline{B}$ contains a member of $\mathfrak{F}$ and thus B is disjoint from a member of $\mathfrak{F}$.

<u>Case 4</u>. $|B \cap F_i| = 1$ <u>for all</u> i.

We may assume $B \cap F_i = \{2i-1\}$ for $i = 1, 2, \ldots t$, and $B \cap F_i = \{2i\}$ for $i = t+1, \ldots, n$. If t is even then B contains $\{1, 3, \ldots, 2t-1, 2t+2, \ldots, 2n\}$, i.e. B contains a member of $\mathfrak{F}$. If t is odd, then, since n is odd, B is disjoint from $\{2, 4, \ldots, 2t, 2t+1, \ldots, 2n-1\}$, i.e., B misses a member of $\mathfrak{F}$ and this is impossible.

Since there are no other possibilities, $\mathfrak{F}$ does not have property $\mathcal{B}$.

Let $\mathfrak{F}'$ be a proper subfamily of $\mathfrak{F}$. We must show that there exists a set B such that $0 < |B \cap F| < n$ for all $F \in \mathfrak{F}'$. Let $A \in \mathfrak{F} - \mathfrak{F}'$. Suppose first that $A = G_1 \cup F_i$, $G_1 \in \mathcal{G}$. Since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup\mathcal{G}$ such that $0 < |B_1 \cap G| < n-2$ for all $G \in \mathcal{G} - \{G_1\}$. Moreover, we may assume $B_1 \supset G_1$ (for if $B_1 \not\supset G_1$ then we must have $B_1 \cap G_1 = \emptyset$ and hence instead of choosing B_1 we choose $\overline{B}_1$). Let $B = B_1 \cup F_i$. Then it is easy to see that $0 < |B \cap F| < n$ for all $F \in \mathfrak{F}$. The only other possibility is that $A \in \mathfrak{F}$. In this case we take $B = A$. Then clearly $|B \cap F| < n$ for all $F \in \mathfrak{F}'$. Moreover $B \cap F \neq \emptyset$ for all $F \in \mathfrak{F}'$ since $B \cap F = \emptyset$ implies $F = \overline{A}$, the complement of A with respect to $\{1, 2, \ldots, 2n\}$. But A contains an even number of odd elements and since n is odd,

$\bar{A}$ must contain an odd number of odd elements and hence $\bar{A} \notin \mathfrak{F}$. Thus $0 < |B \cap F| < n$ for all $F \in \mathfrak{F}'$. Hence $\mathfrak{F}'$ has property $\mathfrak{B}$. This completes the proof of (2).

The inequalities given in Theorem 3 are not very strong for large values of n and N. However, for certain small values of the arguments our results appear to be considerably better than previously known results. Denote by $m(n)$ the least integer for which there exists a family of $m(n)$. sets, each set with n elements, and which does not possess property $\mathfrak{B}$. Thus $m(n) = \min_N m_N^*(n)$. By (3) and the fact that $m_3^*(2) = 3$ we get $m_{11}^*(4) \leq 24$ and hence $m(4) \leq 24$. The best previous result is $m(4) \leq 26$. If we use (2) and $m_7^*(3) = 7$ we get $m_{17}^*(5) \leq 51$ and hence $m(5) \leq 51$. The best previous result is $m(5) \leq 88$. Similarly, we get $m(7) \leq 421$, an upper bound which is considerably smaller than that given by $m(7) \leq 708$, the best previous result.

Since we have not proved (3) and since our claim that $m(4) \leq 24$ is based on (3) we indicate briefly the construction which leads to (3). Let $\mathfrak{G}$ be a family of $m_N^*(n-2)$ sets satisfying (a), (b), (c) and (e). For $i = 1, 2, \ldots, n$ let $F_i = \{2i-1, 2i\}$ and let $\mathfrak{H} = \{F_i : i = 1, 2, \ldots, n\}$. Let $\mathfrak{I}$ be the family of all sets of the form $\{1, a_2, a_3, \ldots, a_n\}$ where $a_i \in F_i$ for $i = 2, 3, \ldots, n$. Let $\mathfrak{A}$ be the family of all sets of the form $\{2, a_2, \ldots, a_n\}$ where $a_i \in F_i$ for $i = 2, 3, \ldots, n$ and a_i is even for an odd number of values of i. Let

$$\mathfrak{F} = \{F : F = G \cup H,\ G \in \mathfrak{G},\ H \in \mathfrak{H} \text{ of } F = L,\ L \in \mathfrak{I}$$
$$\text{or } F = K,\ K \in \mathfrak{A}\}.$$

Then $|\mathfrak{F}| = n\, M_n^*(n-2) + 2^{n-1} + 2^{n-2}$ and one can show by arguments only slightly complicated than those used to prove (2) that $\mathfrak{F}$ satisfies (c) and (e).

In conclusion we mention a problem which we have not been able to settle: for fixed $n \geq 3$, does $\lim_{N \to \infty} \frac{m_N^*(n)}{N}$ exist? It follows from our results that there exist constants a_n and b_n such that for all sufficiently large N, $a_n < \frac{m_N^*(n)}{N} < b_n$.

REFERENCE

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