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# HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM-LIOUVILLE FUNCTIONS. III

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**1. Introduction.** A Sturm-Liouville function is simply a non-trivial solution of the Sturm-Liouville differential equation

$$y'' + f(x)y = 0, \tag{1.1}$$

considered, together with everything else in this study, in the real domain. The associated quantities whose higher monotonicity properties are determined here are defined, for fixed $\lambda > -1$, to be

$$M_k(W; \lambda) = M_k = \int_{x_k}^{x_{k+1}} W(x)|y(x)|^{\lambda}\, dx \qquad (k = 1, 2, \ldots), \tag{1.2}$$

where $y(x)$ is an arbitrary (non-trivial) solution of (1.1) and $x_1, x_2, \ldots$ is any finite or infinite sequence of consecutive zeros of any non-trivial solution $z(x)$ of (1.1) which may or may not be linearly independent of $y(x)$. The condition $\lambda > -1$ is required to assure convergence of the integral defining $M_k$, and the function $W(x)$ is taken subject to the same restriction.

This study continues the type of analysis of higher monotonicity properties initiated in [**12**]. Earlier work concerned itself with simple monotonicity and, for oscillatory Sturm-Liouville equations, was confined to particular cases of (1.2). Leaving aside the specializations $W(x) \equiv 1$, $z(x) \equiv y(x)$, made in [**12**] as well as elsewhere, previous studies (originating in 1836 with Sturm [**22**]) provided information on the increase or decrease of the sequence

$$\{x_{k+1} - x_k\} \qquad (k = 1, 2, \ldots).$$

This corresponds to the case $\lambda = 0$. Occasionally, monotonic properties of areas were discussed. This corresponds to the case $\lambda = 1$. None dealt with higher monotonicity in this context.

The interested reader should consult [**12**, Introduction] for further background, motivation, and references. To the comments made there, it should be added that Watson established [**26**, p. 518], and Hartman and Wintner utilized [**7**; **6**, p. 511], a result which also corresponds to the case $\lambda = 1$ and simple monotonicity. Watson's result has implications also for the case $\lambda > 0$.

Roughly speaking, it is shown here that the conditions imposed on the equation (1.1) in [**12**; **13**] imply that the sequence $\{M_k(W; \lambda)\}$ $(k = 1, 2, \ldots)$

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possesses the same order of higher monotonicity (including complete monotonicity) as that exhibited by $W(x)$. In the previous studies, $W(x) \equiv 1$ and $z(x) \equiv y(x)$.

A similar extension (Theorem 3.4) is provided for results of Vosmanský [**25**] involving $y'(x)$ and its zeros rather than $y(x)$ and its zeros.

The desirability of introducing the function $W(x)$ into the definition (1.2) of $M_k(W;\lambda)$ was suggested principally by two considerations. Both can be illustrated in the Bessel function case.

In applying the general results to the Bessel equation in the self-adjoint form (1.1), $f(x)$ becomes $1 - (\nu^2 - \frac{1}{4})x^{-2}$ and the general solution is $y(x) = x^{\frac{1}{2}}\mathscr{C}_\nu(x) \equiv x^{\frac{1}{2}}[AJ_\nu(x) + BY_\nu(x)]$, where $J_\nu(x)$ and $Y_\nu(x)$ are the usual Bessel functions of the first and second kind, respectively, of order $\nu$, and $A$ and $B$ are arbitrary constants.

When $W(x) \equiv 1, z(x) \equiv y(x)$, and $\lambda = 1$, an earlier result [**12**, Theorem 3.1] implies, in particular, that the sequence of areas

$$\int_{c_{\nu k}}^{c_{\nu,k+1}} x^{\frac{1}{2}}|\mathscr{C}_\nu(x)|\,dx \qquad (k = 1, 2, \ldots) \tag{1.3}$$

between successive positive zeros $c_{\nu k}$, $c_{\nu,k+1}$ of $\mathscr{C}_\nu(x)$ under the graph of $|x^{\frac{1}{2}}\mathscr{C}_\nu(x)|$ form a completely monotonic sequence when $|\nu| > \frac{1}{2}$. But this neither implies nor is implied by the corresponding property for the graph of the Bessel function $|\mathscr{C}_\nu(x)|$ itself. However, the results presented here cover both cases, since taking in (1.2) $W(x) = x^{-\frac{1}{2}}$, a completely monotonic function, deletes the adventitious factor $x^{\frac{1}{2}}$ from (1.3), while putting $W(x) \equiv 1$, also a completely monotonic function, yields again the previous information about (1.3).

Thus, the factor $W(x)$ in (1.2) permits extending to *complete* monotonicity the theorems of Cooke [**2**] and Makai [**17**] on the monotonicity of the areas of the successive arches between non-negative zeros of Bessel functions for $|\nu| > \frac{1}{2}$ (in fact, even for $|\nu| \geqq \frac{1}{3}$; cf. Theorem 5.4 below). (Cooke showed [**2**] that, for $\nu > -1$, the areas under the arches of the graph of $|J_\nu(x)|$ are decreasing, starting with an arch between $x = 0$ and $x = j_{\nu 1}$, even though $J_\nu(0) \neq 0$ when $-1 < \nu \leqq 0$. Our result excludes this arch unless $\nu \geqq \frac{1}{3}$.)

The other initial motivation was a search for a proof of Theorem 6.1 below, establishing an inequality involving Bessel functions. This inequality arose in a problem of numerical analysis [**28**]. A proof valid for $\nu \geqq 3/2$ was found [**15**], but seemed excessively dependent on manipulations. The proof provided in § 6 below, due to the flexibility provided by the presence of $W(x)$ in (1.2), requires little calculation and, moreover, has a greater range of validity, namely $\nu \geqq \frac{1}{3}$.

Still other applications come to hand. In a subsequent paper [**16**] there is an extension to higher monotonicity of Sonin's theorem (which establishes that the squares of the extrema of a Sturm-Liouville function form a decreasing sequence when $f(x)$ is positive and increasing). For this application, $\lambda = 2$

and $W(x)$ must be chosen suitably. (The value $\lambda = 2$ can also be applied to probability densities if $z(x) = y(x)$ is normalized appropriately.)

The scope afforded by $W(x)$ yields results concerning $\mathscr{C}_\nu(x)$ not only for $|\nu| > \frac{1}{2}$, the only range accessible to the earlier work [**12**; **13**; **25**], but also for other values of $\nu$, chiefly $|\nu| \geqq \frac{1}{3}$. Thus, this is a beginning into the problem of proving higher monotonicity properties of $\mathscr{C}_\nu(x)$ for $|\nu| < \frac{1}{2}$ analogous to (although often not identical with) those it possesses when $|\nu| > \frac{1}{2}$. Additional such results will be presented in [**16**].

The difficulties in the range $|\nu| < \frac{1}{2}$ are intrinsically greater than those arising when $|\nu| > \frac{1}{2}$. When $|\nu| < \frac{1}{2}$, the function $f(x) = 1 - (\nu^2 - \frac{1}{4})x^{-2}$ in (1.1) is decreasing in $x$; when $|\nu| > \frac{1}{2}$, $f(x)$ is an increasing function of $x$. Hence, as has been well known since Sturm's work in 1836 [**22**], the distances between consecutive zeros of $\mathscr{C}_\nu(x)$ increase when $|\nu| < \frac{1}{2}$ and decrease when $|\nu| > \frac{1}{2}$. But, as Cooke showed [**2**], there is no such change in behaviour for the areas of the arches, at least when $\mathscr{C}_\nu(x) \equiv J_\nu(x)$, $\nu > -1$. Thus, when $|\nu| > \frac{1}{2}$, both the areas and bases of the arches of $\mathscr{C}_\nu(x)$ decrease, but when $|\nu| < \frac{1}{2}$ the bases increase, while, at least for $\frac{1}{3} \leqq |\nu| < \frac{1}{2}$, the areas decrease.

Theorem 5.4 below shows that this completely monotonic (defined after (1.5)) behaviour of the areas does not change with $\nu$ for general $\mathscr{C}_\nu(x)$, at least in the range $|\nu| \geqq \frac{1}{3}$. In this theorem there arises our first need to restrict the range of $\lambda$; its assertion is not valid for all $\lambda > -1$, but only for a subinterval including $\lambda \geqq 1$, enough to give the areas.

Our information concerning Bessel functions in the range $\frac{1}{3} \leqq |\nu| < \frac{1}{2}$ is derived, on appropriate choice of $W(x)$, from Theorem 5.3 below on generalized Airy functions which are related to Bessel functions.

Finally (§ 8), we construct (and apply to Bessel functions) still other sequences possessing higher monotonicity properties. These arise from $(1 + \lambda)M_k(W; \lambda)$ on letting $\lambda \to -1+$. Consideration of this case was suggested by I.M. Gel'fand when some of the other results of this study were presented to his Seminar at the University of Moscow in 1966.

The notation used throughout is standard.

A function $\varphi(x)$ is said to be *N-times monotonic* (or monotonic of order $N$) on an interval $I$ if

$$(1.4) \qquad (-1)^n \varphi^{(n)}(x) \geqq 0 \qquad (n = 0, 1, \ldots, N; x \in I).$$

If (1.4) holds for $N = \infty$, $\varphi(x)$ is said to be *completely monotonic* on $I$. A sequence $\{M_0, M_1, \ldots\}$ is said to be *N-times monotonic* (or monotonic of order $N$), if

$$(1.5) \qquad (-1)^n \Delta^n M_k \geqq 0 \qquad (n = 0, 1, \ldots, N; k = 0, 1, \ldots).$$

Here $\Delta^0 M_k = M_k$, $\Delta M_k = M_{k+1} - M_k, \ldots, \Delta^n M_k = \Delta(\Delta^{n-1} M_k), \ldots$. If (1.5) holds for $N = \infty$, the sequence $\{M_0, M_1, \ldots\}$ is said to be *completely monotonic*. Our general theorems are stated so as to apply to $N$-times monotonic functions and sequences. Results on complete monotonicity follow from these, on putting $N = \infty$.

**2. Preliminary results.** Two lemmas on $N$-times monotonic functions are required. The first of these is useful in [**16**], where some of the results of the present paper are extended to solutions of the more general self-adjoint equation

$$(h(x)y')' + f(x)y = 0.$$

LEMMA 2.1. *Let $g(x)$ be an $N$-times differentiable function on an interval $I$, satisfying*

$$(2.1) \qquad (-1)^{n+1}g^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N; x \in I).$$

*Let $\varphi(x)$ be an $N$-times differentiable function on $g(I)$ satisfying*

$$(2.2) \qquad (-1)^{n}\varphi^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N; x \in g(I)).$$

*Then*

$$(2.3) \qquad (-1)^{n}D_x^{\,n}\varphi[g(x)] \geqq 0 \qquad (n = 1, 2, \ldots, N; x \in I).$$

*If, in addition, $g'(x) > 0$ and strict inequality holds throughout* (2.2), *or if $\varphi'(x) < 0$ and strict inequality holds throughout* (2.1), *then strict inequality holds throughout* (2.3).

*Proof.* Using the formula of Faa di Bruno (see e.g. [**9**, § 81, pp. 92–93]) for the $n$th derivative of a function of a function, we find, for each $n$,

$$(2.4) \qquad (-1)^{n}D_x^{\,n}\varphi[g(x)] = (-1)^{n}\sum_{k=1}^{n}\{\sum p_r[g'(x)]^{\alpha_1}\ldots[g^{(n)}(x)]^{\alpha_n}\}\varphi^{(k)}(g(x)),$$

where $p_r > 0$ (all $r$), and the summation inside the braces is taken over all non-negative integers $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1 + \ldots + \alpha_n = k, \qquad \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n.$$

Now, the sign of $\varphi^{(k)}(g(x))$ times each term of the sum inside the braces in (2.4) is, on account of (2.1) and (2.2),

$$\begin{aligned}(-1)^{2\alpha_1}(-1)^{3\alpha_2}\ldots(-1)^{(n+1)\alpha_n}(-1)^{k} &= (-1)^{\alpha_1+2\alpha_2+\ldots+n\alpha_n}(-1)^{\alpha_1+\alpha_2+\ldots+\alpha_n}(-1)^{k}\\ &= (-1)^{n+2k} = (-1)^{n}.\end{aligned}$$

Thus, the conclusion (2.3) holds.

If we assume, in addition, that $g'(x) > 0$ and that strict inequality holds throughout (2.2), we find that the right-hand side of (2.4) includes the (strictly) positive term

$$(-1)^{n}[g'(x)]^{n}\varphi^{(n)}(g(x))$$

(obtained by taking $k = n$; $\alpha_1 = n$, $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$). Hence there is strict inequality in (2.3). To see that the same situation prevails when $\varphi'(x) < 0$ and strict inequality holds throughout (2.1), we need only notice that in this case the right-hand side of (2.4) includes the (strictly) positive term

$$(-1)^n g^{(n)}(x)\varphi'(g(x))$$

(obtained by taking $k = 1, \alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0, \alpha_n = 1$).

This completes the proof of Lemma 2.1.

*Remarks.* In the precise form given, Lemma 2.1 appears to be new. The proof is modelled on that used in a slightly different situation [**14**, p. 95]. (In [**14**, p. 95 (15)] there is a misprint: $g^{(k)}(t)$ should be $g^{(k)}\{c_\nu(t)\}$.) Duff [**4**, p. 472] has recently proved the result in the case where $N = \infty$ and $\varphi(x) \geqq 0$. His method is similar to that given here, but does not make explicit use of the formula of Faa di Bruno. Under the additional assumption that $g(0) = 0$ and that $I = [0,\infty)$, the lemma (again with $N = \infty$) was proved by Bochner [**1**, pp. 498–499], using a more complicated method. Subsequently, Schoenberg [**21**, p. 833, Theorem 8] extended Bochner's result and proved a converse.

Next we need a generalization of [**12**, Lemma 2.2], where the special case $W(x) \equiv 1$ was considered. In the proof we use both [**12**, Lemma 2.2] and Lemma 2.1 above. An alternative but longer proof, following the lines of the proof of [**12**, Lemma 2.2], but avoiding the use of our present Lemma 2.1, could also be given.

LEMMA 2.2. *Let $p(x)$ and $W(x)$ be N-times differentiable functions on an interval $I$. Suppose that, for $x \in I$,*

$$(2.5)\qquad (-1)^n p^{(n)}(x) > 0 \qquad (n = 0, 1),$$
$$(-1)^n p^{(n)}(x) \geqq 0 \qquad (n = 2, 3, \ldots, N),$$

$$(2.6)\qquad W(x) > 0, \quad (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N).$$

*Map $I$ onto an interval of a variable $t$, through the relation $x'(t) = p(x)$. Then on this t-interval, we have for any $\sigma > 0$,*

$$(2.7)\qquad (-1)^n D_t^n[W(x)\{p(x)\}^\sigma] > 0 \qquad (n = 0, 1, \ldots, N).$$

*If strict inequality holds throughout* (2.6), *the condition $p'(x) < 0$ may be weakened to $p'(x) \leqq 0$.*

*Proof.* We know, from an extension found in [**18**] of [**12**, Lemma 2.2], that, under the hypothesis (2.5),

$$(2.8)\qquad (-1)^n D_t^n[\{p(x)\}^\sigma] > 0 \qquad (n = 0, 1, \ldots, N).$$

In particular, for $\sigma = 1$,

$$(2.9)\qquad (-1)^{n+1} x^{(n)}(t) > 0 \qquad (n = 1, 2, \ldots, N + 1).$$

Thus, using Lemma 2.1, we have

$$(2.10)\qquad (-1)^k D_t^k[W(x)] \geqq 0 \qquad (k = 0, 1, \ldots, N).$$

Leibniz's formula for the $n$th derivative of a product shows that for each $n = 0, 1, \ldots, N$,

$$(2.11)\quad (-1)^n D_t^n[W(x)\{p(x)\}^\sigma] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_t^k[W(x)](-1)^{n-k} D_t^{n-k}[\{p(x)\}^\sigma].$$

It is clear from (2.8) and (2.10) that the right-hand side of (2.11) consists entirely of non-negative terms and includes the positive term

$$(-1)^n W(x) D_t^n[\{p(x)\}^\sigma].$$

Hence (2.7) holds.

Suppose now that we replace the hypothesis $p'(x) < 0$ by $p'(x) \leqq 0$, and that we replace (2.6) by

$$(2.6')\qquad (-1)^n W^{(n)}(x) > 0 \qquad (n = 0, 1, \ldots, N).$$

The result of [**18**] is not applicable and so we cannot make the assertion (2.8). However,

$$(2.8')\qquad (-1)^n D_t^n[\{p(x)\}^\sigma] \geqq 0 \qquad (n = 0, 1, \ldots, N),$$

from [**12**, Lemma 2.2, and p. 70, Remarks (i) and (ii)]. Hence,

$$(2.9')\quad x'(t) > 0,\ (-1)^{n+1} x^{(n)}(t) \geqq 0 \qquad (n = 2, 3, \ldots, N + 1).$$

Using Lemma 2.1, modified as in its last sentence, we have

$$(2.10')\qquad (-1)^k D_t^k[W(x)] > 0 \qquad (k = 1, 2, \ldots, N).$$

From (2.8′) and (2.10′), we see that the right-hand side of (2.11) is again a sum of non-negative terms and that in the present circumstances it includes the positive term $(-1)^n D_t^n[W(x)]\{p(x)\}^\sigma$. Thus (2.7) again holds and the proof is complete.

**3. The principal results.** Throughout this section we suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$(3.1)\qquad y'' + f(x)y = 0$$

in some open interval $I$. We define

$$p(x) = [y_1(x)]^2 + [y_2(x)]^2,$$

and suppose that for some positive integer $N$, $p^{(N)}(x)$ exists in the open interval $I$. We use $y(x)$ to denote an arbitrary non-trivial solution of (3.1) on $I$, and denote by $\{x_1, x_2, \ldots\}$ any finite or infinite increasing sequence of

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consecutive zeros on $I$ of a non-trivial solution $z(x)$ of (3.1). In most applications of our results, $z(x)$ will be taken to be $y(x)$, as is the case in the earlier papers [**12**; **13**]. Also, $\{x_1', x_2', \ldots\}$ denotes any finite or infinite increasing sequence of consecutive zeros of $z'(x)$ on $I$. The function $W(x)$, defined on $I$, is assumed to be differentiable as many times as may be necessary in the context in which it is used. For fixed $\lambda > -1$ (this restriction is to assure convergence of the integrals) we define (for suitable $W(x)$)

$$(3.2)\qquad M_k = M_k(W;\lambda) = \int_{x_k}^{x_{k+1}} W(x)|y(x)|^{\lambda}\,dx \qquad (k = 1, 2, \ldots).$$

The symbol $M_0$ also occurs frequently in the following; its meaning will be analogous to that of $M_k$ ($k = 1, 2, \ldots$) and obvious from the context.

In case $f(x) > 0$ on $I$, we define, again for fixed $\lambda > -1$ and suitable $W(x)$,

$$(3.3)\qquad M_k' = M_k'(W;\lambda) = \int_{x_k'}^{x_{k+1}'} W(x)|y'(x)[f(x)]^{-\frac{1}{2}}|^{\lambda}\,dx \qquad (k = 1, 2, \ldots).$$

We then have the following result.

THEOREM 3.1.(†) *Suppose that, for $x \in I$,*

$$(3.4)\qquad \begin{aligned} &(-1)^n p^{(n)}(x) > 0 && (n = 0, 1);\\ &(-1)^n p^{(n)}(x) \geqq 0 && (n = 2, 3, \ldots, N),\end{aligned}$$

$$(3.5)\qquad W(x) > 0, \qquad (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N).$$

*Then*

$$(3.6)\qquad (-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

*The conclusion* (3.6) *remains true if the hypotheses* (3.4) *and* (3.5) *are replaced simultaneously by*

$$(3.4')\qquad p(x) > 0,\ (-1)^n p^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N),$$

$$(3.5')\qquad (-1)^n W^{(n)}(x) > 0 \qquad (n = 0, 1, \ldots, N).$$

*Finally, all of the above remains true if the factors $(-1)^n$ are deleted simultaneously from* (3.4), (3.5) (*or* (3.4′), (3.5′)) *and* (3.6).

*Proof.* The proof is similar to that of the first part of [**12**, Theorem 2.1], which deals with the special case where $W(x) \equiv 1$ and $z(x) = y(x)$. We normalize the solutions $y_1(x)$ and $y_2(x)$ so that their Wronskian is 1 and then

†The quantities $M_k$ and $M_k'$, discussed in Theorems 3.1, 3.3, and 3.4, are defined by (3.2) and (3.3), respectively. These involve the functions $y(x)$, explicitly, and $z(x)$, implicitly. These functions, it should be recalled, are *arbitrary* non-trivial solutions of (3.1) and may or may not coincide with one another, at the convenience of the user. Moreover, neither one need have any particular connection with the functions $y_1(x)$ and $y_2(x)$ used in the definition of $p(x)$.

apply [**12**, Lemma 2.3] to show that the change of variables $y(x) = [p(x)]^{\frac{1}{2}}u(t)$, $p(x) = x'(t)$, transforms the differential equation (3.1) into

$$u''(t) + u(t) = 0.$$

Since $p(x) = x'(t) > 0$ on $I$, there is a one-to-one correspondence between the zeros of $z(x)$ and those of $v(t) = [p(x)]^{-\frac{1}{2}}z(x)$. But $v(t) = A\cos(t - b)$, where $A$ and $b$ are constants, so that the consecutive zeros $t_k$ of $v(t)$ are equidistant from one another with $\Delta t_k = \pi$ $(k = 1, 2, \ldots)$, where $t_k$ is the zero of $v(t)$ corresponding to $x_k$. If $y(x) = z(x)$, then $u(t) = v(t)$.

Thus, (3.2) becomes

$$M_k = \int_{t_k}^{t_{k+1}} [W\{x(t)\}][x'(t)]^{1+\frac{1}{2}\lambda}|u(t)|^\lambda\, dt,$$

and, as in [**12**, p. 60], it follows that

$$\Delta^n M_k = \int_{t_k}^{t_{k+1}} \{\Delta_\pi^n\{W[x(t)][x'(t)]^{1+\frac{1}{2}\lambda}\}\}|u(t)|^\lambda\, dt,$$

where $\Delta_\pi F(t) = F(t + \pi) - F(t)$. A mean-value theorem for higher differences and derivatives [**20**, p. 55, no. 98; **9**, p. 74] implies that

$$\Delta^n M_k = \pi^n \int_{t_k}^{t_{k+1}} \{D_t^n\{W[x(t + \theta n\pi)][x'(t + \theta n\pi)]^{1+\frac{1}{2}\lambda}\}\}|u(t)|^\lambda\, dt,$$

where $0 < \theta(t) < 1$. Lemma 2.2 shows now that

$$(-1)^n\Delta^n M_k > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

If the hypotheses (3.4) and (3.5) are replaced by (3.4′) and (3.5′), the same result follows on using the modified form of Lemma 2.2 described in its last sentence.

The final assertion of the theorem follows on making obvious changes in the above proof; modified forms of Lemmas 2.1 and 2.2 in which the factors $(-1)^n$ are deleted and $p'(x) < 0$ $(\leqq 0)$ is replaced by $p'(x) > 0$ $(\geqq 0)$ can be employed.

This completes the proof of Theorem 3.1.

Taking $\lambda = 0$ and $W(x) = w'(x)$, Theorem 3.1 can be stated as follows.

COROLLARY 3.1. *Let $p(x)$ satisfy* (3.4) *on $I$ and suppose that*

$$(3.7)\quad w'(x) > 0, \qquad (-1)^n w^{(n+1)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N;\ x \in I).$$

*Then*

$$(3.8)\quad (-1)^n\Delta^{n+1}w(x_k) > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

*The result remains valid when* (3.4) *is replaced by* (3.4′), *provided strict inequality holds throughout* (3.7).

The following analogous result holds as well.

COROLLARY 3.2. *Let $p(x)$ satisfy* (3.4) *on $I$ and let $w(x)$ be a function on $I$ for which*

$$(3.9) \qquad \begin{aligned} &(-1)^n w^{(n+1)}(x) > 0 && (n = 0, 1), \\ &(-1)^n w^{(n+1)}(x) \geqq 0 && (n = 2, 3, \ldots, N). \end{aligned}$$

*Then*

$$(3.10) \qquad (-1)^n \Delta^n w'(x_k) > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

*The result remains valid when* (3.4) *is replaced by* (3.4′), *provided strict inequality holds throughout* (3.9).

*Proof.* We have $\Delta^0 w'(x_k) > 0$, by hypothesis (3.9). Moreover,

$$-\Delta w'(x_k) = \int_{x_k}^{x_{k+1}} [-w''(x)]|y(x)|^0\,dx,$$

and, since

$$-w''(x) > 0, \qquad (-1)^n D_x^{\,n}[-w''(x)] \geqq 0 \qquad (n = 1, 2, \ldots, N-1),$$

we see, from Theorem 3.1, that

$$(-1)^n \Delta^n(-\Delta w'(x_k)) > 0 \qquad (n = 0, 1, \ldots, N-1).$$

Thus (3.10) holds and the proof is complete.

Corollary 3.2 generalizes a result given by Widder [**27**, p. 158, Theorem 11d] to the effect that if $W(x) = w'(x)$ is completely monotonic on $a \leqq x < \infty$ and if $\delta$ is any positive number, then the sequence $\{W(a + n\delta)\}$ $(n = 0, 1, \ldots)$ is completely monotonic. To obtain Widder's result we need only apply Corollary 3.2 to the differential equation (3.1) in the case in which $f(x) = \pi^2/\delta^2$, where $p(x) = 1$, since (when $N = \infty$) strict inequality prevails throughout (3.9) for non-constant $W(x)$ [**3**, p. 98; **12**, p. 72].

For these corollaries, companion results can be formulated arising from the final sentence of Theorem 3.1.

Next we extend [**12**, (2.9)] which becomes the case $w(x) = x$ of the following theorem.

THEOREM 3.2. *Let $p(x)$ and $W(x)$ satisfy conditions* (3.4) *and* (3.5) (*or, alternatively,* (3.4′) *and* (3.5′)) *for $x \in I$ and suppose that $y(x)$ and $\bar{y}(x)$ are non-trivial solutions of* (3.1) *having respective sequences of consecutive zeros $\{x_1, x_2, \ldots\}$ and $\{\bar{x}_1, \bar{x}_2, \ldots\}$ on $I$ with $x_1 > \bar{x}_1$. If $W(x) = w'(x)$, then*

$$(3.11) \qquad (-1)^n \Delta^n \{w(x_k) - w(\bar{x}_k)\} > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

*The result remains valid if all factors $(-1)^n$ are deleted from* (3.4), (3.5), (*or* (3.4′), (3.5′)) *and* (3.11).

*Proof.* Making the same changes of variable as in the proof of Theorem 3.1, letting $t_1, t_1 + \pi, t_1 + 2\pi, \ldots$ be the sequence of $t$-values corresponding to the zeros $x_1, x_2, x_3, \ldots$, and letting $\bar{t}_1, \bar{t}_1 + \pi, \bar{t}_1 + 2\pi, \ldots$ be the sequence of $t$-values corresponding to $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$, we have $t_k - \bar{t}_k = \eta$, a number independent of $k$. Moreover,

$$\begin{aligned}(-1)^n\Delta^n\{w(x_k) - w(\bar{x}_k)\} \\ &= (-1)^n\Delta_\pi^{\,n}\{w(x(t_k)) - w(x(t_k - \eta))\} \\ &= (-1)^n\pi^n D_t^{\,n}\{w(x(t_k + \theta n\pi)) - w(x(t_k - \eta + \theta n\pi))\}\end{aligned}$$

for some $\theta$, $0 < \theta < 1$, on using the same mean-value theorem as in the proof of Theorem 3.1. This can be rewritten as

$$\begin{aligned}(-1)^n\Delta^n\{w(x_k) - w(\bar{x}_k)\} &= \pi^n \int_{t_k+\theta n\pi-\eta}^{t_k+\theta n\pi} (-1)^n D_t^{\,n+1}[w(x(t))]\,dt \\ &= \pi^n \int_{t_k+\theta n\pi-\eta}^{t_k+\theta n\pi} (-1)^n D_t^{\,n}[\{W(x(t))\}p(x)]\,dt.\end{aligned}$$

As in [**12**, p. 61], we see that $\eta = t_k - \bar{t}_k > 0$, and since the last integrand is positive by Lemma 2.2, the result (3.11) follows.

Again the last sentence of the theorem follows by making obvious changes in the above proof.

COROLLARY 3.3. *The hypotheses of Theorem* 3.2 *imply, for* $w(x) > 0$, *that*

$$(-1)^n\Delta^n\{[w(x_k)/w(\bar{x}_k)]^\alpha\} > 0 \qquad (\alpha > 0;\, n = 0, 1, \ldots, N;\, k = 1, 2, \ldots),$$

*so that, in particular, with* $\bar{x}_k = x_{k-1}$,

$$(-1)^n\Delta^n\{[w(x_{k+1})/w(x_k)]^\alpha\} > 0 \qquad (\alpha > 0;\, n = 0, 1, \ldots, N;\, k = 1, 2, \ldots).$$

*Proof.* Under these conditions, $w(x)$ may be replaced in (3.11) by $\alpha \log w(x)$, as may be seen from Lemma 2.1. Moreover,

$$(-1)^n\Delta^n\varphi(\mu_k) > 0 \qquad (n = 0, 1, \ldots, N;\, k = 1, 2, \ldots)$$

for $\varphi(x)$ *absolutely monotonic* (i.e., its successive derivatives are all non-negative) on a suitable interval, whenever

$$(-1)^n\Delta^n\mu_k > 0 \qquad (n = 0, 1, \ldots, N;\, k = 1, 2, \ldots).$$

(This can be established by minor extensions of known results [**27**, Chapter IV].) Here $\varphi(x) = e^x$.

*Remark.* For the case $\alpha = 1$, the inequalities of Corollary 3.3 can be demonstrated particularly simply. Here

$$\frac{w(x_k)}{w(\bar{x}_k)} = [w(x_k) - w(\bar{x}_k)]\frac{1}{w(\bar{x}_k)} + 1.$$

Now, Lemma 2.1, with $\varphi(x) = 1/x$, in conjunction with Corollary 3.2, shows that $(-1)^n\Delta^n\{1/w(\bar{x}_k)\} > 0$ $(n = 0, 1, \ldots, N;\ k = 1, 2, \ldots)$. The result follows since multiplication and addition preserve these properties.

If the interval $I$ is $(a, \infty)$, the principal result of [**13**] may be extended so as to give an analogue of Theorem 3.1, in which higher monotonic properties of the sequence $\{M_k\}$ are deduced from properties of the function $f(x)$ rather than from properties of $p(x)$. The precise statement follows.

THEOREM 3.3. *Let the interval $I$ be $(a, \infty)(-\infty < a < \infty)$, and let $N \geqq 2$. Let the function $f(x)$ in equation* (3.1) *satisfy* $0 < f(\infty) \leqq \infty$, *and let*

$$(3.12) \qquad f'(x) > 0, \qquad (-1)^n f^{(n+1)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N)$$

*for $x \in (a, \infty)$. Let*

$$(3.13) \qquad W(x) > 0, \qquad (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N; a < x < \infty).$$

*Then*

$$(3.14) \qquad (-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots, N; k = 1, 2, \ldots).$$

*The hypothesis $f'(x) > 0$ may be weakened to $f'(x) \geqq 0$, provided strict inequality holds throughout* (3.13).

*Proof.* As in the proof for the case $W(x) \equiv 1$, $a = 0$ [**13**], we use results of Hartman [**5**, Theorem $18.1_n$ when $f(\infty) < \infty$; Theorem $20.1_n$ when $f(\infty) = \infty$], with $n = N$, to show that, under hypothesis (3.12) (even with $f'(x) \geqq 0$), equation (3.1) has linearly independent solutions $y_1(x)$ and $y_2(x)$ on $(a, \infty)$ such that for $p(x) = [y_1(x)]^2 + [y_2(x)]^2$,

$$(3.15) \qquad p(x) > 0, \qquad (-1)^n p^{(n)}(x) \geqq 0 \qquad (n = 1, \ldots, N).$$

(Hartman's work refers to the interval $(0, \infty)$, but a translation of the variable $x$ shows it to be valid for the interval $(a, \infty)$, for $a$ finite.) Hence the last sentence of the theorem follows from the modified form of Theorem 3.1, in which the hypotheses (3.4′) and (3.5′) are assumed.

The principal assertion of the theorem follows from the principal assertion of Theorem 3.1, on noting that, under hypothesis (3.12), $p'(x) < 0$ in (3.15). For, suppose that there were a point $x_0$ such that $p'(x_0) = 0$. Then, since $p''(x) \geqq 0$, we would find that $p'(x) = 0$ on $[x_0, \infty)$ and so $p(x)$ would be constant on this interval. From [**12**, Appendix 2, p. 72] it follows that $f(x)$ would be constant on $[x_0, \infty)$, contradicting the assumption that $f'(x) > 0$ on $(a, \infty)$. This completes the proof of Theorem 3.3.

Corresponding analogues of Corollaries 3.1 and 3.2 and Theorem 3.2, in which hypotheses on $f(x)$ replace those on $p(x)$, can also be formulated and proved.

Vosmanský [**25**] proved a theorem relating the higher monotonic behaviour of the sequence $\{M_k'\}$ (as defined by (3.3)) in the special case where $W(x) \equiv 1$, and $y(x) = z(x)$ to the higher monotonic behaviour of $f(x)$. His result can be extended to the following.

THEOREM 3.4. *Let the interval $I$ be $(a, \infty)$ and let $N \geqq 4$. For the function $f(x)$ in equation* (3.1), *suppose that*

$$(3.16) \qquad f(x) > 0, f'(x) > 0, \ (-1)^n f^{(n+1)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N), \quad a < x < \infty.$$

*Let*

$$(3.17) \qquad W(x) > 0, \quad (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N-2), a < x < \infty.$$

*Then*

$$(3.18) \qquad (-1)^n \Delta^n M_k' > 0 \qquad (n = 0, 1, \ldots, N-2; k = 1, 2, \ldots).$$

*The hypothesis $f'(x) > 0$ may be weakened to $f'(x) \geqq 0$, provided strict inequality prevails throughout* (3.17).

*Proof.* We use Vosmanský's method but are able to avoid his complicated lemmas [**25**, pp. 105–107, Lemmas 1, 2] by using general results on $N$-times monotonic functions (in particular, Lemma 2.1).

As shown by Vosmanský [**25**], the function $Y(x) = y'(x)[f(x)]^{-\frac{1}{2}}$ satisfies, on $(a, \infty)$,

$$(3.19) \qquad Y'' + Q(x) Y = 0,$$

where

$$(3.20) \qquad Q(x) = f(x) - \tfrac{3}{4}[f'(x)/f(x)]^2 + \tfrac{1}{2} f''(x)/f(x).$$

Lemma 2.1, with its $g(x)$ replaced by $f(x)$ and its $\varphi(x)$ replaced by $1/x$, shows that $1/f(x)$ is $(N+1)$-times monotonic on $(a, \infty)$. Hence $[f'(x)/f(x)]^2$ is $N$-times monotonic, since the product of $N$-times monotonic functions is $N$-times monotonic. Thus $D_x[-\frac{3}{4}\{f'(x)/f(x)\}^2]$ is $(N-1)$-times monotonic. Similarly, we find that $D_x[\frac{1}{2} f''(x)/f(x)]$ is $(N-2)$-times monotonic. Thus, (3.16) implies that

$$(3.21) \qquad Q'(x) > 0, \quad (-1)^n Q^{(n+1)}(x) \geqq 0 \qquad (n = 1, 2, \ldots, N-2)$$

for $a < x < \infty$.

It is clear from (3.16) that $f(\infty) > 0$, and since Vosmanský has shown that $Q(\infty) = f(\infty)$, we have $0 < Q(\infty) \leqq \infty$.

The result (3.18) now follows on applying Theorem 3.3 to solutions of equation (3.19). Similarly, the last sentence of the present theorem follows from the last sentence of Theorem 3.3.

An analogue of Theorem 3.2 for zeros of $y'(x)$ and $\bar{y}'(x)$ can be formulated readily, and proved as was Theorem 3.2.

**4. Remarks.** (i) The results in § 3 have either " $>$ " or " $\geqq$ " in their hypotheses and "$>$" in their conclusions. It is possible to obtain similar

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results with "$\geqq$" in the conclusions if we have only "$\geqq$" throughout the hypotheses. However, we must retain strict positivity of $p(x)$ in Theorems 3.1 and 3.2. (See [**12**, p. 70, Remark (i)].)

(ii) In Theorem 3.1 we dealt with only those zeros of solutions of (3.1) which lie in the open interval $I$. This was primarily for ease of statement. As remarked in [**12**, p. 70, Remark (ii)], we may extend our results to zeros which lie at an end-point of the closure $I^*$ of $I$, provided $p(x)$ is bounded away from 0 as $x$ approaches the end-point. This is to ensure that there is a one-to-one correspondence between the zeros of $z(x)$ and those of $v(t)$. The function $W(x)$ must be chosen in such a way that the integral defining $M_k(W; \lambda)$ converges, in case one of its limits of integration is an end-point zero. An examination of the proof of Theorem 3.1 shows that no other extra hypotheses are needed to extend the results to end-point zeros. For example, in the proof of Theorem 3.1, we use a mean-value theorem for higher derivatives and differences to show that the integrand in

$$\int_{t_k}^{t_{k+1}} (-1)^n [\Delta_\pi^n \{ W[x(t)][x'(t)]^{1+\lambda/2} \}] |u(t)|^\lambda \, dt$$

is positive throughout the interval of integration. It is sufficient that it be positive throughout the interior of this interval. Hence, there is no need to apply the mean-value theorem in the case where (say) $t = t_k$, and no difficulty arises when $t_k$ corresponds to an end-point of the interval $I^*$. In particular, it is not necessary that $p(x)$ or $W(x)$ be continuous at such a point.

(iii) Theorems 3.1 and 3.2 are valid for each $N = 0, 1, 2, \ldots$ . On the other hand, we assume in Theorem 3.3 that $N \geqq 2$. This is because the proof uses the fact that $p''(x) \geqq 0$, a consequence of $f'''(x) \geqq 0$. It is not necessary to use this fact to prove the modification of the theorem noted in its last sentence, so that this form of our result is still valid in case $N = 1$. Thus, if we assume that $f'(x) \geqq 0, f''(x) \leqq 0$, $W(x) > 0$, $W'(x) < 0$, $0 < f(\infty) \leqq \infty$, we obtain $\Delta M_k < 0$. A similar remark applies to the case $N = 3$ in Theorem 3.4.

(iv) Theorem 3.3 implies that the sequence $\{M_k\}$ is completely monotonic when the function $f'(x)$ is positive and completely monotonic on $I$, provided $0 < f(\infty) \leqq \infty$, and when $W(x)$ satisfies (3.13) with $N = \infty$. *The converse is not valid.* To see this we consider the equation

$$y'' + (e^{2x} - \nu^2)y = 0 \qquad (-\infty < x < \infty); \qquad y(x) = \mathscr{C}_\nu(e^x)$$

[**26**, p. 99, (21)], in the notation of § 5. Here, $f'(x) = 2e^{2x}$ is absolutely monotonic on $(-\infty, \infty)$, so that the hypotheses on $f(x)$ in Theorem 3.3 are far from being satisfied. On the other hand, if

$$M_k = \int_{x_k}^{x_{k+1}} W(x) |\mathscr{C}_\nu(e^x)|^\lambda \, dx \qquad (\lambda > -1; k = 1, 2, \ldots),$$

where $x_k = \log c_{\nu k}$, $k = 1, 2, \ldots$, are consecutive zeros of $\mathscr{C}_\nu(e^x)$, then

$$M_k = \int_{c_{\nu k}}^{c_{\nu, k+1}} W(\log t) t^{-1-\frac{1}{2}\lambda} |t^{\frac{1}{2}} \mathscr{C}_\nu(t)|^\lambda \, dt. \tag{4.1}$$

If $W(x)$ is positive and completely monotonic on $(x_1 - \epsilon, \infty)$, $\epsilon > 0$, Lemma 2.1 implies that $W(\log t)t^{-1-\frac{1}{2}\lambda}$ is completely monotonic on $(c_{\nu 1} - \epsilon', \infty)$, for some $\epsilon' > 0$. Thus, if $|\nu| > \frac{1}{2}$, Corollary 5.1 (to be proved in the next section) may be applied to the expression (4.1) for $M_k$ to show that

$$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots ; k = 1, 2, \ldots),$$

although $f(x) = e^{2x} - \nu^2$ does not satisfy the hypotheses of Theorem 3.3.

**5. Applications to Bessel and generalized Airy functions.** Throughout this section $\mathscr{C}_\nu(x)$ is a cylinder (Bessel) function of order $\nu$, whose positive zeros, in increasing order, are $c_{\nu 1}, c_{\nu 2}, \ldots$. The symbols $d_{\nu 1}, d_{\nu 2}, \ldots$ denote the positive zeros in increasing order of any cylinder function of order $\nu$, possibly $\mathscr{C}_\nu(x)$ again. As usual, $j_{\nu k}$ is the $k$th positive zero of $J_\nu(x)$.

The principal result of this section is the following.

THEOREM 5.1. *Suppose that* $|\nu| > \frac{1}{2}$ *and that*

$$W(x) > 0,\ (-1)^n W^{(n)}(x) \geqq 0 \quad (n = 1, 2, \ldots, N),\ \delta < x < \infty, \tag{5.1}$$

*where* $0 \leqq \delta < d_{\nu 1}$.

*Let*

$$M_k = M_k(W; \lambda) = \int_{d_{\nu k}}^{d_{\nu, k+1}} W(x) x^{\lambda/2} |\mathscr{C}_\nu(x)|^\lambda \, dx \qquad (k = 1, 2, \ldots) \tag{5.2}$$

*for some fixed* $\lambda > -1$. *Then*

$$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots, N; k = 1, 2, \ldots). \tag{5.3}$$

*The result* (5.3) *remains true in case* $|\nu| = \frac{1}{2}$, *provided strict inequality holds throughout* (5.1).

*Proof.* The proof is based on [**12**, proof of the first part of Theorem 3.1] to which it reduces in case $W(x) \equiv 1$, $d_{\nu k} = c_{\nu k}$ $(k = 1, 2, \ldots)$ and $N = \infty$. As in [**12**], we consider the differential equation satisfied by $x^{\frac{1}{2}} \mathscr{C}_\nu(x)$ and take

$$p(x) = \tfrac{1}{2}\pi x \{[J_\nu(x)]^2 + [Y_\nu(x)]^2\}. \tag{5.4}$$

It is shown in [**12**, p. 62] that if $|\nu| > \frac{1}{2}$, then

$$(-1)^n p^{(n)}(x) > 0 \qquad (n = 0, 1, \ldots, N, \ldots).$$

Hence, in this case (5.3) follows on applying Theorem 3.1. In case $|\nu| = \frac{1}{2}$ we have $p(x) = 1$, so that (5.3) follows again from Theorem 3.1, under the hypotheses (3.4′) and (3.5′).

If $N \geqq 2$, an alternative and shorter proof can be based on Theorem 3.3.

COROLLARY 5.1. *Let* $|\nu| > \frac{1}{2}$ *and let* $W(x)$ *be a positive, completely monotonic function on* $(\delta, \infty)$, *where* $0 \leqq \delta < d_{\nu 1}$. *Let* $\{M_k\}$ *be defined by* (5.2). *Then*

$$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots ; k = 1, 2, \ldots). \tag{5.5}$$

*The result* (5.5) *remains true in case* $|\nu| = \frac{1}{2}$, *provided* $W(x)$ *is not constant on* $(\delta, \infty)$.

*Proof.* In case $|\nu| > \frac{1}{2}$, the corollary is the case $N = \infty$ in Theorem 5.1. The same is true in case $|\nu| = \frac{1}{2}$, but in this case it is necessary to use the remark that if $W(x)$ is a non-constant completely monotonic function on $(0, \infty)$, then in fact

$$(-1)^n W^{(n)}(x) > 0 \qquad (n = 0, 1, 2, \ldots), \quad 0 < x < \infty.$$

This property of completely monotonic functions is proved in [**12**, p. 72]. It had been established earlier by Dubourdieu [**3**, p. 98].

Remark (ii) of § 4 on end-point zeros may be applied to the case $\mathscr{C}_\nu(x) = J_\nu(x)$ when $\nu \geqq \frac{1}{2}$. For such $\nu$, $p(x)$, as defined by (5.4), is bounded away from 0 as $x \to 0+$. Moreover, for these $\nu$, $x^{\frac{1}{2}} J_\nu(x)$ has a zero at $x_0 = 0$. Denoting this zero by $j_{\nu 0}$, we have the following supplement to Theorem 5.1.

THEOREM 5.2. *Suppose that* $\nu > \frac{1}{2}$, *and that* $\lambda > -1$. *Let* $W(x)$ *be a function on* $(0, \infty)$ *which satisfies* (5.1) *with* $\delta = 0$, *and also* $W(x) = O(x^\epsilon)$, $\epsilon > -1 - (\frac{1}{2} + \nu)\lambda$, *as* $x \to 0+$. *Let*

$$M_k = \int_{j_{\nu k}}^{j_{\nu, k+1}} W(x) x^{\frac{1}{2}\lambda} |J_\nu(x)|^\lambda \, dx \qquad (k = 0, 1, \ldots). \tag{5.6}$$

*Then*

$$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots, N;\ k = 0, 1, \ldots). \tag{5.7}$$

*The result* (5.7) *is valid for* $\nu = \frac{1}{2}$, *provided strict inequality prevails throughout* (5.1), *again with* $\delta = 0$.

Next we apply Theorem 3.3 to certain generalized Airy functions, i.e., solutions of

$$y'' + \beta^2 x^{2\beta - 2} y = 0, \tag{5.8}$$

where $1 < \beta \leqq 3/2$ [**12**, p. 63]. The solutions $y(x)$ of (5.8) are expressible in terms of cylinder functions:

$$y(x) = x^{\frac{1}{2}} \mathscr{C}_{1/(2\beta)}(x^\beta).$$

Theorem 3.3 implies an extension of [**12**, Theorem 4.1] which, in part, can be formulated as follows.

THEOREM 5.3. *Suppose that* $1 < \beta \leqq 3/2$ *and that* $W(x)$ *is positive and completely monotonic on* $(\delta, \infty)$, $0 \leqq \delta < x_1$. *Let*

$$M_k = \int_{x_k}^{x_{k+1}} W(x) x^{\lambda/2} |\mathscr{C}_{1/(2\beta)}(x^\beta)|^\lambda \, dx \qquad (k = 1, 2, \ldots)$$

*for* $\lambda > -1$, *where* $x_k$ *denotes the kth positive zero of some solution of* (5.8) (*e.g.*, $x_k = (c_{\nu k})^{1/\beta}$, *where* $2\beta = 1/\nu$). *Then*

$$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, 1, \ldots;\ k = 1, 2, \ldots).$$

*Remarks.* (i) It is possible to formulate an analogous result in which $W(x)$ is assumed to be $N$-times monotonic.

(ii) In case $\mathscr{C}_{1/(2\beta)}(x) \equiv J_{1/(2\beta)}(x)$, Theorem 5.3 may be extended (by considering the corresponding $p(x)$ and using the remark (ii) of § 4), to include the zero of $J_{1/(2\beta)}(x)$ which occurs at $x = 0$.

(iii) The remaining part of [**12**, Theorem 4.1; namely formula (4.3)] can be generalized by using Theorem 3.2 above.

By combining Theorems 5.1, 5.2, and 5.3 we obtain the following result for Bessel functions of order $\nu$, $|\nu| \geqq \frac{1}{3}$.

THEOREM 5.4. *Suppose that* $|\nu| \geqq \frac{1}{3}$ *and that* $\lambda$ *satisfies both* $\lambda \geqq 0$ *and* $\lambda \geqq |\nu|^{-1} - 2$, *with* $\lambda > 0$ *when* $|\nu| = \frac{1}{2}$. *Then*

$$(5.9)\qquad (-1)^n \Delta^n \left\{ \int_{c_{\nu k}}^{c_{\nu,k+1}} |\mathscr{C}_\nu(x)|^\lambda \, dx \right\} > 0 \qquad (n = 0, 1, \ldots; k = 1, 2, \ldots).$$

*If, in addition,* $\nu \geqq \frac{1}{3}$, *then*

$$(5.10)\qquad (-1)^n \Delta^n \left\{ \int_{j_{\nu k}}^{j_{\nu,k+1}} |J_\nu(x)|^\lambda \, dx \right\} > 0 \qquad (n = 0, 1, \ldots; k = 0, 1, \ldots).$$

*Proof.* For $|\nu| \geqq \frac{1}{2}$, (5.9) follows from Corollary 5.1 on taking

$$d_{\nu k} = c_{\nu k} \qquad (k = 1, 2, \ldots)$$

and $W(x) = x^{-\frac{1}{2}\lambda}$, which is completely monotonic since $\lambda \geqq 0$. Similarly, for $\nu \geqq \frac{1}{2}$, (5.10) follows from Theorem 5.2. For the range $\frac{1}{3} \leqq \nu < \frac{1}{2}$, (5.9) is a consequence of Theorem 5.3, with $\nu = 1/(2\beta)$ and $W(x) = x^{\beta-1-\frac{1}{2}\lambda}$, which is completely monotonic since $\lambda \geqq |\nu|^{-1} - 2$. In the range $-\frac{1}{2} < \nu \leqq -\frac{1}{3}$, $\mathscr{C}_\nu(x) = AJ_\nu(x) + BJ_{-\nu}(x) = BJ_{-\nu}(x) + AJ_\nu(x)$, i.e. $\mathscr{C}_\nu(x)$ is some $\mathscr{C}_{-\nu}(x)$ with different (in fact, interchanged) constants $A, B$. But $\frac{1}{3} \leqq -\nu < \frac{1}{2}$, so that $\mathscr{C}_\nu(x)$, with $-\frac{1}{2} < \nu \leqq -\frac{1}{3}$, becomes a function for which (5.9) has been established. Thus, (5.9) holds also for $\frac{1}{3} \leqq |\nu| < \frac{1}{2}$. In case $\frac{1}{3} \leqq \nu < \frac{1}{2}$, (5.10) follows from remark (ii) following Theorem 5.3.

*Remarks.* Theorem 5.4 (with $\lambda = 1$) shows that the sequence of areas under the arches enclosed between the non-negative zeros of $|\mathscr{C}_\nu(x)|$ is completely monotonic, in case $|\nu| \geqq \frac{1}{3}$. In particular, the sequence of these areas is decreasing. This last fact was proved for $|\nu| > \frac{1}{2}$ by Makai [**17**] and, earlier, in the special case $\mathscr{C}_\nu(x) = J_\nu(x)$ by Cooke [**2**] for the overlapping range $\nu > -1$; (cf. also [**6**, pp. 511–512] and [**7**]).

It is worth noting that (5.9) and (5.10) cannot be extended by considering the respective integrands $|\mathscr{C}_\nu(x)|^\lambda$ and $|J_\nu(x)|^\lambda$ for our usual range $\lambda > -1$. In fact, if $\frac{1}{3} \leqq |\nu| < \frac{1}{2}$, these results are false for $\lambda = 0$, since by the Sturm comparison theorem [**22**, pp. 173–175], $\Delta^2 c_{\nu k} > 0$ for $|\nu| < \frac{1}{2}$, $k = 1, 2, \ldots$.

When $|\nu| = \frac{1}{3}$, our hypotheses require $\lambda \geqq 1$. Thus, the question arises as to the greatest lower bound of the set of values of $\lambda$ for which (5.9) and (5.10) remain valid for given $\nu$. Correspondingly, it would be of interest to determine the greatest lower bound of the set of values of $|\nu|$ for which these inequalities hold for given $\lambda$. This is of particular concern for $\lambda = 1$, involving the areas of the successive arches of $\mathscr{C}_\nu(x)$, since it would indicate how far the results of Cooke [**2**] and Makai [**17**] can be extended.

Finally, we exhibit several additional completely monotonic sequences involving Bessel functions. They are interesting partly because of their simplicity and, in most cases, even more because their range of validity is not the customary range $|\nu| \geqq \frac{1}{2}$. For the first four such sequences, the range is $|\nu| \geqq \frac{1}{3}$.

($*$) *Each of the sequences whose kth term is given by* (5.11), (5.12), (5.13), *and* (5.14), *respectively, is completely monotonic for* $|\nu| \geqq \frac{1}{3}$:

$$(c_{\nu k})^{-\alpha} \qquad (\alpha > 0), \tag{5.11}$$

$$(\log c_{\nu k})^{-\alpha} \quad (\alpha > 0), \text{ provided } c_{\nu 1} > 1, \tag{5.12}$$

$$(c_{\nu,k+1})^{\alpha} - (c_{\nu k})^{\alpha} \qquad (0 < \alpha \leqq \min\{1, 2|\nu|\}), \tag{5.13}$$

$$\log\,(c_{\nu,k+1}/c_{\nu k}), \tag{5.14}$$

$k = 1, 2, \ldots$ .

As applied to (5.11) and (5.12) for the range $|\nu| \geqq \frac{1}{2}$, the statement ($*$) follows from Corollary 3.2 with $W(x) = w'(x) = x^{-\alpha}$ and $W(x) = w'(x) = (\log x)^{-\alpha}$, respectively. (That the second choice of $W(x)$ is completely monotonic is seen from Lemma 2.1, with $g(x) = \log x$ and $\varphi(x) = x^{-\alpha}$.)

For the range $\frac{1}{3} \leqq \nu < \frac{1}{2}$, the assertions concerning (5.11) and (5.12) follow from applying to the differential equation (5.8) the corollary to Theorem 3.3 which is analogous to Corollary 3.2. The function $w'(x)$ is taken to be $x^{-\alpha\beta}$ and $(\log x)^{-\alpha}$, respectively, $\beta > 1$. The transition to the case $-\frac{1}{2} < \nu \leqq -\frac{1}{3}$ can be done as in the proof of Theorem 5.4 and as in the remark (i) below.

To the cases (5.13) and (5.14), Corollary 3.1 can be applied when $|\nu| \geqq \frac{1}{2}$, with $w'(x) = x^{\alpha-1}$, $0 < \alpha \leqq 1$, and $w'(x) = 1/x$, respectively. When $\frac{1}{3} \leqq \nu < \frac{1}{2}$, the analogous consequence of Theorem 3.3 suffices, with $w'(x) = x^{\alpha\beta-1}$, $0 < \alpha \leqq 1/\beta = 2\nu$, for (5.13), and, again, $w'(x) = 1/x$ for (5.14). Again, the range $-\frac{1}{2} < \nu \leqq -\frac{1}{3}$ can be handled as above.

*Remarks.* (i) The sequence

$$\{(\log c_{\nu k})^{-\alpha}\} \qquad (\alpha > 0;\, k = 2, 3, \ldots) \tag{5.12$'$}$$

is completely monotonic when $|\nu| \geqq \frac{1}{3}$. A reading of the proof for (5.12) shows that the present assertion holds if $c_{\nu 2} > 1$ for $|\nu| \geqq \frac{1}{3}$. This inequality obtains, in fact, for *all* $\nu$.

For $\nu \geqq 0$, we have $c_{\nu 2} > j_{\nu 1} \geqq j_{01} > 2.4$, from the interlacing theorem [**26**, § 15.24, p. 481], since $j_{\nu 1}$ is an increasing function of $\nu$ [**26**, p. 508 (2)].

For $|\nu| \geqq \frac{1}{2}$, the Sturm comparison theorem implies that $c_{\nu 2} - c_{\nu 1} \geqq \pi$, whence $c_{\nu 2} > \pi$.

There remains only to consider the range $0 > \nu > -\frac{1}{2}$. Here, an arbitrary cylinder function $\mathscr{C}_\nu(x) \not\equiv J_\nu(x)$ can be expressed as

$$\mathscr{C}_\nu(x) = AJ_\nu(x) + BJ_{-\nu}(x) = BJ_{-\nu}(x) + AJ_{-(-\nu)}(x) = \overline{\mathscr{C}}_{-\nu}(x), \qquad B \neq 0,$$

where $\overline{\mathscr{C}}_{-\nu}(x)$ is a cylinder function of order $-\nu$, $0 < -\nu < \frac{1}{2}$. But for this range it has already been shown that $c_{\nu 2} = \bar{c}_{-\nu,2} > j_{01} > 2.4$. When $\mathscr{C}_\nu(x) \equiv J_\nu(x)$, $-\frac{1}{2} < \nu < 0$, we have $j_{\nu 2} > j_{-\frac{1}{2},2} = (3/2)\pi > 4.6$.

Hence $c_{\nu 2} > 2.4 > 1$ for all $\nu$, and the complete monotonicity of (5.12′) is established.

(ii) The sequence $\{(\log j_{\nu k})^{-\alpha}\}$ $(\alpha > 0; k = 1, 2, \ldots)$ is completely monotonic for $\nu \geqq \frac{1}{3}$ and for $-\frac{1}{2} \leqq \nu \leqq -\frac{1}{3}$, since $j_{\nu 1} \geqq j_{-\frac{1}{2},1} = \frac{1}{2}\pi > 1$ for such $\nu$. Similarly for $\{(\log y_{\nu k})^{-\alpha}\}$ $(\alpha > 0; k = 1, 2, \ldots)$ when $\nu \geqq \frac{1}{3}$, where $y_{\nu k}$ is the $k$th positive zero of $Y_\nu(x)$.

(iii) In (5.13) taking $\alpha > 1$ would destroy the complete monotonicity of the sequence, since (5.13) becomes infinite with $k$ for $\alpha > 1$ [**26**, p. 506].

(iv) The sequence $\{\exp(-\alpha c_{\nu k})\}$ $(\alpha > 0; k = 1, 2, \ldots)$ is completely monotonic for $|\nu| \geqq \frac{1}{2}$. This follows from Corollary 3.2 with $w'(x) = e^{-\alpha x}$.

(v) The asymptotic expression for $c_{\nu k}$ ($\nu$ fixed, $k \to \infty$) suggests that the statement (*) concerning (5.11)–(5.14) may hold for *all* $\nu$, at least for sufficiently large $k$.

(vi) Sequences similar to (5.13) and (5.14), but arising from the zeros of two different Bessel functions of the same order, can be shown to be completely monotonic by choosing $w'(x)$ appropriately in Theorem 3.2. This provides a generalization of [**12**, p. 63, (3.9)]. Thus, e.g., for $|\nu| \geqq \frac{1}{3}$, the following sequences are completely monotonic $(k = 1, 2, \ldots)$:

$$\{j_{\nu k}{}^\alpha - y_{\nu k}{}^\alpha\} \qquad (0 < \alpha \leqq \min\{1, 2|\nu|\})$$

and

$$\{\log(j_{\nu k}/y_{\nu k})\}.$$

This implies the complete monotonicity of $\{(j_{\nu k}/y_{\nu k})^\alpha\}$ $(\alpha > 0)$, $|\nu| \geqq \frac{1}{3}$, as can be shown by an argument similar to the proof of Corollary 3.3. Here $\varphi(x) = e^{\alpha x}$.

Similarly, it follows that the sequence $\{(c_{\nu,k+1}/c_{\nu k})^\alpha\}$ $(\alpha > 0)$ is completely monotonic for $|\nu| \geqq \frac{1}{3}$.

Other choices of $W(x)$, coupled with special properties of particular functions, generate additional completely monotonic sequences. Thus, with

$W(x) = x^{-\frac{1}{2}-\nu}$, $\nu \geqq -\frac{1}{2}$, $\lambda = 1$,

$$
\begin{aligned}
(5.15)\quad M_k(W;1) &= \int_{c_{\nu+1,k}}^{c_{\nu+1,k+1}} W(x)x^{\frac{1}{2}}|\mathscr{C}_{\nu+1}(x)|\,dx \\
&= \left|\int_{c_{\nu+1,k}}^{c_{\nu+1,k+1}} x^{-\nu}\mathscr{C}_{\nu+1}(x)\,dx\right| \\
&= \left|\int_{c_{\nu+1,k}}^{c_{\nu+1,k+1}} D_x\{x^{-\nu}\mathscr{C}_\nu(x)\}\,dx\right| \\
&= |(c_{\nu+1,k+1})^{-\nu}\mathscr{C}_\nu(c_{\nu+1,k+1}) - (c_{\nu+1,k})^{-\nu}\mathscr{C}_\nu(c_{\nu+1,k})|, \\
&\qquad\qquad k = 1, 2, \ldots,
\end{aligned}
$$

yields a completely monotonic sequence. This sequence is of some interest in that it involves values of $\nu$, namely $\frac{1}{2} > \nu > -\frac{1}{2}$, excluded from most previous results on complete monotonicity associated with Bessel functions. (We remark, in passing, that standard asymptotic expansions suggest that even more may be true, namely that the sequence

$$\{|(c_{\nu+1,k})^{-\nu}\mathscr{C}_\nu(c_{\nu+1,k})|\} \qquad (k = 1, 2, \ldots),$$

may be completely monotonic for $\nu \geqq -\frac{1}{2}$.)

Two special cases present quite simply expressible completely monotonic sequences.

For $\nu = 0$, (5.15) becomes the completely monotonic sequence

$$(5.16)\qquad \{|\mathscr{C}_0(c_{1,k+1}) - \mathscr{C}_0(c_{1k})|\} \qquad (k = 1, 2, \ldots).$$

For $\nu = \frac{1}{2}$, we obtain the completely monotonic sequences whose respective $k$th terms are

$$
\begin{aligned}
(5.17)\quad &|\cos(j_{3/2,k+1}) - \cos(j_{3/2,k})| \\
&\qquad = |(1 + j^2_{3/2,k+1})^{-\frac{1}{2}} - (1 + j^2_{3/2,k})^{-\frac{1}{2}}| \qquad (k = 1, 2, \ldots)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.18)\quad &|\sin(y_{3/2,k+1}) - \sin(y_{3/2,k})| \\
&\qquad = |(1 + y^2_{3/2,k+1})^{-\frac{1}{2}} - (1 + y^2_{3/2,k})^{-\frac{1}{2}}| \qquad (k = 1, 2, \ldots),
\end{aligned}
$$

on taking $\mathscr{C}_{\frac{1}{2}}(x)$ to be $J_{\frac{1}{2}}(x)$ and $Y_{\frac{1}{2}}(x)$, respectively, since

$$\tan(j_{3/2,k}) = j_{3/2,k} \qquad \text{and} \qquad -\cot y_{3/2,k} = y_{3/2,k}.$$

**6. A Bessel function inequality.** As an application of Theorems 5.2 and 5.3, we obtain the following extension of an inequality, established in [**15**], which arose in a problem of numerical analysis [**28**].

THEOREM 6.1. *Let $\nu \geqq \frac{1}{3}$ and let $q$ be a number such that $q < \nu + 3/2$. Suppose also that $q > 0$ in case $\nu \geqq \frac{1}{2}$, and that $q \geqq -3\nu + 3/2$ in case $\frac{1}{3} \leqq \nu < \frac{1}{2}$. Then*

$$(6.1)\qquad \int_0^\infty x^{\frac{1}{2}-q}J_\nu(x)\,dx < \int_0^{j_{\nu 1}} x^{\frac{1}{2}-q}J_\nu(x)\,dx < 2\int_0^\infty x^{\frac{1}{2}-q}J_\nu(x)\,dx.$$

*Proof.* The hypothesis $q < \nu + 3/2$ guarantees the existence of the integrals involved in (6.1) at the end-point $x = 0$. Since $q > 0$ for all $\nu$ considered, the infinite integrals converge.

We let

$$a_k = \int_{j_{\nu k}}^{j_{\nu,k+1}} x^{\frac{1}{2}-q} |J_\nu(x)| \, dx \qquad (k = 0, 1, \ldots).$$

Using Theorem 5.2, with $\lambda = 1$, and $W(x) = x^{-q}$, we find that

$$(-1)^n \Delta^n a_k > 0 \qquad (n, k = 0, 1, \ldots) \tag{6.2}$$

in case $\nu \geqq \frac{1}{2}$. (Actually, (6.2) is needed only for $n = 0, 1, 2$, but the proof can be formulated more briefly by using (6.2) for all $n$.) Theorem 5.3, with

$$W(x) = x^{-3/2+3/(4\nu)-q/(2\nu)} \qquad \text{and} \qquad \lambda = 1,$$

and the remark on end-point zeros which follows the statement of Theorem 5.3, show that (6.2) holds also in case $\frac{1}{3} \leqq \nu < \frac{1}{2}$.

Moreover,

$$\lim_{k\to\infty} a_k = 0, \tag{6.3}$$

since $x^{\frac{1}{2}-q} J_\nu(x) \to 0$, as $x \to \infty$, and $j_{\nu,k+1} - j_{\nu k}$ is bounded as $k \to \infty$, for fixed $\nu$. Now

$$s = \sum_{k=0}^{\infty} (-1)^k a_k = \int_0^\infty x^{\frac{1}{2}-q} J_\nu(x) \, dx.$$

Obviously,

$$s < a_0. \tag{6.4}$$

As a consequence of a result given by Knopp [**8**, p. 270, Exercise 119], (6.2) and (6.3) imply that

$$a_0 < 2s. \tag{6.5}$$

The inequalities (6.4) and (6.5) are equivalent to the desired (6.1), so that the proof is complete.

In Knopp's exercise, it is necessary to make an additional assumption for strict inequality to prevail in his result. It suffices in the present instance to have $\Delta a_0 < 0$ and this follows from (6.2). It must be noted that Knopp uses $\Delta a_k$ to denote $a_k - a_{k+1}$, so that our notation differs from his.

The right-hand inequality in (6.1) extends the result of [**15**], where it was proved for $q = 1$ and $\nu \geqq 3/2$.

The inequality arose in a problem on stability of least square smoothing considered by Wilf [**28**]. An alternative approach to this problem, using Legendre polynomials rather than Bessel functions, has been provided by Trench [**23**; **24**]. A conjecture (and numerical supporting evidence) concerning a complete monotonicity property connected with (6.1) is contained in [**15**, § 2].

**7. Other applications.** We may apply Theorems 3.1 and 3.2, in the modified forms where $(-1)^n$ is deleted, to solutions of the Euler-Cauchy equation

$$y'' + (a^2/x^2)y = 0, \qquad 0 < x < \infty,$$

for $a^2 > \frac{1}{4}$ and to solutions of the equation

$$y'' + [n^2x^{-2\nu-2} - (n^2-1)/(4x^2)]y = 0, \qquad 0 < x < \infty,$$

where $n$ is a positive integer. These equations were considered in [**18**], as applications of less general results than those of the present work. The present introduction of the function $W(x)$ into the definition (3.2) of $M_k$ permits the obvious generalizations of these earlier applications.

Theorem 3.3 can be applied to give more information than is found in [**13**] on zeros of solutions of the confluent hypergeometric equation

$$y'' + (a + bx^{-1} + cx^{-2})y = 0, \qquad 0 < x < \infty,$$

still provided $a > 0$, $b \leqq 0$, $c \leqq 0$, $b + c < 0$.

In Theorem 3.4, it must be assumed that $f(x) > 0$ throughout $(a, \infty)$ in contrast to Theorem 3.3 where no such assumption is made. Thus, for the equation

$$y'' + \left\{1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right\}y = 0, \qquad |\nu| > \tfrac{1}{2},$$

Theorem 3.3 gives information on all positive zeros of $x^{\frac{1}{2}}\mathscr{C}_\nu(x)$, whereas Theorem 3.4, like Vosmanský's result [**25**], gives information only on those zeros of $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$ which fall in the interval $((\nu^2 - \frac{1}{4})^{\frac{1}{2}}, \infty)$.

However, this interval contains all, except possibly the first, of the positive zeros of $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$ [**10**, p. 144]. When the first zero of this (differentiated) function is preceded by a zero of $\mathscr{C}_\nu(x)$, the interval contains *all* the positive zeros of $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$, without exception [**10**, p. 144]. This has been shown to be the case when $\mathscr{C}_\nu(x) \equiv J_\nu(x)$ and $\nu \geqq 0$ [**10**, p. 143], but it was pointed out there that it is indeed possible for the first positive zero of $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$ to be less than $(\nu^2 - \frac{1}{4})^{\frac{1}{2}}$. It is pertinent to observe, however, that this does not occur for $\mathscr{C}_\nu(x) \equiv Y_\nu(x)$ when $\nu \geqq \frac{1}{2}$, i.e.:

(**) *All positive zeros of* $D_x\{x^{\frac{1}{2}}Y_\nu(x)\}$ *exceed* $\nu > (\nu^2 - \frac{1}{4})^{\frac{1}{2}}$ *when* $\nu \geqq \frac{1}{2}$, so that all positive zeros of this function are covered by Theorem 3.4.

This assertion follows from the more precise inequalities

$$\mu_{\nu 1} > m_{\nu 1} > j_{\nu 1}' > \nu > (\nu^2 - \tfrac{1}{4})^{\frac{1}{2}}, \qquad \nu > \tfrac{1}{2}, \tag{7.1}$$

where $\mu_{\nu 1}$ is the first positive zero of $D_x\{x^{\frac{1}{2}}Y_\nu(x)\}$, $m_{\nu 1}$, as in [**10**], the first positive zero of $D_x\{x^{\frac{1}{2}}J_\nu(x)\}$ and, as usual, $j_{\nu 1}'$, the first positive zero of $J_\nu'(x)$.

All except the first of these inequalities, i.e., $\mu_{\nu 1} > m_{\nu 1}$, have been established [**10**, p. 143 (4)]. This remaining inequality can be inferred from Nicholson's integral, a much deeper result than the ones required in [**10**]. Using this integral, Watson has shown [**26**, p. 446] that $[x^{\frac{1}{2}}J_\nu(x)]^2 + [x^{\frac{1}{2}}Y_\nu(x)]^2$ is a *decreasing* function of $x$ when $\nu > \frac{1}{2}$ (in fact, that its derivative is strictly negative), for $x > 0$. On the other hand, $[x^{\frac{1}{2}}J_\nu(x)]^2$ is an *increasing* function of $x$, $0 \leqq x \leqq m_{\nu 1}$.

Thus, $[x^{\frac{1}{2}}Y_\nu(x)]^2$ is a *decreasing* function of $x$, with non-vanishing derivative, for $0 < x \leqq m_{\nu 1}$, when $\nu > \frac{1}{2}$. Hence, its first positive extremum (occurring *a priori* either at $x = y_{\nu 1}$ or at $x = m_{\nu 1}$) cannot precede $m_{\nu 1}$. Thus, $\mu_{\nu 1} \geqq m_{\nu 1}$.

If $\mu_{\nu 1} = m_{\nu 1}$, then the Wronskian of $x^{\frac{1}{2}}J_\nu(x)$ and $x^{\frac{1}{2}}Y_\nu(x)$ would equal zero, and this is false.

Therefore, $\mu_{\nu 1} > m_{\nu 1}$ and (7.1) is proved.

As by-products of (7.1), there follow the inequalities

$$y_{\nu 2} > \mu_{\nu 1} > y_{\nu 1}' > y_{\nu 1} > \nu, \qquad \nu > \tfrac{1}{2}, \tag{7.2}$$

also implying (**).

That $y_{\nu 2} > \mu_{\nu 1}$ is clear. Also, $\mu_{\nu 1} \neq y_{\nu 1}$, as can be seen from the differential equation for $y(x) = x^{\frac{1}{2}}Y_\nu(x)$. If $\mu_{\nu 1} < y_{\nu 1}$, it would follow that $y(\mu_{\nu 1})$, a negative quantity, would be a relative maximum, so that the differential equation would imply $\mu_{\nu 1} < (\nu^2 - \frac{1}{4})^{\frac{1}{2}}$, contradicting (7.1). Thus, $\mu_{\nu 1} > y_{\nu 1}$.

Now, $x^{\frac{1}{2}}Y_\nu(x)$ is a positive, increasing function for $y_{\nu 1} < x \leqq y_{\nu 1}'$. Hence, $\mu_{\nu 1} \geqq y_{\nu 1}'$. If $\mu_{\nu 1} = y_{\nu 1}'$, then $y_{\nu 1}$ would equal this common value, as can be seen on differentiating $x^{\frac{1}{2}}Y_\nu(x)$. But $y_{\nu 1}' > y_{\nu 1} > \nu$ [**26**, p. 521] and so (7.2) is proved.

**8. The limiting case of $M_k(W;\lambda)$ as $\lambda \to -1+$.** In the definition (3.2) of $M_k(W;\lambda)$ it is necessary to require that $\lambda > -1$, so as to assure convergence of the integral. However, as I. M. Gel'fand pointed out, it is worthwhile to consider the limiting case, $\lambda \to -1+$. His suggestion leads directly, as shown in this section, to the discovery of interesting new sequences possessing higher monotonicity properties.

The calculation of the relevant limit is facilitated by the following lemma. In it, there is no requirement that $y(x)$ be a solution of a differential equation.

LEMMA 8.1. *Suppose that $y(x)$ is defined over the closed interval $[a, b]$, that it vanishes only for $x = x_k$ and changes sign at $x = x_k$, $a < x_k < b$, that $y'(x)$ exists for $a \leqq x \leqq b$, that $y'(x)$ and $y'(x)g(x)$ are Lebesgue integrable over $[a, b]$, that $g'(x)$ is Lebesgue integrable over $[x_k - \delta, x_k + \delta]$ for some $\delta > 0$. Then*

$$\lim_{\mu \to 0+} \int_a^{x_k} \mu g(x)y'(x)|y(x)|^{\mu-1}\,dx = g(x_k)[\operatorname{sgn} y(b)], \tag{8.1}$$

$$\lim_{\mu \to 0+} \int_{x_k}^{b} \mu g(x)y'(x)|y(x)|^{\mu-1}\,dx = g(x_k)[\operatorname{sgn} y(b)], \tag{8.2}$$

*so that*

$$\lim_{\mu \to 0+} \int_a^{b} \mu g(x)y'(x)|y(x)|^{\mu-1}\,dx = 2g(x_k)[\operatorname{sgn} y(b)]. \tag{8.3}$$

*Proof.* We shall confine ourselves to the proof of (8.1), that of (8.2) being quite similar, and shall consider only the case in which $y(x) > 0$, $a \leqq x < x_k$; the other case follows readily from this. Here $\operatorname{sgn} y(b) = -1$, and so the right member of (8.1) is $-g(x_k)$.

It is helpful to decompose the integral in (8.1) as follows:

$$\int_a^{x_k} = \int_a^c + \int_c^{x_k} \qquad \text{where } c = x_k - \delta.$$

(i) First, concerning the first integral on the right, we note that

$$\lim_{\mu\to 0+} \int_a^c \mu g(x)y'(x)|y(x)|^{\mu-1}\,dx = \left\{\lim_{\mu\to 0+} \mu\right\}\left\{\lim_{\mu\to 0+} \int_a^c g(x)y'(x)[y(x)]^{\mu-1}\,dx\right\}$$
$$= \{0\}\left\{\int_a^c g(x)y'(x)[y(x)]^{-1}\,dx\right\} = 0.$$

The passage to the limit is justified by the Lebesgue dominated convergence theorem, since the integrand $g(x)y'(x)[y(x)]^{\mu-1}$ is less, in absolute value, than a constant multiple of the integrable function $|g(x)y'(x)|$ when $a \leqq x \leqq c$, $y(x)$ being positive and continuous for these $x$.

(ii) In the second integral in the right member, the integrand can be rewritten as $g(x)\ D_x\{[y(x)]^\mu\}$ and integration by parts utilized. Hence

$$\int_c^{x_k} = \int_c^{x_k} g(x)[D_x\{[y(x)]^\mu\}]\,dx$$
$$= -g(c)[y(c)]^\mu - \int_c^{x_k} [y(x)]^\mu g'(x)\,dx$$
$$\to -g(c) - \int_c^{x_k} g'(x)\,dx$$
$$= -g(x_k) \qquad \text{as } \mu \to 0+,$$

since $y(c) > 0$ and $y(x)$ is continuous throughout $[c, x_k]$, vanishing only at $x = x_k$. Again, the Lebesgue dominated convergence theorem justifies the interchange of limit and integral.

This completes the proof of Lemma 8.1.

Lemma 8.1 leads directly to the construction of new higher monotonic sequences. However, it is convenient to modify slightly our earlier notation. Throughout this section, we shall take $M_k(W; \lambda)$ to be

$$\text{(8.4)} \quad M_k(W;\lambda) = \int_{\zeta_k}^{\zeta_{k+1}} W(x)|y(x)|^\lambda\,dx \qquad (\lambda > -1; k = 1, 2, \ldots),$$

where $\zeta_1, \zeta_2, \ldots$ are consecutive zeros (in the open interval $I$) of a solution $z(x)$ of (3.1) *linearly independent of* $y(x)$. The consecutive zeros of $y(x)$ in $I$ are $x_1, x_2, \ldots$ with $\zeta_1 < x_1 < \zeta_2$.

In this notation we can state now the main lemma of this section.

LEMMA 8.2. *Let $M_k(W;\lambda)$ be defined by* (8.4) *and let $W'(x)$ be integrable. Then*

$$\lim_{\lambda\to-1+}(1+\lambda)M_k(W;\lambda) = 2\frac{W(x_k)}{|y'(x_k)|}. \tag{8.5}$$

*Proof.* In (8.3), put $\mu = 1+\lambda$, $g(x) = W(x)/y'(x)$, $a = \zeta_k$, and $b = \zeta_{k+1}$. To establish the existence of $\delta > 0$ such that $g'(x)$ is integrable over $[x_k - \delta, x_k + \delta]$, it is sufficient to note the existence of $\delta > 0$ such that $y'(x) \neq 0$ in this closed interval. This is obvious, because $y'(x)$ is continuous in $[\zeta_k, \zeta_{k+1}] \subset I$, and $y'(x_k) \neq 0$, since the derivative of a non-trivial solution of (3.1) cannot vanish at interior zeros of the solution.

The difference operator being a finite linear combination, Lemma 8.2 implies the following result.

THEOREM 8.1. *If $(-1)^n\Delta^n M_k \geqq 0$ ($n = 0, 1, \ldots, N$; $k = 1, 2, \ldots$), where $M_k$ is defined by* (8.4), *with $W'(x)$ integrable and $W(x) \geqq 0$, then*

$$(-1)^n\Delta^n\left\{\left|\frac{W(x_k)}{y'(x_k)}\right|\right\} \geqq 0 \qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots). \tag{8.6}$$

*If the factor $(-1)^n$ is deleted from the hypothesis, then* (8.6) *holds with the same deletion. In particular, the hypothesis holds (and with it* (8.6)) *e.g., if the hypotheses of Theorems* 3.1 *or* 3.3 *are satisfied.*

*Remark.* It should be noted that strengthening the hypothesis by replacing "$\geqq 0$" by "$>0$" does not appear to permit, in general, a corresponding strengthening of the conclusion (8.6), due to the limit process. However, this improvement can be made for an important class of differential equations of type (3.1), satisfied, e.g., by Bessel functions of order more than $\frac{1}{2}$, Airy functions, Coulomb wave functions, and the confluent hypergeometric function for appropriate values of the parameter. A case in point deals with complete monotonicity, where $N = \infty$, for Sturm-Liouville functions defined over a half-line.

The general result can be put as follows.

THEOREM 8.2. *If the differential equation* (3.1) *is oscillatory, with $b = \infty$, $f'(x)$ continuous and non-negative, $W(x) > 0$, $W'(x) \leqq 0$, $0 < x < \infty$, and if $(-1)^n\Delta^n M_k \geqq 0$ ($k, n = 1, 2, \ldots$), then*

$$(-1)^n\Delta^n\left\{\left|\frac{W(x_k)}{y'(x_k)}\right|\right\} > 0 \qquad (n, k = 1, 2, \ldots), \tag{8.7}$$

*unless $f(x)$ is constant.*

*In particular, if $(-1)^n f^{(n+1)}(x) \geqq 0$, $W(x) > 0$, $(-1)^n W^{(n)}(x) \geqq 0$, $n = 0, 1, \ldots, a < x < \infty$, and if $0 < f(\infty) \leqq \infty$, then* (8.7) *holds, again provided $f(x)$ is not constant.*

*Proof*. To prove this theorem, it suffices to show that its hypotheses, a strengthening of those of Theorem 8.1, imply that equality can never occur in (8.6) when $N = \infty$. It has been shown [**11**] that if there should exist a single pair of values of $n$ and $k$ for which equality occurs in (8.6), when $N = \infty$, then

$$\left|\frac{W(x_k)}{y'(x_k)}\right| = \left|\frac{W(x_{k+1})}{y'(x_{k+1})}\right|$$

for all $k = 2, 3, \ldots$ .

Clearly, $y'(x_k)$ and $y'(x_{k+1})$ are of opposite sign ($k = 1, 2, \ldots$), while $W(x) > 0$, so that the above equality reduces to

$$\frac{W(x_k)}{y'(x_k)} = \frac{W(x_{k+2})}{y'(x_{k+2})} \qquad (k = 2, 3, \ldots). \tag{8.8}$$

It remains to show that the equality (8.8) implies, in the light of our other assumptions, that $f(x)$ is constant.

This follows from a formula of Wiman [**29**, p. 125 (15)] which states, in our notation,

$$[y'(x_{k+2})]^2 - [y'(x_k)]^2 = \int_{x_k}^{x_{k+2}} [y(x)]^2 f'(x)\, dx.$$

The left member cannot be positive, in view of (8.8), since $W(x)$ is positive and non-increasing. But the right member cannot be negative, since $f'(x) \geqq 0$. Hence, they must both be zero. Therefore, $f'(x) = 0$, $x_k < x < x_{k+2}$. Thus, the function $f(x)$ is a constant, as asserted.

*Remarks*. (i) The Wiman formula yields, in this way, a trivial inverse theorem for Sturm-Liouville equations: *If in* (3.1), $f(x)$ *is a continuously differentiable and monotonic function*, $\alpha \leqq x \leqq \beta$, *and there exists a solution* $y(x)$ *such that* $y(\alpha) = y(\beta) = 0$, $|y'(\alpha)| = |y'(\beta)| \neq 0$, *then* $f(x)$ *is a constant*, $\alpha \leqq x \leqq \beta$.

(ii) In case the hypotheses of Theorem 8.2 are satisfied only for $n = 1, 2, \ldots, N$, with $N$ finite, $k = 1, 2, \ldots$, the conclusion holds for $n = 1, 2, \ldots, N - 1$, except that the word "constant" needs to be replaced by the phrase "eventually constant" in both occurrences [**19**].

Similarly, under certain circumstances equality can be deleted from (8.6) when $N$ is finite. It has been shown [**19**] that if equality occurs in (8.6) for some pair of indices $n$, $k$, where $n \leqq N - 1$, $k = 1, 2, \ldots$, that $|W(x_k)/y'(x_k)|$ is eventually constant, i.e., constant for all sufficiently large $k$. This implies that

$$\frac{W(x_k)}{y'(x_k)} = \frac{W(x_{k+2m})}{y'(x_{k+2m})}$$

for a fixed such $k$ and all $m = 1, 2, \ldots$ . A knowledge of the asymptotics of the situation will often show this to be impossible. This would imply strict inequality in (8.6), except possibly for $n = N$.

Applications of Theorem 8.2 follow.

(a) $f(x) = 1 - (\nu^2 - \frac{1}{4})x^{-2}$; $|\nu| > \frac{1}{2}$; $y(x) = x^{\frac{1}{2}}\mathscr{C}_\nu(x)$.

When $W(x) = 1$, (8.7) becomes, for $|\nu| > \frac{1}{2}$,

$$(8.9)\qquad (-1)^n\Delta^n\left\{\left|\frac{1}{c_{\nu k}^{\frac{1}{2}}\mathscr{C}_\nu'(c_{\nu k})}\right|\right\} > 0 \qquad (n, k = 1, 2, \ldots; |\nu| > \tfrac{1}{2}).$$

With $W(x) = x^{-\frac{1}{2}}$, this result becomes

$$(-1)^n\Delta^n\left\{\left|\frac{1}{c_{\nu k}\mathscr{C}_\nu'(c_{\nu k})}\right|\right\} > 0 \qquad (n, k = 1, 2, \ldots; |\nu| > \tfrac{1}{2}).$$

A familiar recursion formula [**26**, p. 83, (3)] permits recasting this last inequality as

$$(8.10)\qquad (-1)^n\Delta^n\left\{\left|\frac{1}{c_{\nu k}\mathscr{C}_{\nu-1}(c_{\nu k})}\right|\right\} > 0 \qquad (n, k = 1, 2, \ldots; |\nu| > \tfrac{1}{2}).$$

One point of interest attaching to this last inequality is that it discloses higher monotonicity properties for Bessel functions of order between $-\frac{1}{2}$ and $\frac{1}{2}$, in contrast with most of our results which exclude precisely this range of $\nu$.

(b) Further results of this character can be found, but for them it is convenient to revert to Lemma 8.1. To this end, let $M_k(W; \lambda)$ for the Bessel equation be considered for $-1 < \lambda < 0$ and rewritten as

$$(8.11)\qquad M_k(W; \lambda) = \int_{d_{\nu k}}^{d_{\nu, k+1}} W(x)x^{(\frac{1}{2}+\nu)\lambda}\,\frac{|\mathscr{C}_\nu(x)|^\lambda}{x^{\nu\lambda}}\,dx,$$

with $d_{\nu k} < c_{\nu k} < d_{\nu, k+1} < c_{\nu, k+1}$. For $\nu < -\frac{1}{2}$ and $-1 < \lambda < 0$, the function $x^{-(\frac{1}{2}+\nu)\lambda}$ is completely monotonic and so may be taken as the definition of the function $W(x)$ for our purposes. The resulting sequence

$$M_k(W; \lambda) = \int_{d_{\nu k}}^{d_{\nu, k+1}} \left|\frac{\mathscr{C}_\nu(x)}{x^\nu}\right|^\lambda dx \qquad (-1 < \lambda < 0; \nu < -\tfrac{1}{2}; k = 1, 2, \ldots)$$

is, then, completely monotonic, from Theorem 5.1. Hence, so too is the sequence

$$\lim_{\lambda \to -1+} (1 + \lambda)M_k(W; \lambda) = 2\left|\frac{1}{D_x[x^{-\nu}\mathscr{C}_\nu(x)]}\right|_{x = c_{\nu k}} \qquad (k = 1, 2, \ldots),$$

where the limit has been evaluated by putting

$$\mu = 1 + \lambda, \qquad y(x) = x^{-\nu}\mathscr{C}_\nu(x), \qquad g(x) = 1/y'(x)$$

in (8.3). But [**26**, p. 83 (4) or (6)], $D_x[x^{-\nu}\mathscr{C}_\nu(x)] = -x^{-\nu}\mathscr{C}_{\nu+1}(x)$.

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Thus, the sequence

$$\left\{\left|\frac{c_{\nu k}{}^{\nu}}{\mathscr{C}_{\nu+1}(c_{\nu k})}\right|\right\} \qquad (\nu < -\tfrac{1}{2}; k = 1, 2, \ldots) \tag{8.12}$$

is completely monotonic.

For $\nu = -1$ this yields the special case $\nu = 1$ of (8.10) since

$$\mathscr{C}_{-1}(x) = -\mathscr{C}_1(x).$$

Finally, we apply Lemma 8.2, *mutatis mutandis*, to a modified form of (3.3), namely

$$M_k'(W; \lambda) = \int_{\zeta_k'}^{\zeta_{k+1}'} W(x)|y'(x)[f(x)]^{-\frac{1}{2}}|^{\lambda}\, dx \qquad (k = 1, 2, \ldots), \tag{8.13}$$

where $\zeta_k' < x_k'$ $(< \zeta_{k+1}' < x_{k+1}')$ are zeros in $I$ of $z'(x)$ and $y'(x)$, respectively, $z(x)$ and $y(x)$ being linearly independent solutions of (3.1), and $f(x) > 0$. This yields, with $W(x) \geqq 0$,

$$\begin{aligned} &\lim_{\lambda \to -1+} (1 + \lambda) M_k'(W; \lambda) \\ &= 2\left|\frac{W(x_k')[f(x_k')]^{\frac{1}{2}}}{y''(x_k')}\right| = 2\left|\frac{W(x_k')}{[f(x_k')]^{\frac{1}{2}} y(x_k')}\right| \qquad (k = 1, 2, \ldots), \end{aligned} \tag{8.14}$$

where the last expression equals its predecessor since $y'' = -f(x)y$.

Theorems 8.1 and 8.2, suitably construed, apply to the sequence in (8.14). Proper choices of appropriate functions $f(x)$ and $W(x)$ then lead to further sequences exhibiting higher monotonicity properties.

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