BULL. AUSTRAL. MATH. SOC. 34C30, 35K55
VOL. 33 (1986), 1-20.

DIRECTIONAL WAVE FRONTS OF REACTION-DIFFUSION SYSTEMS

J. SABINA

In this work, we study types of undulatory solutions, that we term Directional Wave Fronts (DWF), of non scalar reaction diffusion systems. The DWFs are a natural extension of the well known Plane Wave Fronts (PWFs) solutions. However, the DWFs admit a certain type of boundary conditions. In the present work we show, under suitable conditions on the reaction term, that DWFs also exhibit typical behaviour of PWFs: we just prove the existence of heteroclinic, homoclinic and periodic families of DWFs. Essentially, we require the reaction term to be linearly uncoupled. These results are the generalization of a previous work, concerning the scalar case.

0. Introduction

In this article we study a particular type of solution, with propagatory character, of reaction-diffusion systems:

Received 11 April 1985. I wish to thank Professor J. Fraile for introducing me to the topics here developed, as well as for several work sessions on them.

$$\frac{\partial u}{\partial t} = \mathcal{D}\cdot\Delta u + f(u) \qquad u \in \mathbb{R}^m \, , \, m > 1 \, . \tag{0}$$

An important aspect in the theory of R-D systems such as (0), is the study of those particular solutions which generate either stable asymptotic states or transitory states of equation (0) (see [7]). The well-known Plane Wave Fronts (PWFs) solutions belong to this group of particular solutions (see [7], [13], [14]). The PWFs are solutions with propagatory properties and exhibiting - in the most significant cases - a typical behaviour; they connect solutions which are stationary and homogeneous states of (0). Moreover, in unidimensional media, those connections are stable asymptotic states of certain initial value problems (see [2], [7], [8]). In other cases, the relevant property of PWFs in applications is the periodicity in space and time (see [10], [13]).

The type of solutions analyzed in this work are the Directional Wave Fronts (DWFs), which are, to some extend, a generalization of PWFs. However, the DWFs have the interesting property of admitting a certain type of boundary condition (see Section 1).

The objective of this paper is to extend, to the non-scalar case (m>1), the analysis of the existence and asymptotic behaviour of small amplitude DWFs, developed in previous works for the scalar case (see [9], [16]).

In Section 1, we define the homogeneous Dirichlet and Neumann problems for DWFs. Section 4 contains the main results. There we establish the existence of homoclinic, heteroclinic and periodic DWFs - with small amplitude - for the homogeneous Neumann problem. However, it requires certain structure conditions on the reaction term f. Essentially, we ask that the linearization of equation (0) be uncoupled (see Section 2).

In Section 3 we describe the method - a centre manifold theorem - used in the analysis of Section 4. This method provides a reduction in the dimension of the problem. It is important to point out that the presence in (0) of drift terms (see [7]), gives an interesting asymptotic behaviour. In this case, such asymptotic behaviour is governed by an ODE which is reversible in the time variable (see Section 3).

Specifically, in this work we deal with the equation:

$$(1) \qquad \frac{\partial u}{\partial t} = \mathcal{D}\cdot Lu + f(u) \quad ,$$

where

$$L = \Delta + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \quad b_i \in \mathbb{R} \,,$$

and where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n - \{0\}$. In the equation

(1) $\begin{array}{l} u : \Omega \times \mathbb{R} \to \mathbb{R} \quad \mathbb{R}^m \\ (x,t) \to u(x,t) \end{array}$, $m > 1$, Ω being a C^2 bounded domain in $\mathbb{R}^m$.

The diffusivities matrix $\mathcal{D}$ is symmetric and positive definite. L acts coordinate to coordinate: $Lu = (Lu_i)_{1 \le i \le m}$ and the reaction term f is a smooth nonlinear function of u (see Section 2).

Some relevant information has been obtained when $b = 0$ in (1), and this equation is scalar (see Section 5c), but we do not discuss this case here.

1. Directional wave fronts. Boundary conditions

A Directional Wave Front (DWF) is a solution $u = u(x,t)$ of (1) with the form:

$$(1.1) \qquad u(x,t) = v(Kx - ct) \qquad x \in \Omega \,,$$

where the matrix $K = \operatorname{diag}[k_1, \ldots, k_n]$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. Thus, $y = Kx - ct \in \mathbb{R}^n$ if $x \in \Omega$ and $t \in \mathbb{R}$.

It is easy to see that DWFs represent perturbations, which propagate with constant velocity in the direction $K^{-1}c$. If $n = 1$, PWFs and DWFs are the same type of solution of (1).

In the sequel we assume, for simplicity, that $\mathcal{D} = 1$. So every DWF $v = v(y)$ of (1) satisfies the equation:

$$(1.2) \qquad \sum_{i=1}^{n} k_i^2 \frac{\partial^2 v}{\partial x_i^2} + \sum_{i=1}^{n} (k_i b_i + c_i) \frac{\partial v}{\partial x_i} + f(v) = 0$$

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

where the variable y ranges over the unbounded domain:

$$W = \bigcup_{t\in\mathbb{R}} (K\Omega - ct) ,$$

if $x \in \Omega$ and $t \in \mathbb{R}$.

We are interested just in bounded solutions of (1.2) defined on W. However, establishing the existence of those solutions and determining their asymptotic behaviour in W, may be very hard.

In this work, the approach to the problem consists in transforming it into an abstract evolution problem. In fact, there exists a non singular linear transformation $\begin{array}{c} S : \mathbb{R}^n \to \mathbb{R}^n \\ y' \mapsto y = Sy' \end{array}$ (see [9], [16]), which puts (1.2) into the form:

$$\frac{\partial^2 v}{\partial s^2} + \Delta_z v + f(v) = 0 \quad s \in \mathbb{R} , z \in D .$$

In that equation we write $y' = (s,z)$ with $z \in \mathbb{R}^{n-1}$. We are supposing also that the identity $Kb + c = 0$ holds, hence, the condition $b \neq 0$ is essential. If $y \in W$ then $z \in D$, a domain in $\mathbb{R}^{n-1}$. The regularity of D does not depend only on the intrinsic regularity of Ω, but also on c and K (see [9], [16]). Here, we suppose that Ω, K and c have been chosen in order to obtain a regular domain $D \subset \mathbb{R}^{n-1}$.

As a consequence of the above discussion, it is possible to impose boundary conditions on DWFs. In fact, by using the transformations S and K, the following fact holds (see [9], [16]):

$$\exists \Gamma \subset \delta\Omega \text{ such that } x \in \Gamma \Leftrightarrow z \in \partial D .$$

We define the homogeneous Dirichlet problem for DWFs as the finding of solutions $u = u(x,t)$ of (1), with the form (1.1), satisfying:

(1.3) $$u(x,t) = 0 \quad \text{on } x \in \Gamma , t \in \mathbb{R} .$$

On the other hand, a suitable field $\nu = \nu(x)$ can be defined on Γ (see [9], [16]) such that, for a DWF $u = u(x,t)$:

$$\frac{\partial u}{\partial \nu} (x,t) = 0 \quad \text{on } x \in \Gamma \text{ and } t \in \mathbb{R} \Leftrightarrow$$

$$\frac{\partial v}{\partial n}(s,z) = 0 \quad \text{on} \quad z \in \partial D \quad \text{and} \quad s \in \mathbb{R} .$$

Above $n = n(z)$ denotes the outer unitary normal at $z \in \partial D$. Thus, the homogeneous Neumann problem for DWFs is to find $u = u(x,t)$, of type (1.1), such that:

$$(1.4) \qquad \frac{\partial u}{\partial \nu}(x,t) = 0 \quad \text{on} \quad x \in \Gamma , \; t \in \mathbb{R} .$$

Accordingly, the Dirichlet and Neumann problems for DWFs are, respectively, equivalent to the evolution problems:

$$(P_D) \qquad \begin{cases} \dfrac{\partial^2 v}{\partial s^2} + \Delta_z v + f(v) = 0 & z \in D \\[2ex] v = 0 & z \in \partial D \end{cases}$$

$$(P_N) \qquad \begin{cases} \dfrac{\partial^2 v}{\partial s^2} + \Delta_z v + f(v) = 0 & z \in D \\[2ex] \dfrac{\partial v}{\partial n} = 0 & z \in \partial D . \end{cases}$$

In this work we shall analyze the Neumann problem. The corresponding conclusions for (P_D) follow in a similar way (see Section 5b).

Finally, observe that the only PWFs solutions of (1) satisfying the condition (1.3) are the trivial ones: $u \equiv 0$. Analogously, if a PWF satisfies (1.4) it must be constant in space and time.

2. The reaction term structure. Consequences.

We are going to consider the analysis of bounded DWFs as a bifurcation problem, with respect to the solution $u = 0$ and a real parameter λ. Thus, we shall suppose that $\underset{(\lambda,u)\to f(\lambda,u)}{f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m}$ is a C^{r+2} function, which satisfies:

(i) $f(\lambda,0) = 0$, $\lambda \in \mathbb{R}$.

Hence, $f = f(\lambda,u)$ can be written in the form: $f(\lambda,u) = A(\lambda)u + g(\lambda,u)$, $A(\lambda)$ being an $m \times m$ matrix and $g(\lambda,u) = 0(|u|^2)$ when $|u| \to 0$, uniformly in λ, $|\lambda|$ small. We suppose also that:

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

(ii) $\exists a > 0$ such that $A(\lambda) = \operatorname{diag}[d_1(\lambda),\ldots,d_m(\lambda)]$, $d_1(\lambda) > d_j(\lambda)$ for $j \in \{2,\ldots,m\}$ and $|\lambda| < a$, with $d_1(0) = 0$ and $d_1'(0) > 0$.

(iii) $g_1(\lambda,u) = \sum_{|\alpha|=k} a_k^{\alpha}(\lambda)u + r(\lambda,u)$, $k \le r+1$, where $r(\lambda,u) = 0(|u|^{k+1})$, when $|u| \to 0$, uniformly in λ. If $\alpha_0 = (k,0,\ldots,0) \in N^m$ then $a_k^{\alpha_0}(\lambda) \neq 0$ for $|\lambda| < a$.

REMARKS. Amongst (i), (ii), (iii) the essential hypothesis is (ii). There, we impose the condition that (1) is linearly uncoupled. This condition holds, for example if $f(\lambda,u) = \frac{\partial\phi}{\partial u}(\lambda,u)$, with $\phi = \phi(\lambda,u)$ a scalar smooth function. In general, (i) always implies (iii) with $k = 2$. In (i), (ii), (iii) we endow the first equation of (1) with a structure which is analogous to the structure that generates small amplitude DWFs in the scalar case (see [9], [16]).

On the other hand, the presence of small amplitude stationary-homogeneous solutions of (1), suggests the existence of DWFs whose α and ω-limit sets consist of those solutions. The following result ensures the existence of that type of solution.

PROPOSITION 1. *Let us consider the kinetic equation associated with* (1):

$$\frac{du}{dt} = f(\lambda,u) \quad , \tag{2.1}$$

where f satisfies (i), (ii), (iii). *Then:*

(1) $(\lambda,u) = (0,0)$ *is a bifurcation point of stationary solutions of* (2.1) *with respect to $u = 0$. The bifurcated branch has the form* $(\lambda,u) = (\lambda(u_1),\phi(u_1))$, u_1 *small*, ϕ *and* λ *being* C^{r+1} *functions*, $\phi_1(u_1) = u_1$, $\phi_j(u_1) = 0(u_1^2)$ *for* $2 \le j \le m$, *and*

$$\lambda(u_1) = -\frac{a_k^{\alpha_0}(0)}{d_1'(0)}\, u_1^{k-1} + 0(|u_1|^k) \;;$$

(2) *at* $(\lambda,u) = (0,0)$ *a stability interchange happens between the bifurcated branches.*

Proof. The proof of (1) is an immediate consequence of Theorem 1 in

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

[5]. However, because of considerations in Section 4, we give a proof using the Lyapunov-Schmidt method. Let us put $\hat{u} = (u_2,\ldots,u_m)$ and $\hat{A}(\lambda) = \operatorname{diag}[d_2(\lambda),\ldots,d_m(\lambda)]$, then equation $f = 0$ can be written in the form

$$\begin{cases} d_1(\lambda)u_1 + g_1(\lambda,u_1,\hat{u}) = 0 , \\ \hat{A}(\lambda)\hat{u} + \hat{g}(\lambda,u_1,\hat{u}) = 0 . \end{cases} \tag{2.2}$$

The implicit function theorem, applied to the second equation, implies that $\hat{u} = \psi(\lambda,u_1)$, where ψ is a C^{r+2} function satisfying: $\psi(\lambda,u_1) = O(|u_1|^2)$, when $|u_1| \to 0$ and $|\lambda|$ small.

Hence, (2.2) is equivalent to:

$$d_1(\lambda)u_1 + g_1(\lambda,u_1,\psi(\lambda,u_1)) = 0 , \tag{2.3}$$

and nontrivial solutions of (2.2) are just $(\lambda,u) = (\lambda(u_1),(u_1,\psi(\lambda,u_1)))$, where $\lambda(\cdot)$ is the C^{r+1} function defined by the equation:

$$d_1(\lambda) + a_k^{\alpha_0}(\lambda)u_1^{k-1} + \tilde{r}(\lambda,u_1) = 0 , \tag{2.4}$$

where $\tilde{r}(\lambda,u_1) = O(|u_1|^k)$, when $|u_1| \to 0$, $|\lambda|$ small.

(2) is a well known fact from bifurcation theory (see [11]). #

The abstract approach to the evolution problem (P_N) in a Hilbert space frame, requires an additional condition, on f namely:

(iv) f together with its derivatives up to the order $r + 1$, are polynomially bounded. Also we assume that $n \le 3$.

From (iv) the function f is smooth, considered as a Nemitskii operator in $(H^1(D))^m$ with values in $(L^2(D))^m$:

PROPOSITION 2. *Assume that* $f \in C^{r+2}(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ *and suppose that* (iv) *holds, that is,* $n \le 3$ *and:*

$$|\partial^\gamma f(\lambda,u)| \le b_\gamma(\lambda) + c_\gamma(\lambda)\sum_{j=1}^m |u_j|^{s_{\gamma,j}} ,$$

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

with $\gamma \in \mathbb{N}^{m+1}$, $0 \le |\gamma| \le r+1$, $s_{\gamma,j} \ge 1$, $b_\gamma(\cdot)$ *and* $c_\gamma(\cdot)$ *continuous. Then* :

$$f \in C^{r+1}(\mathbb{R} \times (H^1(D))^m , (L^2(D))^m) .$$

Proof. If $\Omega \subset \mathbb{R}^3$ then $D \subset \mathbb{R}^2$. Thus, Sobolev Immersions implies that $(H^1(D))^m \subset (L^p(D))^m$, $\forall p > 1$, with compact embedding. Then, it suffices to prove that $f_i \in C^{r+1}(\mathbb{R} \times (H^1(D))^m , L^p(D))$, for some $p > 2$. Let us take $u = (u_1,\dots,u_m)$ and $h = (h_1,\dots,h_m) \in (H^1(D))^m$. At every $z \in D$ we can write:

$$f_i(\lambda+\hat{\lambda},u(z)+h(z)) = \sum_{\ell=0}^{r+1} \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (\partial_\lambda^{\ell-k} \partial_u^k f_i(\lambda,u(z))h^k(z))\hat{\lambda}^{\ell-k} + \rho(\hat{\lambda},h(z)) ,$$

where

$$\rho(\hat{\lambda},h(z)) = \frac{1}{(r+2)!}\sum_{k=0}^{r+2} \binom{r+2}{k} \partial_\lambda^{r+2-k}(\partial_u^k f_i(\lambda+\theta(z)\hat{\lambda},(u+\theta h)(z))h^k(z))\hat{\lambda}^{r+2-k}$$

and where $0 < \theta(z) < 1$, *a.e.* in D .

Because of the polynomial growth of $\partial^\gamma f(\lambda,u)$, $|\gamma| = k \le r \le 1$, then $\partial_u f_i(\cdot,\cdot) \in C(\mathbb{R} \times (L^p(D))^m , L^k((L^p(D))^m , L^2(D)))$, for $p \ge \max_{\substack{|\gamma|=k \\ 1\le j\le m}} \{2s_{\gamma,j}\}$ (see [3]). For $X = (L^p(D))^m$ and $Y = L^2(D)$, $L^k(X,Y)$ denotes the space of the continuous k-linear maps from X to Y .

On the other hand:

$$\rho(\hat{\lambda},h) = o((|\hat{\lambda}|+|h|_p)^{r+2}) ,$$

where $|h|_p = |h_1|_{L^p(D)} + \dots + |h_m|_{L^p(D)}$. Thus, the conclusion follows from the Converse Taylor's Theorem (see [1]).#

3. A reduction in the dimension of the problem

In the study of small amplitude solutions of (P_N) , a reduction in the dimension can be introduced. Here, the information about those solutions, is furnished by a two-dimensional O.D.E. Let us describe how

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

such a reduction can be performed.

Firstly, the problem (P_N) can be rewritten in the form:

$$v'' - T(\lambda)v + g(\lambda,v) = 0 \qquad ' = \frac{d}{ds}\,, \tag{3.1}$$

where $T(\lambda) = -\Delta_z - A(\lambda)$ is defined in $(L^2(D))^m$, with domain:

$$D(T) = \{v\in (H^2(D))^m / \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial D\}$$

The operator $T(\lambda)$ has compact resolvent, so its spectrum

$\Sigma(T(\lambda)) = \bigcup_{i=1}^{m} \{\sigma_k - d_i(\lambda)\}_{k\in N}$, $\{\sigma_k\}_{k\in N}$ being the eigenvalue sequence of $-\Delta$ defined on $L^2(D)$, with domain

$$D = \{v\in H^2(D) / \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial D\}\ .$$

Thus, for $|\lambda|$ small enough, $\mu = 0$ is the first eigenvalue of $T(\lambda)$, with corresponding eigenfunction $\psi = 1_D\, e_1$, $e_1 = (1,0,\ldots 0) \in \mathbb{R}^m$, $1_D(z)=1$, $\forall z \in D$. If $H_0 = \text{span}\ \{\psi\}$, $H_1 = (H_0)^{\perp}_{(L^2(D))^m}$ then $L^2(D) = H_0 \oplus H_1$.

Denoting by P_i the projection onto H_i , and writing $T_i(\lambda) = P_i \circ T(\lambda)|_{H_i}$, $i = 0,1$, we can put (3.1) in the form:

$$\text{(3.2)} \qquad \begin{cases} v_0'' + d_1(\lambda)v_0 + \tilde{g}_0(\lambda,v_0,v_1) = 0 & (3.2.1) \\ v_1'' - T_1(\lambda)v_1 + \tilde{g}_1(\lambda,v_0,v_1) = 0 & (3.2.2) \end{cases}$$

Above, for $v \in (L^2(D))^m$, $v = v_0 e_1 + v_1 = P_0 v + P_1 v$ and, for example, $P_0 g(\lambda,v_0 e_1 + v_1) = \tilde{g}_0(\lambda,v_0,v_1)\ e_1$, where:

$$\tilde{g}_0(\lambda,v_0,v_1) = \frac{1}{|D|}\int_D g_1(\lambda,v_0(z)e_1+v_1(z))\,dz\ ,$$

with $v \in (H^1(D))^m$ and $|D|$ being the Lebesgue measure of $D \subset \mathbb{R}^2$

Equation (3.2) exhibits a coupling between a bidimensional equation ((3.2.1)) and a infinite dimensional one ((3.2.2)). We shall show how

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

(3.2.1) it is just the result of the behaviour of small amplitude solutions of (3.1).

Intuitively, observe that, for λ small:

$$\exists\alpha > 0 : (T_1(\lambda)\ v,v) \geq \alpha|v|^2_{(L^2(D))^m}\ ,\ v \in D(T_1)\ ,$$

where $(\cdot,\cdot)$ denotes the usual $(L^2(D))^m$ inner product. Therefore, the linear part of (3.2.2) has its spectrum bounded away from $\mathrm{Re}z = 0$. However, the corresponding spectrum of (3.2.1) is $\{\pm\sqrt{d_1(\lambda)}\}$ which collapses to $\{0\}$ when $\lambda \to 0$. This fact suggests the existence of a Centre Manifold (see [4]). In fact, a theorem of existence of Center Manifolds, due to Kirchgassner (see [12]), for abstract evolution equations of type (3.1) is applicable. This result implies the existence of ε_0, ε_1, ε_2, positive numbers and a smooth function:

$$h \in C^r((-\varepsilon_0,\varepsilon_0) \times B^2_{\varepsilon_1}(0),\ B'_{\varepsilon_2}(0))\ ,$$

where $B^2_{\varepsilon_1}(0)$ is the open ball in $\mathbb{R}^2$, centred at 0 with radius $\varepsilon_1 > 0$, and $B'_{\varepsilon_2}(0)$ the corresponding ball in the space $(H^1(D))^m \cap H_1$.

The function h - known in the literature as a Centre Manifold (see [4]) - has the following invariance property: If $v = v(s)$ is a solution of (3.1), $|\lambda| < \varepsilon_0$, with the regularity $v \in C^2(\mathbb{R},(L^2(D))^m) \cap C^1(\mathbb{R},(H^1(D))^m) \cap C(\mathbb{R},(H^2(D))^m)$ - see [6 Chap. 6, §5] and [12] - and satisfies

$$\sup_{s\in\mathbb{R}}(v_0^2(s)+v_0'^2(s)) < \varepsilon_1^2,\ \sup_{s\in\mathbb{R}}|v_1(s)|_{(H^1(D))^m} < \varepsilon_2\ , \tag{3.3}$$

then, such a solution satisfies:

$$v_1(s) = h(\lambda,v_0(s)\ ,\ v_0'(s)) \quad \forall s \in \mathbb{R}\,. \tag{3.4}$$

As a consequence, every small amplitude solution of (3.1), regular enough, will has the form: $v(s) = v_0(s)e_1 + h(\lambda,v_0(s),\ v_0'(s))$, where $v_0 = v_0(s)$ is a solution of the equation:

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

(3.5) $$v_0'' + d_1(\lambda)v_0 + \tilde{g}_0(\lambda, v_0 e_1 + h(\lambda, v_0, v_0')) = 0 .$$

Hence, (3.5) governs the small amplitude solutions of (P_N) .

Furthermore, function h exhibits the properties (see [4], [12]):

(3.6) $$h(\lambda, v_0, v_0') = 0(v_0^2 + v_0'^2)$$

(3.7) $$h(\lambda, v_0, -v_0') = h(\lambda, v_0, v_0') .$$

Hence, some information about the nonlinearity in (3.5) can be deduced. In fact, some calculations show that (3.5) can be put in the form:

(3.8) $$v_0'' + a(\lambda)\ _0 + b(\lambda)v_0^k + r_1(\lambda, v_0, v_0') = 0$$

where $a(\lambda) = d_1(\lambda)$, $b(\lambda) = a_k^{\alpha_0}(\lambda)$ and $r_1(\lambda, v_0, -v_0') = r_1(\lambda, v_0, v_0')$ with $r = 0(|v_0|^{k+1} + |v_0'|^2\ |v_0|^{k-1})$.

REMARKS.

1. The regularity required on g for obtaining C^r class in h is just $g \in C^{r+1}(\mathbb{R} \times (H'(D))^m, (L^2(D))^m)$ (see [12]). Thus, if $f = f(\lambda, h)$ satisfies (i),..., (iv), that condition holds.

2. Every small solution $v_0 = v_0(s)$ of (3.8) generates a DWF $u = u(x,t)$ of equation (3.4), which is solution of the Neumann problem. The regularity of $v(s) = v_0(s)e_1 + h(\lambda, v_0(s), v_0'(s))$ implies that $u(\cdot,t) \in H^2(\Omega)\ \forall t \in I\!R$, and $u \in H^2_{loc}(\Omega \times \mathbb{R})$. Hence, $u(\cdot,t) \in C^0(\bar{\Omega})\ \forall t \in I\!R\ (\Omega \subset \mathbb{R}^3)$.

4. Existence of small amplitude DWFs for the Neumann problem

Let us begin studying the small amplitude solutions of (3.8). If we consider the truncated equation:

(4.1) $$v'' + a(\lambda)v_0 + b(\lambda)v_0^k = 0 ,$$

via the first integral $V(v_0, v_0') = \frac{1}{2} v_0'^2 + a(\lambda)\ \frac{v_0^2}{2} + b(\lambda)\frac{v_0^{k+1}}{k+1}$, is easily seen that:

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

(a) If k is even and $\lambda < 0$; then $(v_0, v_0') = (0,0)$ is a saddle point of (4.1) connected with itself by a homoclinic orbit, which encloses to the nonlinear center $(-(a(\lambda)/b(\lambda))^{1/k-1}, 0)$. Thus, a one parameter family of periodic solutions of (4.1) arises.

If $\lambda > 0$, points $(0,0)$ and $(-(a(\lambda)/b(\lambda))^{1/k-1}, 0)$ interchange their behaviour (see Figure 1).

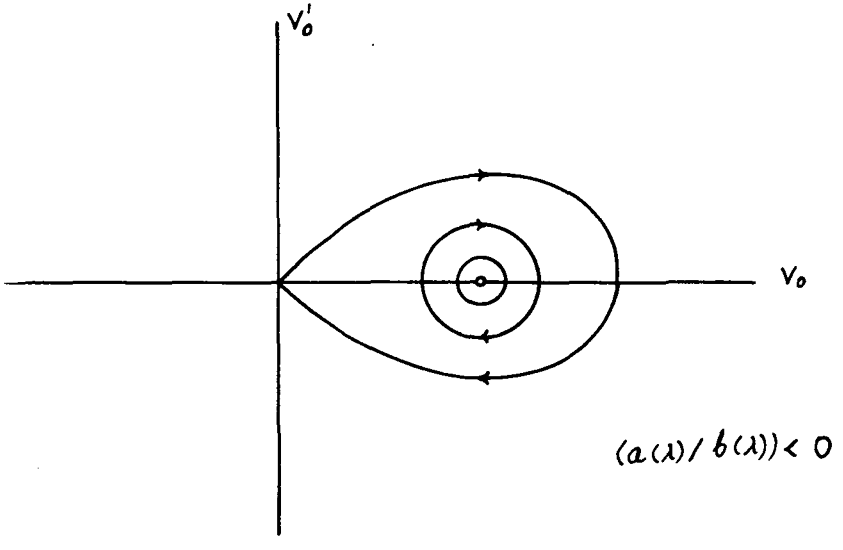

Figure 1.

(b) If k is odd, $\lambda b(\lambda) < 0$ and $\lambda < 0$, $(0,0)$ is a saddle point connected with itself by two homoclinic orbits, enclosing the nonlinear centres $(\pm (-a(\lambda)/b(\lambda))^{1/k-1}, 0)$. If $\lambda > 0$, $(\pm(-a(\lambda)/b(\lambda))^{1/k-1}, 0)$ are saddle points which are connected by two heteroclinic orbits, bounding a domain containing the nonlinear centre $(0,0)$ (see Figure 2).

Let us observe that orbits in a) and b), $\lambda > 0$, are all symmetric with respect to the reflection R_1 , with $R_1(v_0, v_0') = (v_0, -v_0')$. Property (3.7) of h implies that (3.8) inherits that symmetry. The results of Renardy in [15] are then applicable and (3.8) inherits also the phase plane configuration of (4.1), with respect to small solutions.

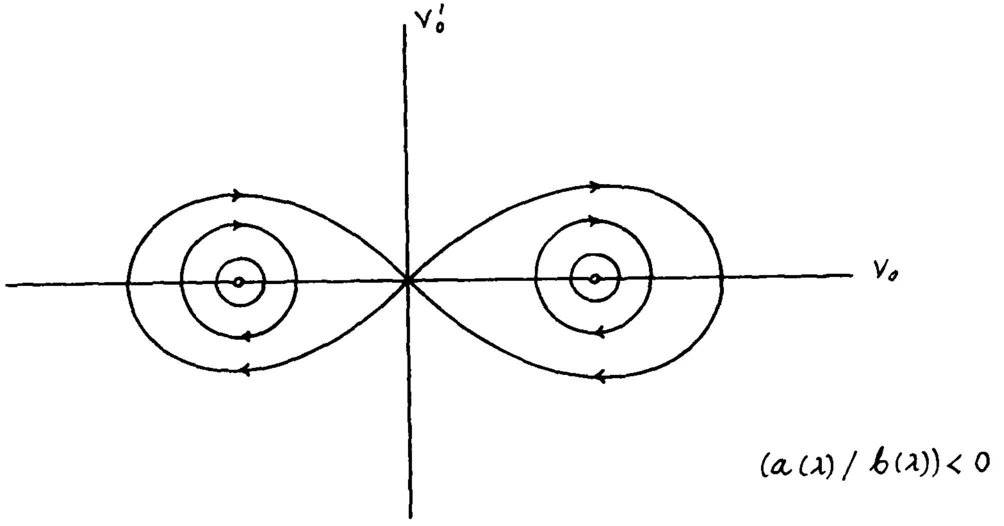

Figure 2.

In the case b), $\lambda > 0$, the orbits in (4.1) are R_2-invariants, with $R_2(v_0,v_0') = (-v_0,v_0')$. Suppose that $f = f(\lambda,u)$ is odd in u, in equation (1). Then h inherits this property: $h(\lambda,v_0,-v_0') = -h(\lambda,v_0,v_0')$ (see [12]). Hence, (3.8) has this symmetry and Renardy's results ([15]) are again applicable. So, (3.8) has the same plane configuration as (4.1), with respect to small solutions.

The above considerations allow us to ensure, for $|\lambda|$ small enough, the existence of several types of DWFs for the Neumann problem. We summarize their properties in the next theorem:

THEOREM 3. *Assume that conditions* (i) ..., (iv) *holds and* $\lambda \in (-\varepsilon,\varepsilon)$, $\lambda \neq 0$, *small enough. Then the Neumann problem for* DWFs *admits solutions* $u = u(x,t)$, *with the following properties:*

(i) *If* k *is even (respectively odd,* $\lambda b(\lambda) < 0$ *and* $\lambda < 0$), *there exists* $u = u(\lambda)$, *a solution of* (2.1) *and a* DWF $u = u(\lambda,x,t)$ *such that:*

(4.2) $$|u(\lambda,\cdot,t) - u(\lambda)|_{(C^0(\bar{\Omega}))^m} \leq Ce^{-\beta|t|} \quad t \in \mathbb{R},$$

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

where $C > 0$, $\beta > 0$ *and* $u(\lambda) = 0$ *in* $\lambda < 0$.

(ii) *If* k *is odd,* $f = f(\lambda,u)$ *is odd in* u *and* $\lambda b(\lambda) < 0$ *then, for* $\lambda \in (0,\varepsilon)$ *there exist* $u_+(\lambda)$ *and* $u_-(\lambda)$ *solutions of* (2.1), *and a* DWF $u = u(\lambda,x,t)$ *such that:*

$$|u(\lambda,\cdot,t) - u_+(\lambda)|_{(C^0(\bar{\Omega}))^m} \le C_+ e^{-\beta_1 t} \qquad t \to \infty,$$

$$|u(\lambda,\cdot,t) - u_-(\lambda)|_{(C^0(\bar{\Omega}))^m} \le C_- e^{\beta_2 t} \qquad t \to -\infty,$$

where C_+ , C_- , β_1 *and* β_2 *are positive constants.*

(iii) *Furthermore, in the cases* (i) *and* (ii) *there exists a small amplitude one parameter family of* t-*periodic* DWFs.

Proof. First of all, observe that (3.8) admits a branch of stationary solutions $(\lambda,v_0) = (\lambda(v_0),v_0)$, $|v_0|$ small, $\lambda(\cdot)$ being a C^r function of the form:

$$\lambda = \frac{b(0)}{a'(0)} v_0^{k-1} + 0(v_0^k) ,$$

and satisfies:

$$a(\lambda) + b(\lambda)v_0^{k-1} + \frac{r_1(\lambda,v_0,0)}{v_0} = 0 . \tag{4.4}$$

Thus, $\exists\varepsilon > 0$ such that: a) If k is even, $\forall\lambda \in (-\varepsilon,\varepsilon)$, $\exists v_0(\lambda) \neq 0$ stationary solution of (3.8). b) If k is odd, $\forall\lambda \in (-\varepsilon,\varepsilon)$ such that $\lambda b(\lambda) > 0$, $\exists v_0^+(\lambda)$ and $v_0^-(\lambda)$ which are non zero stationary solutions of (3.8). Furthermore, the stability and connection properties of $(0,0)$, $(v_0(\lambda),0)$, $(v_0^{\pm}(\lambda),0)$ are those of the equation (4.1). On the other hand each of the solutions $w = w(\lambda)$ generates a stationary DWF $v_\lambda = v_\lambda(z)$:

$$v_\lambda = w(\lambda)e_1 + h(\lambda,w(\lambda),0) .$$

Let us prove the point (iii). If $p_\mu = p(\mu,\lambda,s)$ denotes the family of periodic solutions, with period $\omega(\mu,\lambda)$, associated to a nonlinear centre, then we get the family of DWFs:

$$v_\mu(s) = p_\mu(s)e_1 + h(\lambda, p_\mu(s), p_\mu'(s))$$

On the other hand: $\exists \eta > 0$, $\kappa_1, \ldots, \kappa_n \in \mathbb{R}^n$ such that, every DWF of (1) is written as: $u = u(x,t) = v(s,z)$, with $s = \kappa_1 x - \eta t$, $z_j = \kappa_j x$, $j \in \{2, \ldots, m\}$; η defined by the equality: $Sc = \eta e_1$. Thus, $u_\mu(\lambda, x, t) = v(s,z)$ is a t-periodic DWF, with period $-\frac{\omega(\mu,\lambda)}{\eta}$.

For proving (i), let $|\lambda|$ small and let $p = p(s)$ be a homoclinic of (3.8) satisfying:

$$\lim_{|s|\to\infty} p(s) = w(\lambda) \,,$$

where $w(\lambda) = 0$ if $\lambda < 0$, $w(\lambda) = v_0(\lambda)$ if $\lambda > 0$. Then, $\exists C_0 > 0$ and $\beta_0 > 0$ such that:

$$|p(s) - w(\lambda)| + |p'(s)| + |p''(s)| \le C_0 e^{-\beta_0 |s|} \qquad s \in \mathbb{R} \,.$$

If $p_1(s) = h(\lambda, p(s), p'(s))$, boundedness of $\frac{\partial h}{\partial v_0}$, $\frac{\partial h}{\partial v_0'}$ on $B^2_{\varepsilon_2}(0)$ implies

$$\|p_1(s) - w(\lambda)\|_m + \|p_1'(s)\|_m \le C_1 e^{-\beta_0 |s|} \qquad s \in \mathbb{R} \,,$$

where, for $\phi \in (H^1(D))^m$, $\|\phi\|_m = |\phi|_{(H^1(D))^m}$. If $v(s) = p(s)e_1 + p_1(s)$ then:

$$\max\{\|v(s) - v_\lambda\|_m, \|v'(s)\|_m\} \le C_2 e^{-\beta_0 s} \qquad s \in \mathbb{R} \,,$$

where $C_2 > 0$.

Let us put $u_\lambda(x) = v_\lambda(\kappa_2 x, \ldots, \kappa_n x)$, we are going to estimate:

$$|u(\cdot, t) - u_\lambda(\cdot)|_{(H'(\Omega))^m} \,.$$

By using the transformation S (see §1): $W = \bigcup_{t \in \mathbb{R}} W_t$ with $W_t = S(K\Omega - ct)$. To estimate the above norm, we must estimate

$|v(s)-v_\lambda|_{(H^1(W_t))^m}$ or, equivalently, the norm:

$$|v(s)-v_\lambda|_{(H^1(I(t)\times D))^m} ,$$

where $I(t) = [\alpha_0-\eta t,\alpha_1-\eta t]$ and:

$$\alpha_0 \text{ (respectively } \alpha_1) = \inf(\sup)\ \{s \in \mathbb{R} / (s,z) \in SK\Omega \subset \mathbb{R}^n\} .$$

Thus, we have that:

$$|v(s,\cdot)-v_\lambda|^2_{(H^1(I(t)\times D))^m} = \int_{I(t)} \|v(s)-v_\lambda\|^2_m ds + \int_{I(t)}\int_D |\frac{dv}{ds}(s,z)|^2 dzds$$

$$\leq \int_{I(t)} (\|v(s)-v_\lambda\|^2_m + \|v'(s)\|^2_m) ds \leq 2C_2^2 \int_{I(t)} e^{-2\beta_0|s|} ds .$$

And the last term is bounded by:

$$\frac{C^2{}_2}{\beta} \max\{(e^{2\beta_0\alpha_1}-e^{2\beta_0\alpha_0}) , (e^{-2\beta_0\alpha_0}-e^{-2\beta_0\alpha_1})\} .$$

Thus, there exists $C_3 > 0$ such that:

$$|v(s)-v_\lambda|_{(H^1(W_t))^m} \leq C_3 e^{-\beta_0\eta|t|} \qquad t \in \mathbb{R} .$$

If $u = u(x,t)$ is the DWF associated to $v = v(s)$ then we can put $u(x,t) = \tilde{v}(y)$, where $\tilde{v}(y) = v(S^{-1}y)$ with $y = Kx - ct$. If $y' = S^{-1}y$:

$$\int_{K\Omega-ct} |\tilde{v}(y)-\tilde{v}_\lambda|^2 dy = |\det S| \int_{W_t} |v(y')-v_\lambda|^2 dy' ,$$

$$\int_{K\Omega-ct} |\nabla_y \tilde{v}_i(y)|^2 dy = |\det S| \int_{W_t} |S^{-1}\nabla_{y'} v_i(y')|^2 dy' , \quad 1 \leq i \leq m .$$

Taking $C_4 = C_3 |\det S|^{\frac{1}{2}} \max\{1, \|S^{-1}\|\}$:

$$|\tilde{v}(\cdot)-v_\lambda|_{(H^1(K\Omega-ct))^m} \leq C_4 e^{-\beta_0\eta|t|} \quad \forall t \in \mathbb{R} ,$$

hence

$$(4.5) \qquad |u(\cdot,t)-u_\lambda(\cdot)|_{(H^1(\Omega))^m} \le C_5 e^{-\beta_0\eta|t|} \qquad \forall t \in \mathbb{R} ,$$

where $C_5 = C_4|\det K|^{-\frac{1}{2}} (\min\{1,A\})^{\frac{1}{2}}$, $A = \min\limits_{1\le i\le m} \{\frac{1}{k_i^2}\}$.

On the other hand, boundedness of $\dfrac{\partial^2 h}{\partial v_0^2}$, $\dfrac{\partial^2 h}{\partial v_0 \partial v_0'}$ and $\dfrac{\partial^2 h}{\partial v_0'^2}$ in $B_2^2(0)$ implies that:

$$\|p_1''(s)\|_m \le C_1' e^{-2\beta_0|s|} \qquad s \in \mathbb{R} ,$$

with $C_1' > 0$. So, $\exists C_2' > 0$ such that:

$$|v''(s)|_{(L^2(D))^m} \le \|v(s)\|_m \le C_2' e^{-\beta_0|s|} \qquad s \in \mathbb{R} .$$

On the other hand:

$$\Delta_z v_{\lambda,i} = -f_i(\lambda,v_\lambda) \qquad 1 \le i \le m ,$$

then, for $1 \le i \le m$:

$$\Delta_z(v_i(s)-v_{\lambda,i}) = -d_i(\lambda)(v_i(s)-v_{\lambda,i}) - (g_i(\lambda,v)-g_i(\lambda,v_\lambda)) - v_i''(s) ,$$

where $s \in \mathbb{R}$. Thus:

$$|\Delta_z(v_i-v_{\lambda,i})|_{(L^2(D))^m} \le |v_i''(s)|_{L^2(D)} + \sup_{|\lambda|<\varepsilon} |d_i(\lambda)|\, |v_i(s)-v_{\lambda,i}|_{H^1(D)}$$

$$+ |g_i(\lambda,v)-g_i(\lambda,v_\lambda)| ,$$

Considering that $g_i \in C^r((H'(D))^m , L^2(D))$ holds, then:

$$|\Delta_z(v_i(s)-v_{\lambda,i})_{L^2(D)} \le C_3' e^{-\min\{1,\eta\}\beta_0|s|} \qquad s \in \mathbb{R} ,$$

with $C_3' > 0$. Because $(v_i(s)-v_{\lambda,i}) \in H^2(D)$ and the ellipticity of (4.5), we get the estimate:

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

$$|v_i(s)-v_{\lambda,i}|_{H^2(D)} \leq (|v_i(s)-v_{\lambda,i}|_{L^2(D)} + |\Delta_z(v_i(s)-v_{\lambda,i})|_{L^2(D)}) ,$$

thus (4.2) follows from the above inequality and the continuous inclusion $H^2(D) \subsetneq C^0(D)$.

Finally, let us show that $u_\lambda = u_\lambda(x)$ is homogeneous, that is is a stationary solution of (2.1). In fact, if $u(\lambda)$ is a small solution of (2.1) given by Proposition 1, then $u(\lambda)$ must lie on the centre manifold (see [4]). Therefore, $\exists \tilde{w}(\lambda) \in \mathbb{R}$ a stationary solution of (3.8) such that: $u(\lambda) = \tilde{w}(\lambda)e_1 + h(\lambda,\tilde{w}(\lambda),0)$, where $h(\lambda,\tilde{w}(\lambda),0)$ is homogeneous. Because of the local uniqueness of bifurcated branches of stationary solutions of (3.8) at $(\lambda,v_0) = (0,0)$ (compare the equations (4.4) and (2.4)), it follows that $\tilde{w}(\lambda) = w(\lambda)$.

Let us observe that periodic DWFs obtained above are also small in the $(C^0(\Omega))^m$ norm. The proof in the case (ii) is analogous.#

5. Final remarks

(A) Theorem 3 settles the existence of homo- and heteroclinic DWFs of (1), in (i) and (ii) respectively. Moreover, such DWFs connect stable solutions of (2.1). On the other hand, such DWFs approximate - in the $(H^1(\Omega))^m$ norm - to PWFs. Specifically, in the case (i) for example, it is easy to show that:

$$|u(\cdot,t) - z(\cdot,t)|_{(H^1(\Omega))^m} \leq C'e^{-\beta|t|} \qquad t \in \mathbb{R} ,$$

where

$$z(x,t) = v_0(\kappa_1 x-\eta t)e_1 + h(\lambda,w(\lambda),0) = \emptyset\ (\kappa_1 x-t) .$$

Observe that $z = z(x,t)$ is a PWF which also exhibits a homoclinic character: it connects the same solution of (2.1) as $u = u(x,t)$. However $z = z(x,t)$ is not a solution of the equation (1).

(B) The Dirichlet problem can be analyzed with the techniques employed in the Neumann problem. Essentially, $(H_0'(D))^m$ must replace to $(H'(D))^m$ in §3, and in hypothesis (i), (ii), (iii) must the positiveness be considered

https://doi.org/10.1017/S0004972700002823 Published online by Cambridge University Press

of the first eigenvalue σ_0 of $-\Delta_z$ in $H_0^1(D)$. In this way, the conclusions of Theorem 3 still hold for the Dirichlet problem. Observe that, the DWFs of points (i) and (ii) now converge to stationary, but not necessarily homogeneous, solutions of (1) (see [16]).

(C) Condition $b \neq 0$ in (1) is essential for the reversibility in s of (P_D) and (P_N) : no terms in v' appear in those problems. If $b = 0$ the identity $Kb + c = 0$ is not possible with K and c non-zero.

However, when $b = 0$ and $m = 1$ (that is, the scalar case) we have obtained the existence of heteroclinic DWFs for (P_N) and (P_D). The reaction term structure is:

$$f(\lambda,u) = a(\lambda)u + a_k(\lambda)u^k + 0(|u|^{k+1}) ,$$

where $a(0) = 0$, $\frac{du}{d}(0) \neq 0$ and $a_k(0) = 0$. This structure in f is precisely the right one for obtaining, in the scalar case, the results of Theorem 3 (see [8] and [16]). However, to eliminate the restriction $b \neq 0$ in the case of non scalar equations, is an open problem.

References

[1] R. Abraham and J. Robin, *Transversal Mappings and Flows,* (W.A.Benjamin, New York, 1967).

[2] D. Aronson and H. Weinberger, *Nonlinear diffusion in population genetrics, combustion, and nerve impulse propagation* (Lecture Notes in Maths. 446, 5-49, Springer, New York, 1975).

[3] M.S. Berger, *Nonlinearity and Functional Analysis,* (Academic Press, New York, 1977).

[4] S.N. Chow and J.K. Hale, *Methods of Bifurcation Theory,* (Grund. der Math. 251 Springer, Berlin, 1982).

[5] M.G. Crandall, "An introduction to constructive aspects of bifurcation theory". In *Applications of Bifurcation Theory,* Rabinowitz Ed. (Academic Press, New York, 1977).

[6] H.O. Fatorini, "The Cauchy Problem", *Encyclopedia of Maths. and Appls.* 18, (Addison-Wesley Pub. Co., Reading, Massachusetts, 1983).

[7] P.C. Fife, *Mathematical aspects of reacting and diffusing systems* (Lecture Notes in Biomaths. 28 Springer, New York, 1979).

[8] P.C. Fife and J.B. McLeod, "A phase plane discussion of convergence to travelling fronts for nonlinear diffusion", *Arch. for Rat. Mech. and Analysis* 75 (1981), 281-314.

[9] J. Fraile and J. Sabina, "Boundary value conditions for wave solutions of reaction-diffusion systems", *Proceedings of the Royal Society of Edinburgh* 99A (1984), 127-136.

[10] L.N. Howard, "Chemical wave-trains and related structures". In *Applications of Bifurcation Theory*, P. Rabinowitz Ed.,(Academic Press, New York, 1977).

[11] G. Iooss and D. Joseph, "Elementary Stability and Bifurcation Theory", *Undergrad. Texts in Maths.*, Springer, Berlin, 1980.

[12] K. Kirchgassner, *Homoclinic bifurcation of perturbed reversible systems* (Lecture Notes in Maths., 1017 (1982).)

[13] N. Kopell and L.N. Howard, "Plane wave solutions to reaction-diffusion equations, *Stud. Appl. Math.* 52 (1973), 291-328.

[14] J.D. Murray, Lectures on Nonlinear Differential Equation Models in Biology, (Clarendon Press, Oxford, 1977).

[15] M. Renardy, "Bifurcation of singular and transient solutions". In *Recent Contributions to Nonlinear Partial Differential Equations*, (Berestycki-Brezis Eds., Pitman, London, 1981).

[16] J. Sabina, Doctoral Thesis. Universidad Complutense de Madrid, Madrid, (1984).

Dpto. de Ecuaciones Funcionales,
Facultad de Matemáticas, U. Complutense
28040-Madrid

and

Dpto. de Matemáticas I,
ETSI de Telecomunicación, U. Politecnica de Madrid
28040-Madrid.