Canad. Math. Bull. Vol. **32** (3), 1989

ON TAKAGI FRACTAL SURFACES

BY
BENOIT DUBUC

ABSTRACT. This paper presents a new type of fractal surfaces called the *Takagi surfaces*. These are obtained by summing up pyramids of increasing (doubling) frequencies scaled by a geometric ratio b. The fractal dimension (box dimension) of the graph of these functions is shown to be log $8b$/log 2.

1. **Introduction.** The study of fractal dimension has become a key area of research in chemistry, geology, metallurgy, and computer vision among others. While various classes of fractal surfaces have been used to model natural phenomena, the value of their fractal dimension has only been conjectured. In this paper, a new class of fractal surfaces, called *Takagi surfaces*, is presented together with the theoretical evaluation of their fractal dimension. To my knowledge, this is the first time a surface model with theoretically determined fractal dimension is presented.

The Takagi surfaces can be thought of as an extension of the Modified Takagi curves which constitute a simple example of a nowhere differentiable curve (6). This paper first presents a variation on the Modified Takagi curves that allows the generation of simple random curves. The theoretical fractal dimension (box dimension) of these objects is stated, and, based on this background, we present the Takagi surfaces. The main result of the paper (Theorem 4.1) shows that these have fractal dimension $<(G_f) = \log 8b / \log 2$.

The construction presented is simple and constitutes a first step in the determination of the fractal dimension of a class of surfaces. The ultimate goal of this study would be to generalize, in some sort, the proof of Theorem 4.1 to more complex surfaces such as the Weierstrass-Mandelbrot surfaces (1) or the Brownian surfaces (7). Finally let us mention that the Takagi surfaces were first used by Dubuc, Zucker, Tricot, Quiniou and Wehbi (3) as a testbed for a new class of algorithms for estimating the fractal dimension of graphs of functions.

2. **Modified Takagi curves.** A Takagi curve is constructed from a superposition of sawtooth functions with frequency and amplitude scaled geometrically:

$$f(x) = \sum_{n=1}^{\infty} b^n \psi(2^{n-1}x) \tag{2. 1}$$

Received by the editors May 25, 1988 and, in revised form, September 15, 1988.
AMS (1980) Classification: 26B35.

$$1/2 < b < 1, x \in I = [0,1].$$

where

$$\psi(x) = 2D(x, \mathbf{Z}) \tag{2.2}$$

and $D(x, \mathbf{Z})$ is the distance to the nearest integer. The function $\psi(\cdot)$ is called the generating kernel and consists here in a sawtooth function. An example of a Modified Takagi curve with $b = 0.707$ is shown in Fig. 2.1a.

3. **Binary random Takagi curves.** Despite their irregularity, the Takagi curves have a completely ordered structure (observe Fig. 2.1a), and hence are not appropriate for modeling natural phenomena, even to a first, intuitive approximation. In this section we present a variation on the Takagi functions that introduces a new class of random curves that look more natural. We call them the *Binary Random Takagi curves (BRTC)*, and they are defined by adding a random term as follows:

$$f(x) = \sum_{n=1}^{\infty} b^n \psi(2^{n-1}x)\phi_n(2^{n-1}x) \tag{3.1}$$

with $1/2 < b < 1$ and

$$\phi_n(x) = \sum_{i=0}^{2^n-1} X_{i,n}\delta_i(x)$$

where $X_{i,n}$ are independent and identically distributed binary random variables taking a value in $\{+1, -1\}$ with $p(+1) = p(-1) = 0.5$, and

$$\delta_i(x) = \begin{cases} 1 & \text{for } x \in [i, i+1] \\ 0 & \text{otherwise.} \end{cases}$$

Thus, each triangle in the sawtooth function can either be rightside up or upside down. A typical Binary Random Takagi curve (with $b = 0.707$) is presented in Fig. 2.1b, and, as shown there, the result looks much more natural.

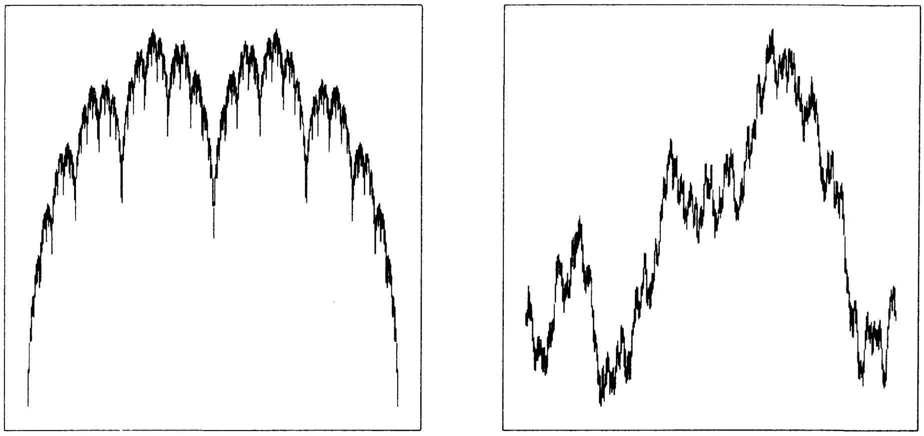
From plots of different Takagi curves made by varying b, observe that there is a direct relationship between the parameter b and the complexity (fractal dimension) of the (deterministic or binary random) Takagi curves. Indeed we have the previously proved:

THEOREM 3.1. *Let f be as defined in Eq. 3.1, with $1/2 < b < 1$ and $x \in I$. If G_f is the graph of f, $G_f = \{(x, f(x)) : x \in I\}$, then the box dimension is*

$$\Delta(G_f) = \log 4b / \log 2.$$

The proof can be found in (4).

https://doi.org/10.4153/CMB-1989-055-3 Published online by Cambridge University Press



(a) (b)

FIGURE 2.1. Examples of Takagi functions. (a) Modified Takagai curve (deterministic) with $b = 0.707$. (b) Binary Random Takagi curve with $b = 0.707$.

4. **Deterministic and Binary Random Takagi surfaces.** With this background, Takagi surfaces can now be defined analogously using surface generating kernels $\psi(\cdot)$. Let

$$f(x) = \sum_{n=1}^{\infty} b^n \psi(2^{n-1}x) \tag{4.1}$$

where $1/2 < b < 1$, and $x \in I^2$. The generating kernel will be a pyramid on I^2 with peak at $(1/2, 1/2)$; i.e., $\psi : \mathbf{R}^2 \longmapsto I$ is as follows

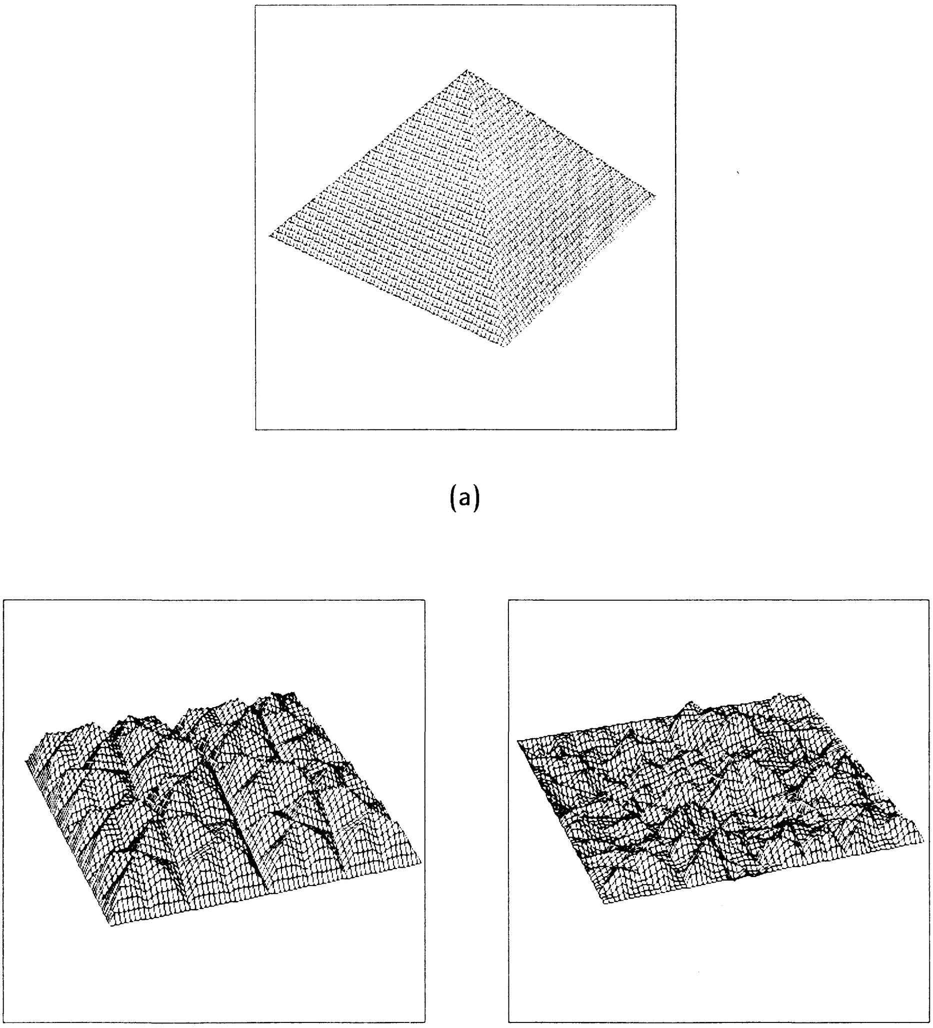
$$\psi(x) = 2D(x, F)$$
$$\text{with } F = (\mathbf{Z} \times \mathbf{R}) \cup (\mathbf{R} \times \mathbf{Z})$$
$$\text{and } d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$$

REMARK 4.1. For $x = (x_1, x_2)$ and $x + \Delta x = (x_1 + \Delta x_1, x_2 + \Delta x_2)$, we have

$$|\psi(x + \Delta x) - \psi(x)| \leqq 2 \max(|\Delta x_1|, |\Delta x_2|).$$

This is shown as follows:

$$\begin{aligned} &|\psi(x + \Delta x) - \psi(x)| \\ &\quad = 2|D(x + \Delta x, F) - D(x, F)| \\ &\quad \leqq 2d(x + \Delta x, x) \\ &\quad = 2\max(|\Delta x_1|, |\Delta x_2|) \end{aligned}$$



(a)

(b) (c)

FIGURE 4.1. The Takagi surfaces, (a) shows the generating kernel $\psi(\cdot)$ on I^2. In (b) we have a deterministic Takagi surface with $b = 0.707$ an in (c), a Binary Random Takagi surface with $b = 0.707$.

Fig. 4.1a shows a 3d-plot of the generating kernel ψ on I^2. An example of a (deterministic) Takagi surface is shown in Fig. 4.1b. Again, for realism, we add a binary random variation term:

$$f(x) = \sum_{n=1}^{\infty} b^n \phi_n(2^{n-1}x)\psi(2^{n-1}x) \tag{4.3}$$

with

$$\phi_n(x) = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} X_{i,j,n}\delta_{i,j}(x),$$

where $X_{i,j,n}$ is a random variable taking values +1 or −1 with $p(+1) = p(-1) = 0.5$, and

$$\delta_{i,j}(x) = \begin{cases} 1 & \text{if } x_1 \in [i, i+1] \wedge x_2 \in [j, j+1] \\ 0 & \text{otherwise} \end{cases}$$

The resulting surfaces are named *Binary Random Takagi* surfaces and an example with $b = 0.707$ is shown in Fig. 4.1c. The surface version of Theorem 3.1, determining the true fractal dimension of the Takagi surfaces, can then be stated as follows:

THEOREM 4.1. *Let G_f be the graph of a Takagi surface f as defined in Eq.* 4.3 *with* $1/2 < b < 1$ *and $x \in I^2$. Then the box dimension is* $\Delta(G_f) = \log 8b/\log 2$.

We begin the proof with:

LEMMA 4.1. *If p is an integer greater than* 1, *then the box dimension of a set $E, \Delta(E)$, is as follows*

$$\Delta(G_f) = \inf\{\alpha : p^{-n\alpha} w(E, p^{-n}) \underset{n\to\infty}{\longrightarrow} 0\},$$

where $w(E, p^{-n})$ is the number of boxes with side p^{-n} that intersect the set E.

A proof of this result can be found in (5).

LEMMA 4.2. *Given a function f and its graph G_f, if $c_1(2^n)^d \leqq w(G_f, 2^{-n}) \leqq c_2(2^n)^d$ where c_1 and c_2 are positive constants that do not depend on n, then $\Delta(G_f) = d$.*

PROOF. Using the result of Lemma 4.1, it is sufficient to show that

$$d = \inf\{\alpha : 2^{-n\alpha} w(G_f, 2^{-n}) \underset{n\to\infty}{\longrightarrow} 0\}.$$

Suppose now that $c_1(2^n)^d \leqq w(G_f, 2^{-n}) \leqq c_2(2^n)^d$, if we multiply by $2^{-n\alpha}$, then we get

$$c_1 2^{n(d-\alpha)} \leqq 2^{-n\alpha} w(G_f, 2^{-n}) \leqq c_2 2^{n(d-\alpha)}.$$

Let $\alpha > \Delta(G_f)$, then $2^{-n\alpha} w(G_f, 2^{-n}) \underset{n\to\infty}{\longrightarrow} 0$, which implies that $\alpha > d$, and therefore $\Delta(G_f) \geqq d$.

In order to show the other side, if we take $\alpha > d$, then $c_1 2^{n(d-\alpha)} \underset{n\to\infty}{\longrightarrow} 0$ and

$$2^{-n\alpha} w(G_f, 2^{-n}) \underset{n\to\infty}{\longrightarrow} 0,$$

which implies that $\alpha > \Delta(G_f)$. Therefore $\Delta(G_f) \geqq d$, and this completes the proof.

LEMMA 4.3. *If f is a Binary Random Takagi surface as defined in Eq.* 4.3 *and if* $I_{k,l} = [k/2^n, (k+1)/2^n] \times [l/2^n, (l+1)/2^n]$, $k, l = 0, 1, \ldots, 2^n - 1$, $n > 0$, *then*

$$c_1 b^n \leqq \max_{I_{k,l}} f(x) - \min_{I_{k,l}} f(x) \leqq c_2 b^n,$$

https://doi.org/10.4153/CMB-1989-055-3 Published online by Cambridge University Press

where c_1 and c_2 are positive constants just depending on b.

PROOF OF LEMMA 4.3. Let us define

$$f_n(x) = \sum_{m=1}^{n} b^m \phi_m(2^{m-1}x)\psi(2^{m-1}x). \tag{4.4}$$

If M is the center point of the region $I_{k,l}$ and $P_i, i = 1, \ldots, 4$, are the 4 corners of $I_{k,l}$ then

fractal dimension) was easier because of the linearity of the kernel involved. Second, instead of the binary variation term, a random phase for $\psi(x)$, similar to what has been done for the Weierstrass-Mandelbrot function (2), could also be added; i.e., we could consider the function

$$f(x) = \sum_{n=0}^{\infty} b^n \psi(2^{n-1}x + \gamma_n).$$

where γ_n is a uniformly distributed random variable on (0, 1). The objects that would result could then be used in physics and materials science as a model for rough surfaces.

ACKNOWLEDGEMENT. The author wishes to thank Steve Zucker for revising this text and Claude Tricot for his contribution in the proof of Theorem 3.1.

REFERENCES

1. M. Ausloos and D. H. Berman, *A Multivariate Weierstrass-Mandelbrot Function*, Proc. R. Soc. Lond A. vol. **400**, pp. 331–350, 1985.

2. M. V. Berry and Z. V. Lewis, *On the Weierstrass-Mandelbrot Fractal Function*, Proc. R. Soc. Lond. A. vol. **370**, pp. 459–484, 1980.

3. B. Dubuc, S. W. Zucker, C. Tricot, J. F. Quiniou and D. Wehbi, *Evaluating the fractal dimension of surfaces*, Technical Report CIM-87-19, McGill Research Center for Intelligent Machines, Montreal, Canada, Proc. Roy. Soc. London, A, 1989 (to appear).

4. B. Dubuc, *On estimating fractal dimension*, MEng Thesis, Montreal: Dept of Electrical Engineering, McGill University, 1988.

5. S. Dubuc and Z. Elqortobi, *Valeurs extrêmes de fonctions fractales*, Cahiers Centre Etudes Rech. Oper., vol. **30**, 1988.

6. M. Hata, *Fractals in Mathematics in Pattern and Waves-Qualitative Analysis of Non-linear Differential Equations*, 1986, pp. 259–278.

7. R. F. Voss, *Random Fractal Forgeries in Fundamental Algorithms for Computer Graphics*, edited by R. A. Earnshaw, Springler-Verlag, 1985, pp. 805–835.

Dept. of Electrical Engineering
Computer Vision and Robotics Lab.
McGill University
Montreal, PQ, H3A 2A7

Transposing this result in Eq. 4.6 gives us

$$\max_{I_{k,l}} f(x) - \min_{I_{k,l}} f(x) \leqq \frac{b}{(2b-1)(1-b)} b^n$$
$$= c_2 b^n.$$

PROOF OF THEOREM 4.1. First let us define

$$I_{k,l} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \left[\frac{l}{2^n}, \frac{l+1}{2^n}\right], k, l = 0, 1, \ldots, 2^n - 1$$
$$E_{k,l} = I_{k,l} \times \mathbf{R}.$$

The number of boxes intersecting $G_f \cap E_{k,l}$ is bounded as follows

$$2^n \left(\max_{I_{k,l}} f(x) - \min_{I_{k,l}} f(x)\right) \leqq w(G_f \cap E_{k,l}, 2^{-n}) \leqq 2^n \left(\max_{I_{k,l}} f(x) - \min_{I_{k,l}} f(x)\right) + 2.$$

Using this and the result of Lemma 4.3 and since $b < 1$, we get

$$c_1(2b)^n \leqq w(G_f \cap E_{k,l}, 2^{-n}) \leqq c_3(2b)^n.$$

The total number of boxes of size 2^{-n} intersecting G_f is obtained as follows

$$w(G_f, 2^{-n}) = \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} w(G_f \cap E_{k,l}, 2^{-n}), \tag{4.8}$$

then

$$c_1(8b)^n \leqq w(G_f, 2^{-n}) \leqq c_3(8b)^n,$$

which can be rewritten as follows

$$c_1 2^{n(\log 8b/\log 2)} \leqq w(G_f, 2^{-n}) \leqq c_3 2^{n(\log 8b \log 2)}. \tag{4.9}$$

Eq. 4.9 and Lemma 4.2 leads to the final result $\Delta(G_f) = \log 8b/\log 2$. □

5. **Conclusion.** In this paper, a new class of fractal objects has been described. These surfaces, that we called *Deterministic and Binary Random Takagi surfaces*, were shown to have fractal dimension $\Delta(G_f) = \log 8b/\log 2$ (Theorem 4.1). Since it is possible to develop a fast algorithm to generate Takagi surfaces, they have been used to validate various algorithms for evaluating the fractal dimension of digitized surfaces (3).

There are many variations on the Takagi functions that remain to be investigated. First, it is possible to have different generating kernels, a separable cosine function for instance. One would then have to determine a class of generating kernels for which the result of Theorem 4.1 would still hold. In our case, the problem (of estimating the fractal dimension) was easier because of the linearity of the kernel involved. Second, instead of the binary variation term, a random phase for $\psi(x)$, similar to what has been done for the Weierstrass-Mandelbrot function (2), could also be added; i.e., we could consider the function

$$f(x) = \sum_{n=0}^{\infty} b^n \psi(2^{n-1}x + \gamma_n).$$

where γ_n is a uniformly distributed random variable on (0, 1). The objects that would result could then be used in physics and materials science as a model for rough surfaces.

ACKNOWLEDGEMENT. The author wishes to thank Steve Zucker for revising this text and Claude Tricot for his contribution in the proof of Theorem 3.1.

REFERENCES

1. M. Ausloos and D. H. Berman, *A Multivariate Weierstrass-Mandelbrot Function*, Proc. R. Soc. Lond A. vol. **400**, pp. 331–350, 1985.

2. M. V. Berry and Z. V. Lewis, *On the Weierstrass-Mandelbrot Fractal Function*, Proc. R. Soc. Lond. A. vol. **370**, pp. 459–484, 1980.

3. B. Dubuc, S. W. Zucker, C. Tricot, J. F. Quiniou and D. Wehbi, *Evaluating the fractal dimension of surfaces*, Technical Report CIM-87-19, McGill Research Center for Intelligent Machines, Montreal, Canada, Proc. Roy. Soc. London, A, 1989 (to appear).

4. B. Dubuc, *On estimating fractal dimension*, MEng Thesis, Montreal: Dept of Electrical Engineering, McGill University, 1988.

5. S. Dubuc and Z. Elqortobi, *Valeurs extrêmes de fonctions fractales*, Cahiers Centre Etudes Rech. Oper., vol. **30**, 1988.

6. M. Hata, *Fractals in Mathematics in Pattern and Waves-Qualitative Analysis of Non-linear Differential Equations*, 1986, pp. 259–278.

7. R. F. Voss, *Random Fractal Forgeries in Fundamental Algorithms for Computer Graphics*, edited by R. A. Earnshaw, Springler-Verlag, 1985, pp. 805–835.

Dept. of Electrical Engineering
Computer Vision and Robotics Lab.
McGill University
Montreal, PQ, H3A 2A7