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NEIGHBOR RELATION AND NEIGHBOR HOMOMORPHISM OF HJELMSLEV GROUPS

FRIEDER KNÜPPEL AND MICHAEL KUNZE

The geometry of Hjelmslev groups is a comprehensive plane metric geometry. It supplies, for example, an approach to euclidean, hyperbolic, elliptic, Minkowskian and Galilean geometry.

The subject of this geometry are Hjelmslev groups and their group planes. (The definition of Hjelmslev groups and some basic concepts and propositions of this theory can be found in the second edition of Bachmann's book [**1**] (pages 318–328). A similar report in English is the lecture [**3**]. A comprehensive introduction is developed in the key work [**2**]. The first part of this work was translated by Garner [**4**]. An abstract is given in Math. Rev. *52*, 9066 (1976).) The group plane arises from giving geometric names to some group theoretical facts. The group plane of a Hjelmslev group is an incidence structure with orthogonality, and the Hjelmslev group acts on this plane as a group of motions.

Within the geometry of Hjelmslev groups ideas from J. Hjelmslev's "Allgemeine Kongruenzlehre" (AKL) appear in a more general shape. In group planes of Hjelmslev groups, the first or the second or both the following classical axioms can be false.

(V) *Any two points are incident with at least one line;*

(E) *Any two distinct points are incident with at most one line.*

A good deal of Hjelmslev's work, mainly the third and the fifth communication of AKL is devoted to the study of a neighbor relation for points and, subsequently, also for lines. These relations serve to make planes not satisfying (E) more accessible. Hjelmslev suggests to merge points which are neighbors and also lines which are neighbors, in order to obtain a geometric structure satisfying (E).

The goal of our article is the definition and study of a neighbor relation for Hjelmslev groups (more precisely: for the group plane of Hjelmslev groups). Diverging from Hjelmslev's definition we shall define: two points A, B are called *neighbors* if and only if there is a rotation fixing A and B but not every point. With this we shall continue: two lines a, b are called *neighbors* if and only if the product of the reflection in the line a and the reflection in the line b moves every point X into a neighbor of X. As a first result we shall, for example, show that orthogonal projections of the point set preserve the neighbor relation.

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It is natural to ask for a description of Hjelmslev groups with transitive neighbor relation. In our Theorem 1 we shall show that a Hjelmslev group has transitive neighbor relation if the following three properties are fulfilled.

(Vf*) Any two points which are non-neighbors are incident with at least one line; there exist non-neighbor points.

(W) There are orthogonal lines a, b and orthogonal lines c, d such that any two of the lines a, b, c, d intersect in just one point.

(Z) Let a, b, c be lines through a point C and suppose a is orthogonal to b; then C is the unique intersection point of a, c, or C is the unique intersection point of b, c.

Each of these additional axioms was already introduced in earlier studies. (The first additional axiom was originally introduced in a somewhat stronger version denoted (Vf); this version claims not only existence of non-neighbor points but instead demands that to each line g there is a pair of non-neighbor points lying on g.)

The geometry of Hjelmslev groups subsumes the plane absolute geometry which is discussed in the main part of Bachmann's book [**1**]. The groups of plane absolute geometry defined by the system of axioms applied there ([**1**] § 3.2; see also § 20.3) are called AGS groups in agreement with the title of the book. They are Hjelmslev groups which fulfill (V) and (E).

Given a Hjelmslev group (with more than one point) satisfying (Vf*), (W) and (Z) we shall canonically construct a Hjelmslev homomorphism onto an AGS group such that points and also lines have the same image if and only if they are neighbors (Theorem 4).

Supplementary to Theorem 1 we shall discuss two special situations in Section 2*. Theorem 2 applies if (V) is valid and Theorem 3 applies to Hjelmslev groups having an ordered group plane. Finally we shall construct a Hjelmslev group over a local ring and give an algebraical description for its neighbor relation and the homomorphism given by Theorem 4.

Remark. In the above mentioned third communication of AKL Hjelmslev presupposed (V). Hjelmslev called two points neighbors if and only if they are incident with more than one line. He proved the transitivity of this neighbor relation in the same article. In this communication strong axioms were assumed (order, free mobility) in addition to (V). In his paper "Euklidische Ebenen mit Nachbarelementen" (Euclidean planes with neighboring elements) W. Klingenberg also assumed validity of (V) and used Hjelmslev's neighbor relation. From propositions of this article transitivity of Hjelmslev's neighbor relation can be deduced for Hjelmslev groups satisfying (V), (W) and (Z).

Note. Hjelmslev's definition implies: 1. If any two points are incident with more than one line then any two points are neighbors. 2. Unjoinable points are non-neighbors. These consequences may indicate an inflexibility of Hjelmslev's neighbor relation, and that its efficiency seems to be linked to the assumption of (V).

1. Definition and elementary properties of the neighbor relation. Throughout this article "Hjelmslev group" means "non-elliptic Hjelmslev group". Let (G, S) be a Hjelmslev group and P its point set. Let S^{even} (S^{odd}) be the set of products of an even (odd) number of lines. If $\alpha \in S^{\text{even}}$ and if the set

$$\mathrm{F}(\alpha) := \{A \in P: A^\alpha = A\}$$

of points which are fixed by α is not empty, then α is called a *rotation*. More exactly we might say: the motion $X \mapsto X^\alpha$, $x \mapsto x^\alpha$ of the group plane of (G, S) is called the rotation induced by α. If $A \in \mathrm{F}(\alpha)$ then there exist lines a and b passing through A such that $\alpha = ab$. Let $\mathrm{D}(A)$ denote the set of rotations fixing A,

$$\mathrm{D}(A) := \{\alpha \in S^{\text{even}}: A \in \mathrm{F}(\alpha)\}.$$

There may exist elements $\alpha \in S^{\text{even}} \backslash \{1\}$ such that $\mathrm{F}(\alpha) = P$. The center $Z(S^{\text{even}})$ of the subgroup S^{even} of G consists of exactly those elements of S^{even} which fix every point.

For a rotation α let

$$\mathfrak{S}(\alpha) := \{c \in S: \alpha c \in S\}.$$

Then $(\mathrm{N}_G(\mathrm{F}(\alpha)), \mathrm{S}(\alpha))$ (where $\mathrm{N}_G(\mathrm{F}(\alpha))$ is the normalizer of $\mathrm{F}(\alpha)$ in G) is a Hjelmslev subgroup of (G, S) and $\mathrm{F}(\alpha)$ is its set of points. We call this Hjelmslev subgroup the *spot* of the rotation α. A spot of a rotation is locally complete: if $A \in \mathrm{F}(\alpha)$ and $b \in S$ with $b|A$ then $b \in \mathrm{S}(\alpha)$. Therefore a spot of a rotation has the

"Thales property": *Let A, B be points of a spot of a rotation and $a|A$ and $b|B, a$. Then ab is also a point of the spot.*

A and B are called *neighbors* if and only if there exists a rotation fixing A and B but not every point. This relation will be denoted by $A \supset B$. Thus,

Definition. Let (G, S) be a Hjelmslev group and P its point set. For $A, B \in P$ define $A \supset B$ if and only if

$$\exists\ \alpha \in S^{\text{even}}: A, B \in \mathrm{F}(\alpha) \neq P.$$

Or, equivalently, if and only if

$$\mathrm{D}(A) \cap \mathrm{D}(B) \backslash \mathrm{Z}(S^{\text{even}}) \neq \emptyset.$$

If A, B are non-neighbors then we say A *distant* B according to the definition of Salow [**10**]. Hence A and B are neighbors if and only if they are points of a spot of a rotation which is not the entire group. If A, B are neighbors in the

Hjelmslev group (G, S) and if (G, S) is a Hjelmslev subgroup of the Hjelmslev group (G', S'), then A, B are neighbors in (G', S'). Our definition of $A \supset B$ depends not only on the set of lines joining A to B but also on the entire Hjelmslev group (G, S).

Examples. In a Hjelmslev group (with more than one point) satisfying (E) two points are neighbors if and only if they are equal. If in a Hjelmslev group (V) holds and $Z(S^{\text{even}}) = \{1\}$ then two points are neighbors if and only if they are incident with more than one line.

Motions of the group plane preserve the neighbor relation:

(*1*) *If $A \supset B$ then $A^\alpha \supset B^\alpha$ for points A, B and $\alpha \in G$.*

The relation $\supset$ is symmetric by definition:

(*2*) *If $A \supset B$ then $B \supset A$.*

Furthermore the relation is also reflexive:

(*3*) *$A \supset A$ for each point A, except for the trivial case that P consists of one point only.*

A main concern of this article is to find sufficient conditions for the transitivity of $\supset$.

If A, B are points in the same spot F of a rotation and if A, B possess a mid-point M (this means $A^M = B$) then M is also a point of F. From this it follows that

(*4*) *$A \supset M$ if and only if $A \supset A^M$.*

If a point A can be moved into a point B, that is if there is an $\alpha \in G$ satisfying $A^\alpha = B$, then A, B have exactly one mid-point, denoted $M_{A,B}$. Therefore it follows from (4) that a point A is a neighbor of A^α if and only if A is a neighbor of the mid-point M_{A,A^α}

A consequence of the Thales property of spots is

(Th) *Suppose that $a|A$; $b|B$, a and $A \supset B$. Then $A, B \supset ab$.*

Orthogonal projections preserve the neighbor relation. In what follows (A, g) denotes the unique perpendicular through A to g; $(A, g)g$ is the foot.

(*5*) (*a*) *If $A \supset B$ then $(A, g)g \supset (B, g)g$ for any line g.*

(*b*) *If $(A, g)g \supset (B, g)g$ then there exists a point $E|(B, g)$ satisfying $A \supset E$; one such point is $E := (A, (B, g))(B, g)$.*

Proof (see also Salow [**12**], (2.3)). Let A, B, g be given and $b|B, g$; $d|A, g$; $C := bg$, $D := dg$; $a|A, b$ and $E := ab$ (see figure). Then for all lines e, h satisfying $e|A$ and $h|D, e$ the following conditions are equivalent:

$E \in \mathrm{F}(de)$;

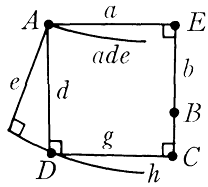
$E|ade$ (because $E|a$);

$E\ ade \in S$;

$CDe \in S$ (because $Ead = bd = CD$);

$C|h$;

$C \in \mathrm{F}(gh)$.

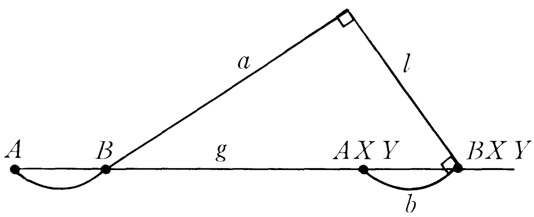


(a) Suppose that $A \supset B$. We have to show that $C \supset D$. From (Th) we know $A \supset E$. Hence there is an $\alpha = \mathrm{D}(A) \cap \mathrm{D}(E) \backslash Z(S^{\mathrm{even}})$. Let $e := d\alpha$ and $h := (D, e)$. From $E \in \mathrm{F}(\alpha) = \mathrm{F}(de)$ and the above equivalences we obtain $C, D \in \mathrm{F}(gh)$. Assuming $\mathrm{F}(gh) = P$ we would have $A = A^{gh} = A^{dgh} = A^{Dh}$ and consequently $A, h|e, Dh$. Therefore $e = Dh$, $eh = D = dg$ and hence $\mathrm{F}(\alpha) = \mathrm{F}(de) = \mathrm{F}(gh) = P$ which is a contradiction. Consequently $\mathrm{F}(gh) \neq P$ and $C \supset D$.

(b) Now suppose that $C \supset D$. Then there is an $\alpha \in \mathrm{D}(C) \cap \mathrm{D}(D) \backslash Z(S^{\mathrm{even}})$. Let $h := g\alpha$ and $e := (A, h)$. Then $C \in \mathrm{F}(\alpha) = \mathrm{F}(gh)$ and by the above equivalences $A, E \in \mathrm{F}(de)$. Assuming $\mathrm{F}(de) = P$ we would have $D|e$ and thus $D|e|h$. $D = eh = dg$, but $\mathrm{F}(de) = \mathrm{F}(gh) = P$ is a contradiction.

(6) *Let A, B, X, Y be points on a line g and $A \supset B$. Then $AXY \supset BXY$.*

Proof. By hypothesis we have a rotation $ag \in \mathfrak{D}(A) \cap \mathfrak{D}(B) \backslash Z(S^{\mathrm{even}})$. It follows that $a|A, B$. Let $l := (BXY, a)$ and $b := lBXY$. By [**2**, 3.6] we have



$b|A \cdot B \cdot BXY = AXY$ and therefore $bg \in \mathrm{D}(AXY) \cap \mathrm{D}(BXY)$. Assuming that $bg \in Z(S^{\mathrm{even}})$ we would have $b|B$ and $a, b|B, l$, hence $a = b$. But $ag \in Z(S^{\mathrm{even}})$ is a contradiction.

(7) *Let $\alpha \in G$ and $A \supset B$. If $\alpha \in S^{\mathrm{odd}}$ or if A, B have a joining line then $M_{A,A^\alpha} \supset M_{B,B^\alpha}$.*

Proof. Suppose that $\alpha \in S^{\mathrm{odd}}$. According to the representation theorem ([**2**], 3*.2) we can write $\alpha = gab$ where $g|a, b$. Thus for each point X we have $M_{X,X^\alpha} = (X, g)g\, ab$. $A \supset B$ and (5) imply $(A, g)g \supset (B, g)g$, and from (6) we obtain $(A, g)gab = (A, g)g \cdot ag \cdot gb \supset (B, g)g \cdot ag \cdot gb = (B, g)gab$, i.e. $M_{A,A^\alpha} \supset M_{B,B^\alpha}$.

Finally, if $\alpha \in S^{\mathrm{even}}$ and A is joined to B by a line c we have $A^\alpha = A^{c\alpha}$ and $B^\alpha = B^{c\alpha}$. Therefore we may repeat the above proof writing $c\alpha$ instead of α.

A special case of (7) is

(8) *Suppose A and B have a joining line. If $A \supset B$ and $AB = CD$ then $C \supset D$.*

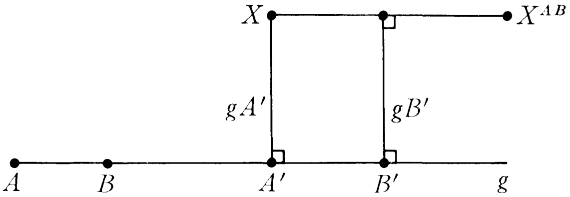
Proof. This follows from $M_{A,A^{BD}} = C$, $M_{B,B^{BD}} = D$ and (7).

(9) *Suppose A is joined to B.*

(a) *If $A \supset B$ then $X \supset X^{AB}$ for each point X.*

(b) *If there is a point X such that $X \supset X^{AB}$ then $A \supset B$.*

Proof. Let $g|A, B$ and $A' := (X, g)g$ and $B' := A'AB$ where X is any point. Then $X^{AB} = X^{A'B'} = X^{gA'gB'} = X^{gB'}$. Hence $(X, gB')gB'$ is the mid-point of X and X^{AB}. From (8), (5) and (4) we conclude that the following statements are equivalent: $A \supset B$; $A' \supset B'$; $X \supset (X, gB')gB'$; $X \supset X^{AB}$.



Proposition (9) suggests the following definition.

Definition. The line a is called a *neighbor* of the line b, denoted by $a \supset b$, if $X \supset X^{ab}$ for every point X. If a is a non-neighbor of b we write a *distant* b.

We now state some properties of the relation $\supset$ on the set of lines.

(10) *If $a \supset b$ then $a^\alpha \supset b^\alpha$ for each $\alpha \in G$.*
$a \supset a$ if there exists more than one point.
If $a \supset b$ then $b \supset a$.
If $ab = cd$ and $a \supset b$ then $c \supset d$.
Let $ab = AB$. Then $a \supset b$ if and only if $A \supset B$.
If $a, b|g$, then $a \supset b$ if and only if $ag \supset bg$.
If $a, m|g$, then $a \supset m$ if and only if $a \supset a^m$.

Proof. The first four assertions are obvious. To prove the fifth one, we notice that $ab = AB$ implies that A is joined to B ([**2**], 3.1) and we can apply (9). The sixth assertion follows immediately, and to prove the last one we use (4) and $(ag)^{mg} = a^m g$.

The following lemma is proved in [**2**, § 5, (*xi*)] and also in [**10**, 3.2].

LEMMA. *Let α be a rotation and $a, b \in \mathrm{S}(\alpha)$ such that $\mathrm{F}(ab) = \{C\}$. Then $C \in \mathrm{F}(\alpha)$.*

A consequence from this lemma is

(11) *Assume $a|A$ and $b|B$. If $A \supset B$ and $\mathrm{F}(ab) = \{C\}$ then $A, B \supset C$.*

2. Transitivity of the neighbor relation. In the following we shall confine our attention mainly to Hjelmslev groups which satisfy two additional axioms called (W) and (Vf).

(W) There exist lines a, b, c, d such that $a|b$ and $c|d$ holds and such that the pairs a, c; a, d; b, c and b, d each intersect in just one point.

A Hjelmslev group satisfying (W) has the following two additional properties. (See [**9**], Anhang. In [**2**], (W) is called the "star axiom".)

(W′) Let $a|b$. There exist lines c, d for which $ab|c|d$ and $\mathrm{F}(ac) = \mathrm{F}(bd) = \mathrm{F}(ad) = \mathrm{F}(bc) = \{ab\}$.

(Kl) If $A|b, c$ then $\mathrm{F}(bc) = \{A\}$ if and only if b, c have a unique point of intersection.

Our second additional axiom is

(Vf) If A distant B then there is a line joining A to B. To each line g there is a pair of distant points lying on g.

If (Vf) is valid and $\mathrm{Z}(S^{\mathrm{even}}) = \{1\}$, then A *distant* B is equivalent to "A and B have a unique joining line". If in a Hjelmslev group there exist two points with just one joining line then $\mathrm{Z}(S^{\mathrm{even}}) = \{1\}$.

LEMMA. *Given a Hjelmslev group, suppose* (W) *and* (Vf) *are valid. Then, for an arbitrary point C and line g, the following conditions are equivalent:*

(*i*) *C distant $(C, g)g$.*

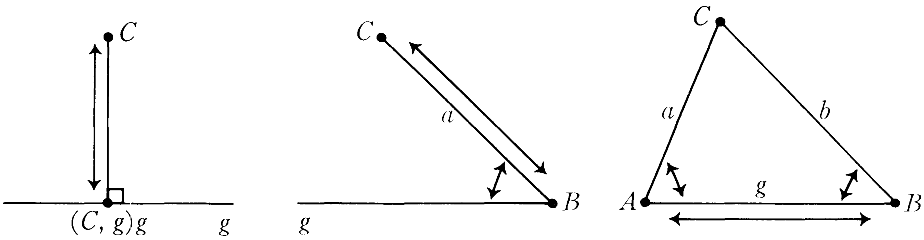
(*ii*) *There exists a line a and a point B such that $a|B$, C and $B|g$ and B distant C, and B is the unique intersection point of a and g.*

(*iii*) *For each point B on g one has B distant C, and if $a|B$, C then B is the unique intersection point of a and g.*

(*iv*) *There exist points A, B and lines a, b such that $a|A$, C and $b|B$, C and A distant B and g has a unique intersection point with both a and b.*

(*v*) *D distant C for each point D on g.*

We illustrate some of the statements as follows.



Proof. (*v*) → (*i*) and (*i*) → (*ii*) are evident. (*ii*) → (*iii*) ([**10**], Lemma 6): According to (*ii*) there is a point $B|g$ and a line $a|B$, C such that B *distant* C and $\mathrm{F}(ag) = \{B\}$. Suppose $A|g$. By (11) A *distant* C, and (Vf) implies the existence of at least one line b joining A and C. The assumption $\{A\} \neq \mathrm{F}(bg) = \mathrm{F}(Ab\,Ag)$ will lead to a contradiction. From (Kl) we obtain a point $D|Ab, Ag$ for which $D \neq A$. Because of A *distant* C it follows from (*5*) that C *distant* D. Hence there is a line $d|C, D$. Now $dba \in S$ (since $d, b, a|C$) and $dbg \in S$ (from $d, Ab, Ag|D$). Furthermore $\mathrm{F}(ag) = \{B\}$, and applying the lemma in Section 1 we find that $B \in \mathrm{F}(db)$. $B, C \in \mathrm{F}(db)$ and B *distant* C imply that $\mathrm{F}(db) = P$,

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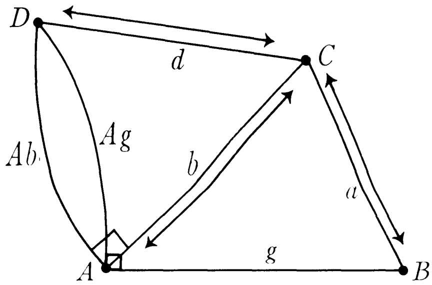
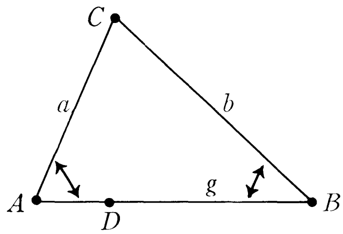


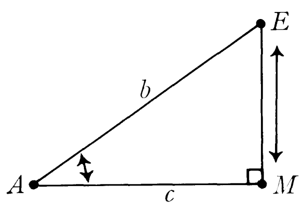
especially $D^{ab} = D$, and since $d|D$ we have $D|b$. Hence $D|b|Ab$ and therefore $A = D$, which is a contradiction. $(iii) \to (iv)$ is clear, since (Vf) yields two points on g which are distant. $(iv) \to (v)$. Consider the triangle $A|a, g$; $B|b, g$; $C|a, b$, where A, B and a, b are chosen according to (iv). Let D be a point on g and α a rotation fixing C and D. Then $a, b, g \in \mathrm{S}(\alpha)$ and $\mathrm{F}(ag) = \{A\}$ and $\mathrm{F}(bg) = \{B\}$. Now the lemma in Section 1 yields $A, B \in \mathrm{F}(\alpha)$ and therefor $\mathrm{F}(\alpha) = P$.



(*12*) *Given a Hjelmslev group which satisfies* (W) *and* (Vf), *and that* $A|b, c$. *Then* $b \supset c$ *if and only if* $\mathrm{F}(bc) \neq \{A\}$.

Thus, two intersecting lines are neighbors if and only if they meet in more than one point.

Proof. Suppose $\mathrm{F}(bc) = \{A\}$. According to (Vf) there are points C, D on b such that C *distant* D. Hence A *distant* $E := DCA$ by (*6*), and from $(ii) \to (i)$ we have E *distant* $M := (E, c)c$. Therefore E *distant* $E^M = E^{bc}$. Thus, by



definition, b *distant* c. Conversely, let $\mathrm{F}(bc) \neq \{A\}$ and let X be any point. We shall prove $X \supset X^{bc}$. If a line $d|A, X$ exists then $X \supset M := (X, dbc)dbc$; otherwise $(i) \to (iii)$ would imply $\mathrm{F}(ddbc) = \mathrm{F}(bc) = \{A\}$. Hence $X \supset X^M = X^{dbc} = X^{bc}$ follows from (4). Now suppose there is no line joining A to X.

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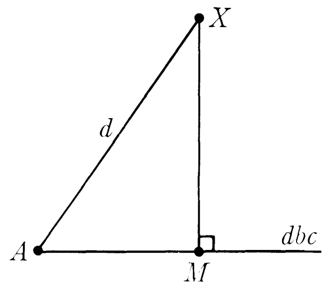


By (Vf) there is a rotation α satisfying $A, X \in \mathrm{F}(\alpha) \neq P$. Thus we have $b, c \in \mathrm{S}(\alpha)$ and $X^{bc} \in \mathrm{F}(\alpha)$, since $(\mathrm{N}_G(\mathrm{F}(\alpha)), \mathrm{S}(\alpha))$ is a locally complete Hjelmslev subgroup of (G, S) with the point set $\mathrm{F}(\alpha)$. Therefore $X \supset X^{bc}$.

For the class of Hjelmslev groups which satisfy (W) and (Vf) the following axiom will prove decisive for the transitivity of the neighbor relation.

(Z) *If $a|b$ and $g|ab$ then a and g, or b and g intersect uniquely.*

In [**2**] this is called the "Gitteraxiom". It prevents a line through a point ab from intersecting both a and b in some other points. In other words it prohibits twisting through a right angle.

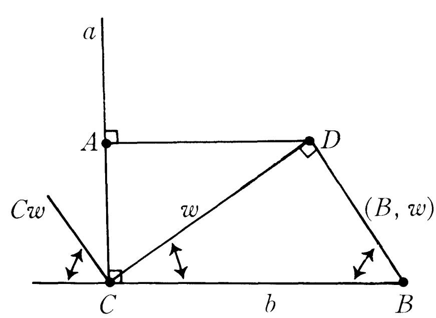
Under (W) and (Z) the axiom (Vf) may be weakened as follows.

(*13*) *Given a Hjelmslev group satisfying* (W), (Z) *and*
(Vf*). *If A distant B then there is a line joining A to B. There exist two points which are distant.*
Then (Vf) *is fulfilled.*

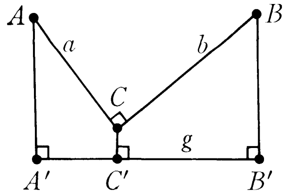
Proof. By assumption there is a line b and points B, $C|b$ where B *distant* C.

Let $a := Cb$. First of all we shall construct a point $A|a$ such that A *distant* C.

By (W′) there exists a line $w|C$ such that w, b and also Cw, b have only one common point. Consider $D := (B, w)w$ and $A := (D, a)a$. Since $w|Cw$, (B, w) and since Cw, b have only one common point we can apply [**2**, 3.7] ("Transversalensatz") from which we conclude that (B, w) and b have only one common point. A similar argument yields that w and (D, a) have only one common point. The assumption $A \supset C$ leads to a contradiction, since (*11*) would imply $D \supset C$, and a second application of (*11*) would yield $B \supset C$.



In order to prove (Vf) let g be a given line. We have to construct two points on g which are distant. Let $A' := (A, g)g$, $B' := (B, g)g$ and $C' := (C, g)g$. According to (Z), (C, g) and a, or (C, g) and b intersect uniquely. Hence, from



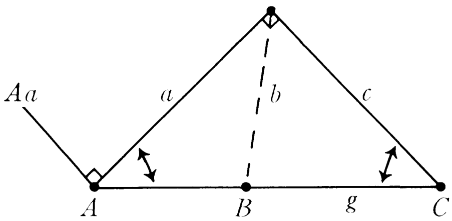
(Kl) we have $F(a(C, g)) = \{C\}$ or $F(b(C, g)) = \{C\}$. The assumption $A' \supset C'$ and $B' \supset C'$ leads to a contradiction: if $F(a(C, g)) = \{C\}$ then *(5)* yields a point $E|(C, g)$ which is a neighbor of A. But then $A \supset C$ by *(11)*. Similarly $F(b(C, g)) = \{C\}$ would imply $B \supset C$.

If the relation $\supset$ on the set of points is transitive then the relation $\supset$ on the set of lines is also transitive. Because, if $X^{ab} \supset X$ and $X^{bc} \supset X$ for every point X then $X^{ab} \supset X^{cb}$, and, by *(1)*, $X^{ac} \supset X$. Therefore talking about transitivity of the neighbor relation cannot cause ambiguity.

(14) *Given a Hjelmslev group which satisfies* (Vf), (W) *and* (Z). *Then* $\supset$ is *transitive.*

Proof. Let A, B, C be points such that $A \supset B$ and $B \supset C$. We have to show $A \supset C$.

First suppose there is a line g passing through A, B, C. From (W′) we obtain a line $a|A$ such that $F(ag) = \{A\} = F(Aag)$. Let $c|C, a$. Then $a|Aa, c$ and from [**2**, 3.7] ("Transversalensatz") we conclude $F(cg) = \{C\}$. We shall see that the assumption A *distant* C leads to a contradiction. *(iv)* → *(v)* in the lemma of this section yields A, B, C *distant* ac. (Vf) provides a line $b|B, ac$.



Axiom (Z) guarantees $F(ab) = \{ac\}$ or $F(bc) = \{ac\}$. In the first case, *(ii)* → *(v)* in the lemma implies A *distant* B, and in the second case C *distant* B. Now consider the general case and suppose A *distant* C. Then we have a line g joining A to C. By *(5)(a)* we have $A \supset (B, g)g$ and $C \supset (B, g)g$. $A \supset C$ now follows since the first part of the proof may be applied to the collinear points A, $(B, g)g$, C.

THEOREM 1. *Given a Hjelmslev group which satisfies* (W) *and* (Vf*). *Then the neighbor relation* $\supset$ *is transitive if and only if* (Z) *is valid.*

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That is, if (Z) holds we deduce (Vf) from (*13*); now transitivity of $\circ$ follows from (14). For the converse we use the following two propositions.

First we shall show that (Vf) may also be replaced by (Vf*) if axiom (W) holds and if the neighbor relation is transitive:

(*13′*) *Given a Hjelmslev group which satisfies* (W) *and* (Vf*). *Suppose that the neighbor relation is transitive. Then* (Vf) *is valid.*

Proof. Assume (Vf) is not valid. Then there is a line g such that any two points on g are neighbors. From (W) and the first part in the proof of (*13*) it follows that the same assertion is true for each line $h|g$. Let $h|g$. We shall show that $A \circ gh$ for any point A. From $(A, g)g \circ gh$ and (*5*)(*b*) it follows that $A \circ (A, h)h$. Furthermore we have $(A, h)h \circ gh$. The transitivity now yields $A \circ gh$.

(*15*) *Given a Hjelmslev group with transitive neighbor relation. Suppose* (Vf) *and* (W) *are fulfilled. If a, b, c are lines through a point D and a, b as well as b, c intersect in more than one point then a, c also intersect in more than one point.*

Proof. From the assumption and (*12*) we obtain $a \circ b$ and $b \circ c$. Therefore $a \circ c$ and from (*12*) it follows that a and c have more than one point of intersection.

Applying (*15*) to the case $a|c$ we obtain

(*14′*) *Consider a Hjelmslev group satisfying* (W) *and* (Vf) *and suppose $\circ$ is transitive. Then* (Z) *is valid.*

Now we shall finish the proof of Theorem 1. Given a Hjelmslev group satisfying (W) and (Vf*). Suppose its neighbor relation is transitive. From (*13′*) we deduce that (Vf) holds, and applying (*14′*) we see that (Z) holds.

We close this section by restating propositions (*7*), (*8*) and (*9*) more plainly under the additional assumption that $\circ$ is transitive.

(*7′*) *Suppose $\circ$ is transitive and let $\alpha \in G$ and $A \circ B$. Then $M_{A,A^\alpha} \circ M_{B,B^\alpha}$.*

Proof. Choose $a|A$ and $b|B, a$. Then by (*5*)(*a*) $A \circ C$ and $C \circ B$, where $C := ab$. Now $M_{A,A^\alpha} \circ M_{C,C^\alpha}$ and $M_{C,C^\alpha} \circ M_{B,B^\alpha}$ by (*7*), and the assertion follows from the transitivity of $\circ$.

(8′) *Suppose $\circ$ is transitive. If $A \circ B$ and $AB = CD$ then $C \circ D$.*

Proof. As is shown in the proof of (*8*) this is a consequence of (*7′*).

(9′) *Suppose $\circ$ is transitive.*

(*a*) *Let A, B be any points. Then $A \circ B$ if and only if $X \circ X^{AB}$ for any point X.*

(*b*) *Let* (Vf*) *be valid and let A, B, X be points such that $X \circ X^{AB}$. Then $A \circ B$.*

Proof. (a) Suppose $A \circ B$. In the same way as in the proof of ($7'$) we construct a point $C \circ A, B$ which is joined to A as well as to B. From (9) it follows that $X \circ X^{AC}$ and $X \circ X^{BC}$ for any point X and thus $X^{AC} \circ X^{BC}$. Therefore $X^{AB} = X^{AC\ CB} \circ X$ by (1). Conversely if $X \circ X^{AB}$ for every point X then especially $A \circ A^{AB} = A^B$ and by (4) $A \circ B$.

(b) If A *distant* B then by (Vf*) there exists a line $c|A, B$ and (9)(b) leads to a contradiction.

2*. Two more theorems deducing transitivity of $\circ$. The results of this supplementary section will not be required in the further sections. Theorem 2 is due to Salow [**12**]; the idea of its proof can be traced back to J. Hjelmslev (see especially third communication, 15.). This theorem applies to Hjelmslev groups with the property that any two points are incident with at least one line, whereas our Theorem 1 also applies to certain Hjelmslev groups admitting pairs of unjoinable points (for an example see Section 4). The second theorem of this section will deduce transitivity of the neighbor relation from an order of the group plane.

Again, let (G, S) be a Hjelmslev group and let P be its point set. If $A \in P$ let $\mathscr{F}_A$ be the set of spots of rotations fixing A:

$$\begin{aligned}\mathscr{F}_A &= \{(\mathrm{N}_G(\mathrm{F}(\alpha)), \mathrm{S}(\alpha)) : \alpha \in \mathrm{D}(A)\}\\ &= \{(\mathrm{N}_G(\mathrm{F}(\alpha)), \mathrm{S}(\alpha)) : \alpha \in S^{\text{even}} \text{ and } A \in \mathrm{F}(\alpha)\}.\end{aligned}$$

$\mathscr{F}_A$ is partially ordered by $\subseteq$, and the structure $\mathscr{F}_A, \subseteq$ is independent of the choice of A: if A and B are points then the partially ordered sets $\mathscr{F}_A, \subseteq$ and $\mathscr{F}_B, \subseteq$ are isomorphic ([**2**], 6.7). Especially, if $\mathscr{F}_A, \subseteq$ is a chain (i.e. totally ordered) for some point A then $\mathscr{F}_B, \subseteq$ is a chain for any point B.

(16) *If* $\mathscr{F}_A, \subseteq$ *is a chain then the relation* $\circ$ *is transitive.*

Proof. Suppose $B \circ C$ and $C \circ D$. From the definition of $\circ$ we have spots F_1, F_2 such that B, C are points of F_1 and C, D are points of F_2 with $F_1 \neq (G, S) \neq F_2$. As $\mathscr{F}_C, \subseteq$ is a chain we may assume that F_1 is a Hjelmslev subgroup of F_2. Therefore B and D are both points of F_2, which means that $B \circ D$.

THEOREM 2 ([**12**]; [**2**] 8.6, 9.4). *Given a Hjelmslev group. Suppose that* (W) *holds and that any two points are incident with at least one line. Let A be a point. Then* $\mathscr{F}_A, \subseteq$ *is a chain if and only if* (Z) *holds.*

COROLLARY. *If, in addition, there is a pair of distant points then transitivity of the neighbor relation is equivalent to axiom* (Z), *and also to the statement that* $\mathscr{F}_A, \subseteq$ *is a chain.*

The next theorem will deduce the property "$\mathscr{F}_A, \subseteq$ is a chain" from an order of the group plane. Such an order is studied in [**9**].

Definition. Let (G, S) be a Hjelmslev group satisfying axiom (W). Suppose

that to each line g we have an order relation $(\,.\,.\,)_g$ defined on the set $\mathrm{P}(g)$ of points lying on g, whose defining properties are as follows.
$(A.B.C)_g \Rightarrow (C.B.A)_g$
$(A.B.C)_g \vee (A.C.B)_g \vee (C.A.B)_g$ for $A, B, C|g$
$(A.B.C)_g \wedge (A.C.B)_g \Rightarrow B = C$
$(A.B.C)_g \wedge X|g \Rightarrow (A.B.X)_g \vee (X.B.C)_g$.
Suppose that $(A.B.C)_g$ implies $((A, h)h.(B, h)h.(C, h)h)_h$, for any two lines g, h. Then the set $\{(\,.\,.\,)_g : g \in S\}$ is called an *order* of the group plane.

THEOREM 3. *Given a Hjelmslev group. Suppose that* (W) *holds and that an order is given on the group plane. Then* $\mathscr{F}_A, \subseteq$ *is a chain for every point* A.

Proof. Let α and β be rotations such that $A \in \mathrm{F}(\alpha), \mathrm{F}(\beta)$ and suppose $\mathrm{F}(\beta) \nsubseteq \mathrm{F}(\alpha)$.

From (W) we can deduce that for any line c through A the statements "$\mathrm{F}(\alpha) \cap \mathrm{P}(c) \subseteq \mathrm{F}(\beta) \cap \mathrm{P}(c)$" and "$\mathrm{F}(\alpha) \subseteq \mathrm{F}(\beta)$" are equivalent (see [**2**], 9.1). Furthermore we need $\mathrm{F}(\alpha) \cap \mathrm{P}(c)$ and $\mathrm{F}(\beta) \cap \mathrm{P}(c)$ are convex sets (with respect to the relation $(\,.\,.\,)_c$; for the easy proof see [**9**], 4.4). Since $\mathrm{F}(\beta) \nsubseteq \mathrm{F}(\alpha)$ there exists a point $B \in \mathrm{F}(\beta) \cap \mathrm{P}(c) \backslash \mathrm{F}(\alpha)$. Now suppose $C \in \mathrm{F}(\alpha) \cap \mathrm{P}(c)$. We have to show that $C \in \mathrm{F}(\beta)$. From $A \in \mathrm{F}(\alpha) \cap \mathrm{F}(\beta)$ and $B \in \mathrm{F}(\beta) \backslash \mathrm{F}(\alpha)$ follows $B^A \in \mathrm{F}(\beta) \backslash \mathrm{F}(\alpha)$. In the case of $(A.B.C)_c$ or $(A.B^A.C)_c$ we would conclude $B \in \mathrm{F}(\alpha)$ or $B^A \in \mathrm{F}(\alpha)$ (by the convexity of $\mathrm{F}(\alpha)$) which is a contradiction. Therefore $(B.C.B^A)_c$ is true (by [**9**], (2.3)) and therefore the convexity of $\mathrm{F}(\beta)$ implies that $C \in \mathrm{F}(\beta)$.

3. The neighbor homomorphism. If, in a Hjelmslev group with more than one point, the neighbor relation $\circlearrowright$ is transitive, then it is an equivalence relation, and there is a partition on the set of points given by the equivalence classes of $\circlearrowright$. Regarding the equivalence classes as new points the situation suggests the construction of a new plane which is expected to satisfy (E). This idea is due to Hjelmslev. The new plane is called the *Gross-Geometrie.* In order to formalize Hjelmslev's idea within the theory of Hjelmslev groups we shall use the concept of Hjelmslev homomorphism as it is defined in [**2**] and outlined in [**1**] and [**3**]. In addition to (W) and the transitivity of the neighbor relation we suppose that (Vf*) holds in order to obtain (V) for our Gross-Geometrie. Thus we derive a connection between the geometry of Hjelmslev groups and classical geometry.

Definition. Let (G, S) and $(G\varphi, S\varphi)$ be Hjelmslev groups, where φ is a group homomorphism from G to $G\varphi$. φ is called a *Hjelmslev homomorphism* if $a|b$ implies $a\varphi|b\varphi$ for all $a, b \in S$.

If P is the set of points of (G, S) then it is easy to see that $P\varphi$ is the se t of points of $(G\varphi, S\varphi)$.

For this section let (G, S) be a Hjelmslev group with more than one point which satisfies (W), (Vf*) and (Z). Then $\circlearrowright$ is transitive. Let $\bar{P}$ be the set of equivalence classes on P in respect to the relation $\supset$ with $\bar{C}$ denoting the class of $C \in P$. With each $\bar{C} \in \bar{P}$ associate the set $\mathrm{S}(\bar{C})$ of lines incident with at least one point $B \in \bar{C}$.

(*17*) *Each $\bar{C} \in \bar{P}$ has the properties* (α) *If $a, b \in \mathrm{S}(\bar{C})$ and $a|b$ then $ab \in \bar{C}$.* (β) *If $A, B \in \bar{C}$ and $A^M = B$ for some point M then $M \in \bar{C}$.* (γ) *$\bar{C}^\alpha = \overline{C^\alpha}$ for each $\alpha \in G$.* (δ) *Let $U, V, W \in \bar{C}$ and $Z := UVW \in P$. Then $Z \in \bar{C}$.*

Proof. (α) is an immediate consequence of (*5*)(a), and (β) follows from (*4*). For (γ) let $C \in P$ and $\alpha \in G$. Then for each point X the statements $X \in \bar{C}$; $X \supset C$; $X^\alpha \supset C^\alpha$; $X^\alpha \in \overline{C^\alpha}$ are equivalent. Now let the hypothesis of (δ) be satisfied. Then $V \supset W$ and by (*9'*)(a) $U \supset U^{VW} = U^{UZ} = U^Z$. From (*4*) we obtain $U \supset Z$, and this means $Z \in \bar{C}$.

From (*17*) and [**2**, § 6, 1-2] we derive the following important conclusion.

(*18*) *Let $N_G(\bar{C})$ be the normalizer of $\bar{C} \in \bar{P}$. Then $(N_G(\bar{C}), S(\bar{C}))$ is a Hjelmslev group and $\bar{C}$ is its set of points. Furthermore,*

$$N := \{\alpha \in S^{\text{even}}\colon \bar{C}^\alpha = \bar{C} \text{ for each } \bar{C} \in \bar{P}\}$$

is a normal subgroup of G.

Actually we need only the last assertion which may easily be derived from (γ): It is obvious that N is a subgroup of G. To prove invariance suppose $\alpha \in N$ and $\beta \in G$. From $\alpha \in N$ we have

$$X^{\beta^{-1}} \supset X^{\beta^{-1}\alpha}$$

and consequently $X \supset X^{\beta^{-1}\alpha\beta}$ for each point X, by (*1*). By (γ) this means $\beta^{-1}\alpha\beta \in N$.

Let $\varphi\colon G \to G/N$ be the canonical homomorphism. We contend that $(G\varphi, S\varphi)$ is a (non-elliptic) AGS group, i.e. a Hjelmslev group with the properties: Any two distinct points are incident with just one line; and there exist three non-collinear points.

(*19*) *Let $A, B \in P$ and $a, b \in S$. Then $A\varphi = B\varphi$ if and only if $A \supset B$ and $a\varphi = b\varphi$ if and only if $a \supset b$.*

Proof. For each pair of points A, B the following statements are equivalent: $A \supset B$; $X \supset X^{AB}$ for every point X (by (*9'*)); $\bar{X} = \overline{X^{AB}} = \bar{X}^{AB}$ for every point X; $AB \in N$. By definition, $a \supset b$ is equivalent to $X \supset X^{ab}$ for each point X, and this means $ab \in N$.

(*20*) *Assume $A|b, c$ and $b \supset b^c$. Then $b \supset c$ or $b \supset Ac$.*

Proof. We need Lemma 1 of [**2**, § 9], which is valid in any Hjelmslev group:

(+) Let A be a point and $\alpha \in \mathrm{D}(A)$. If $\mathrm{F}(\alpha) = \{A\} = \mathrm{F}(A\alpha)$ then $\mathrm{F}(\alpha^2) = \{A\}$.

Now suppose, $b, c|A$ and $\alpha = bc$. Then our assertion follows from (*12*) and (+).

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N has the following properties:

(*N1*) $N \subseteq S^{\text{even}}$.
(*N2*) $N \cap P = \emptyset$.
(*N3*) *If* A, B, C *are points and* $A \cdot A^{BC} \in N$ *then* $BC \in N$.
(*N4*) *If* A *is a point and* $\alpha \in \mathrm{D}(A)$ *such that* $\alpha^2 \in N$ *then* $\alpha \in N$ *or* $A\alpha \in N$.

Proof. (*N1*) is clear from the definition of N. (*N2*). Given any point A. Using (Vf*) and the transitivity of $\supset$ a point B which is distant to A can be constructed. B is distant to B^A by (*4*) and thus $A \notin N$. (*N3*) follows from (*9'*)(*b*) by (*19*), and (*N4*) follows from (*19*) and (*20*).

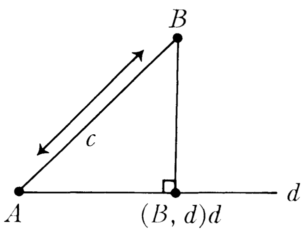
According to [**1**], page 327, or [**3**], page 476, N is the kernel of a Hjelmslev homomorphism.

THEOREM 4. *Let* (G, S) *be a Hjelmslev group with more than one point. Suppose that* (W), (Vf*) *and* (Z) *hold. Then*

$$N := \{\alpha \in S^{\text{even}}: A^\alpha \supset A \text{ for each point } A\}$$

is the kernel of a Hjelmslev homomorphism φ *of* (G, S) *such that* $(G\varphi, S\varphi)$ *is an* AGS *group. For points* A, B *and lines* a, b *one has* $A\varphi = B\varphi$ *if and only if* $A \supset B$, *and* $a\varphi = b\varphi$ *if and only if* $a \supset b$; $A\varphi | b\varphi$ *holds if and only if* $A \supset (A, b)b$; $a\varphi | b\varphi$ *holds if and only if there is a perpendicular* c *to* b *which meets* a *in more than one point.*

Proof. We have already seen that the canonical homomorphism with kernel N is a Hjelmslev homomorphism of (G, S). Consider two distinct points $A\varphi, B\varphi$ of $(G\varphi, S\varphi)$ (where A, B are points of (G, S)). From (19) and (Vf*) we have a line $c|A, B$ and hence $c\varphi | A\varphi, B\varphi$. Now assume there is a line $d \in S$ such that $d\varphi | A\varphi, B\varphi$. We shall see that $d\varphi = c\varphi$. Since $d\varphi = ((A, d)A)\varphi$ we may assume $d|A$; and from $d\varphi | B\varphi$ it follows that $B\varphi = ((B, d)d)\varphi$. By (*19*) B is a neighbor of $(B, d)d$ and A *distant* B. We now apply (*13*) and the lemma



(*ii*) $\rightarrow$ (*i*) and obtain $c \supset d$ and therefore $c\varphi = d\varphi$. In order to show that $(G\varphi, S\varphi)$ has three noncollinear points, we choose lines $b, c \in S$ such that $b|c$. Let $A := bc$. By (Vf) there are distant points C, D on b, and (*8*) implies that $E := ACD$ is a point distant to A which lies on b. Analogously there is a point F which lies on c and is distant to A. (*19*) and the uniqueness of the joining line

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through two different points of $(G\varphi, S\varphi)$ yield that $E\varphi, A\varphi, F\varphi$ are not collinear.

The other assertions follow easily from simple properties of Hjelmslev homomorphisms.

Suppose φ is a Hjelmslev homomorphism of an arbitrary Hjelmslev group (G, S) and the following two properties are valid.

1. $A \circ B \Leftrightarrow A\varphi \circ B\varphi$, for any two points A, B, and
2. $(G\varphi, S\varphi)$ is an AGS group.

These two properties determine φ up to composition with an isomorphism. We shall therefore speak somewhat loosely of the *neighbor homomorphism of* (G, S), provided such a homomorphism exists.

4. Example. The following construction will provide a Hjelmslev group which satisfies the axioms (W), (Vf) and (Z); hence from Theorem 1 we can deduce transitivity of $\circ$. But $\mathscr{F}_A$, $\subseteq$ will not be a chain. We shall find pairs of unjoinable points. The Hjelmslev group will be singular: if a, b, c, d are lines such that $a|b$ and $b|c$ and $c|d$ holds then also $a|d$. This means that P^2 is a group.

Let R be a local (commutative) ring with $1 \in R$. Let J be its maximal ideal. Suppose that all non-units of R are zero-divisors and that the following condition holds.

(*) If $\lambda, \mu \in R$ and $\lambda^2 + \mu^2 \in J$ then $\lambda, \mu \in J$.

Furthermore let V be the free 3-dimensional metric R-module whose symmetric bilinear form is given by 1, 1, 0 with respect to an orthogonal basis. This basis will be fixed.

Let EV be the "euclidean plane" over R: its points are the following 1-dimensional subspaces of V^*, the dual of V.

$R(\alpha, \beta, \gamma)$ where $\alpha, \beta \in R$ and $\gamma \in R\backslash J$.

The lines are the 1-dimensional subspaces of V of the form

$R[\alpha', \beta', \gamma']$ where $\alpha', \beta', \gamma' \in R$ and $\{\alpha', \beta'\} \not\subseteq J$.

As usual, incidence is defined by

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0.$$

A symmetric orthogonality relation on the set of lines is given by the metric form of V. EV is the structure consisting of these points and lines, together with this incidence and orthogonality.

Let η be the homomorphism of EV onto the euclidean plane EV' over the field R/J of residue classes, which maps the points $R(\alpha, \beta, \gamma)$ onto $R/J(\alpha + J, \beta + J, \gamma + J)$, and the lines $R[\alpha', \beta', \gamma']$ onto $R/J[\alpha' + J, \beta' + J, \gamma' + J]$. We omit R and R/J in the following. The discriminant of each line a of EV

contains an R-unit (because of (*)) and therefore the reflection in a, σ_a: $V \to V$ is well-defined. Let

$$S := \{\sigma_a: a \text{ is a line of } EV\}.$$

Writing compositions of reflections σ_a as products one obtains a group G which is generated by S. If A is a point of EV and a, b are orthogonal lines through A let $\sigma_A := \sigma_a\sigma_b$. σ_A is well-defined, and the discussion in [**5**] leads to the following results.

(*a*) (G, S) is a singular Hjelmslev group, and $P := \{\sigma_A: A \text{ is a point of } EV\}$ is the point set of (G, S). $A \mapsto \sigma_a$, $a \mapsto \sigma_a$, is an isomorphism of EV onto the group plane assigned to (G, S).

(*b*) Two points A, B of EV satisfy $A\eta \neq B\eta$ if and only if A, B have exactly one common line. Two intersecting lines a, b of EV satisfy $a\eta \neq b\eta$ if and only if they have not more than one common point.

(*c*) (G, S) satisfies the axioms (W) and (Z). $Z(S^{\text{even}}) = \{1\}$.

In addition we now request that

(**) $\operatorname{Ann}(J) \neq \{0\}$; i.e. there is some $\lambda \in R\backslash\{0\}$ such that $\lambda\mu = 0$ for all $\mu \in J$.

With this assumption (**) we shall prove

(*d*) Let A, B be points of EV. Then $A\eta = B\eta$ is equivalent to $\sigma_A \supset \sigma_B$ in (G, S).

Proof of (*d*). If A, B have more than one common line then by (*a*) and (*c*) $\sigma_A \supset \sigma_B$ in (G, S). If A, B have exactly one common line then σ_A *distant* σ_B in (G, S). If A, B have no joining line then we have to show that there is a rotation $\neq 1$ of (G, S) which fixes σ_A and σ_B. Applying (*b*) we may assume that $A = (0, 0, 1)$ and $B = (\lambda_1, \lambda_2, 1)$ where $\lambda_1, \lambda_2 \in J$. Let $\lambda \in \operatorname{Ann}(J)\backslash\{0\}$ and consider the lines $a := [1, \lambda, 0]$, $b := [-\lambda, 1, -\lambda_2]$, $a' := [1, 0, 0]$ and $b' := [0, 1, -\lambda_2]$. Then the lines a, a' pass through A and b, b' pass through B. a is orthogonal to b, and a' is orthogonal to b'. Also a, b, a', b' meet in the point $C = (0, \lambda_2, 1)$. Therefore we have $\sigma_a\sigma_b = \sigma_C = \sigma_{a'}\sigma_{b'}$ and $\sigma_{a'}\sigma_a = \sigma_{b'}\sigma_b$ is a rotation fixing σ_A and σ_B; it is not the identity because $a \neq a'$.

It is obvious from (*d*) and (*b*) that

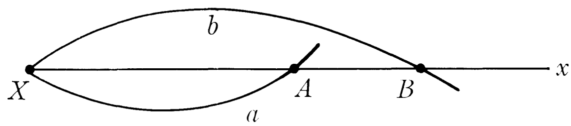
(*e*) The relation $\supset$ is transitive in (G, S). (G, S) satisfies (Vf).

Now assume

(***) There are $\mu, \nu, \epsilon, \omega \in R$ such that $\mu\epsilon = 0 = \nu\omega$, but $\mu\omega \neq 0$ and $\nu\epsilon \neq 0$.

Under this additional assumption we prove (*f*) The spots of rotations with center $\sigma_{(0,0,1)}$ do not form a chain.

Proof. Let $X := (0, 0, 1)$, $A := (0, \mu, 1)$, $B := (0, \nu, 1)$, $a := [1, \epsilon, 0]$, $b := [1, \omega, 0]$, $x := [1, 0, 0]$. Then, by (*a*), $\sigma_x\sigma_a$ fixes σ_X and σ_A but not σ_B, and $\sigma_x\sigma_b$ fixes σ_X and σ_B but not σ_A. Hence the spot of $\sigma_x\sigma_a$ is not contained in the spot of $\sigma_x\sigma_b$ and conversely.



Now we give an example of a local ring R such that (*), (**) and (***) are valid and all non-units are zero-divisors.

For this purpose let K be a commutative field such that $a^2 + b^2 \neq 0$ for each pair $a, b \in K\backslash\{0\}$. Let T be the ideal generated by $\{x^2, y^2\}$ in the ring $K[x, y]$ and consider $R := K[x, y]/T$. We represent R by the set of representatives

$$\{a + bx + cy + dxy\colon a, b, c, d \in K\}.$$

It is easy to see that the R-ideal $J = \{bx + cy + dxy\colon b, c, d \in K\}$ consists of the non-units of R. Thus R is a local ring and J is its maximal ideal. Since $xy \in \mathrm{Ann}(J)\backslash\{0\}$, (**) is satisfied. To verify (***) let $\mu := \epsilon := x$ and $\nu := \omega := y$. (*) follows easily from the requested property of K.

Remark 1. We have seen that the canonical homomorphism $R \to R/J$ provides a homomorphism $EV \to EV'$. Similarly there is a homomorphism

$$\psi\colon GL(V) \to GL(V'),\ (\alpha_{ij}) \mapsto (\alpha_{ij} + J)$$

(in terms of matrices) which is studied in [**5**] and [**9**]. Then the following diagram is commutative:

$$\begin{tikzcd} EV \arrow[r, "\sigma"] \arrow[d, "\eta"'] & (G, S) \arrow[d, "\psi"] \\ EV' \arrow[r, "\sigma"] & (G', S') \end{tikzcd}$$

and the restriction $\psi|_G$ is exactly the neighbor homomorphism of (G, S):

$$(G', S') \cong (G\varphi, S\varphi)$$

where φ is the Hjelmslev homomorphism of Theorem 4.

Remark 2. Let (G, S) be the Hjelmslev group over the above special local ring. Then the neighbor relation of (G, S) is transitive, but the neighbor relation of the spot F which belongs to the rotation $\sigma_{[0,1,0]}\sigma_{[xy,1,0]}$ is not transitive. F satisfies the axioms (W) and (Z), but it fails to satisfy (Vf*): There exist two points, namely for example $\sigma_{(0,0,1)}$ and $\sigma_{(x,y,1)}$, which are distant in F and which have no common line. F is the Hjelmslev subgroup $(\mathrm{N}_G(\overline{\sigma_{(0,0,1)}}), \mathrm{S}(\overline{\sigma_{(0,0,1)}}))$ which is considered in (18).

Proof of *Remark* 2. (W) is valid in F because F is locally complete in (G, S), and (Z) is fulfilled in F because there is no "twisting line" in (G, S).

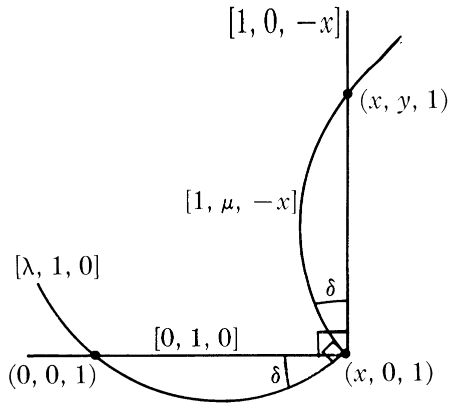
(α) $\sigma_{(0,0,1)}$ *distant* $\sigma_{(x,y,1)}$ in F. Let δ be a rotation of F which fixes both points. Then δ may be written as

$$\delta = \sigma_{[0,1,0]}\sigma_{[\lambda,1,0]},$$

where $\lambda \in \mathrm{Ann}(x)$, and

$$\delta = \sigma_{[1,0,-x]}\ \sigma_{[1,\mu,-x]}$$

where $\mu \in \mathrm{Ann}(y)$, and the two lines $[\lambda, 1, 0]$, $[1, \mu, -x]$ must be orthogonal ([**2**], 4.3). Therefore $\lambda = -\mu \in \mathrm{Ann}(\{x, y\}) = Rxy$, and δ is a rotation of F which fixes each point of F on the line $\sigma_{[0,1,0]}$. From (W) we now conclude that δ fixes each point of F.



(β) $\sigma_{(0,0,1)} \supset \sigma_{(x,0,1)}$ and $\sigma_{(x,0,1)} \supset \sigma_{(x,y,1)}$ in F. $\sigma_{[0,1,0]}\ \sigma_{[x,1,0]}$ fixes $\sigma_{(0,0,1)}$ and $\sigma_{(x,0,1)}$ but not the point $\sigma_{(y,0,1)}$ which also belongs to F. Similarly $\sigma_{[1,0,-x]}$ $\sigma_{[1,y,-x]}$ fixes $\sigma_{(x,0,1)}$ and $\sigma_{(x,y,1)}$ but not the point $\sigma_{(x,x,1)}$.

Remark 2 shows that (Vf*) in Theorem 1 is essential.

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Universität Kiel,
Kiel, W. Germany;
Technische Hochschule Darmstadt,
Darmstadt, W. Germany