ON THE PROJECTION OF THE REGULAR POLYTOPE $\{5, 3, 3\}$ INTO A REGULAR TRIACONTAGON

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1. Introduction. The polytope $\{5, 3, 3\}$ is one of the six convex regular four-dimensional polytopes, and in some ways is the most complicated of the six, being bounded by 120 dodecahedra. The symbol $\{p, q, r\}$ to denote a regular figure in four dimensions originated with L. Schläfli: p is the number of edges bounding each face, q is the number of edges of a cell which meet at a vertex of this cell, and r is the number of cells that share each edge. The Schläfli symbol can be generalized to denote regular figures in any number of dimensions; in particular, a regular p-gon is symbolized by $\{p\}$, and a dodecahedron by $\{5, 3\}$.

It is possible to project the 120-cell (as the polytope $\{5, 3, 3\}$ is "popularly" called) onto a plane so that the projection thus formed is bounded by a regular 30-gon. This polygon arises as the plane projection of a Petrie polygon of the 120-cell. The Petrie polygon of an n-dimensional regular polytope (Coxeter [2], p. 223) is defined inductively as a skew polygon such that any $n - 1$ consecutive sides, but no n, belong to a Petrie polygon of a cell. The induction begins with the case $n = 3$, when "a Petrie polygon of a cell" has to be replaced by "a face".

Because of the way in which the projection arises, its symmetry group will be that of the regular triacontagon, namely, D_{30}, the dihedral group of order 60.

The other five regular convex four-dimensional polytopes can also be projected into regular polygons of various sizes; their projections have already been extensively studied, as well as depicted. A number of excellent drawings of projections of the reciprocal polytope $\{3, 3, 5\}$ were made by S. L. van Oss.

2. Characteristics of the projection of $\{5, 3, 3\}$. In the projection, the 600 vertices can be classified in 12 sets: each set consists of either of 30 or of 60 vertices which lie on a circle concentric with the outer triacontagon. (Coxeter [2], p. 250.) Let these sets be represented by the 12 letters A, B, C, D, ..., L, with A denoting the outermost set, and the other letters denoting sets that lie on successively smaller circles, so that L denotes the innermost set.

(These letters will also be used, when it is deemed helpful, to represent the circles on which the vertex-projections lie: thus, we shall sometimes refer to "vertex-ring K", etc. The context should make clear which signification we are giving to the letter.)

Sets A, C, J, and L consist of 30 vertices each; the vertices of each set are evenly spaced, and thus are the vertices of a regular 30-gon. However, A is the only set in which the vertices are joined so that the sides of the 30-gon are present also. The vertices of L are joined in such a way that a regular $\{\frac{30}{11}\}$ is formed (each vertex is joined to two other vertices 11 steps away). The other sets contain 60 vertices each. The vertices of each of these sets are not evenly spaced, but if one selects a particular vertex and goes around the ring, skipping every second vertex, the set of vertices remaining will form a regular triacontagon (as will the set of vertices omitted). Thus all 60 of the vertices are the vertices of two concentric regular triacontagons or of a centrally-symmetrical equiangular 60-gon with symmetry group D_{30}.

For later convenience, let us adopt a system for labeling the vertices in the projection.

Establish a horizontal line, and a point on this line to serve as an "origin". Take the projection (assumed to be already drawn); locate it so that its center of symmetry coincides with the origin, and so that two vertices of the outer triacontagon lie on the horizontal line, one to the left, the other to the right of the origin. Call the right-hand vertex A_0.

Then, label the vertices of the outer triacontagon in the counterclockwise direction as follows:

$$A_0, A_2, A_4, \ldots, A_{56}, A_{58}.$$

Here "A" refers to the set of vertices under consideration, and the subscripts single out the individual vertices of the set. The innermost vertices can be labelled in the same manner:

$$L_0, L_2, L_4, \ldots, L_{56}, L_{58}.$$

Sets C and J also have 30 vertices each; however, they will not have any vertices lying on the horizontal line. Their vertices are "staggered" with respect to those of A and L. One can choose, in each case, the vertex lying immediately above the line to the right of the origin. Denote the two vertices chosen by C_1 and J_1.

Then, numbering counterclockwise, we obtain:

$$C_1, C_3, C_5, \ldots, C_{57}, C_{59},$$

and

$$J_1, J_3, J_5, \ldots, J_{57}, J_{59}.$$

The remaining eight rings have sixty vertices each. Let us choose a typical ring of this type, say B. No vertices of B lie on the horizontal line; neither do any lie on the line determined by C_1 and J_1. There is a vertex lying between these two lines, however. It could be labelled either B_0 or B_1; let us choose B_0. Then, proceeding counterclockwise and labelling successive vertices, we obtain:

$$B_0, B_1, B_2, \ldots, B_{58}, B_{59}.$$

Exactly the same procedure can be used for the remaining seven sets of vertices.

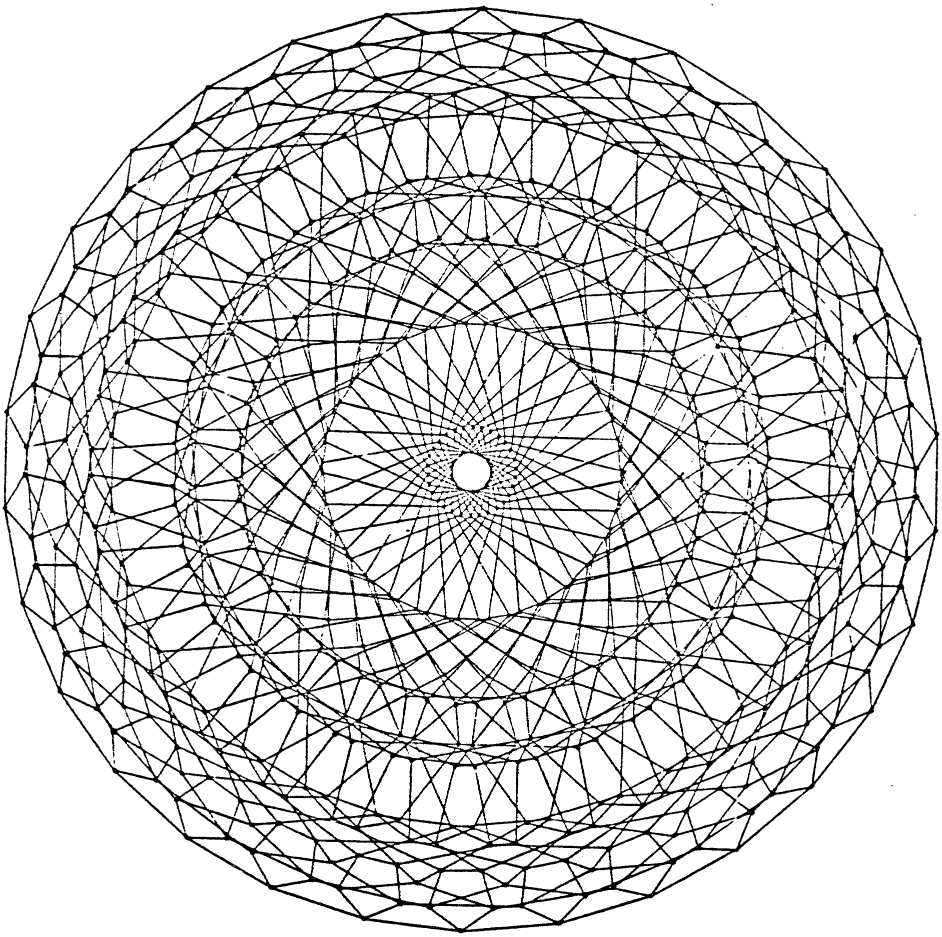

Figure 1

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Returning to our description of the projection, we next consider the projections of the edges of the 120-cell. (Above, we spoke of "vertices", when in reality we referred to the projections of vertices; so here also, we shall refer to the various parts of the projection as "edges", "faces", and "cells".)

The projection contains 1200 edges; these can be classified into four sets, using their lengths. Each set contains 300 of the edges. We label these sets e_1, e_2, e_3, e_4. (These symbols will also be used to refer to the lengths of the edges in each set.) We perform the labelling so that $e_1 < e_2 < e_3 < e_4$. Then, by computation and geometric considerations, these lengths are found to be in the following proportions to e_1:

$$\frac{e_2}{e_1} = \frac{\tau}{2} \csc 24^\circ = 1.9890 ;$$

$$\frac{e_3}{e_1} = \frac{1}{2} \csc 12^\circ = 2.4048 ;$$

$$\frac{e_4}{e_1} = 2\tau \cos 24^\circ = 2.9563 ,$$

where $\tau = \dfrac{\sqrt{5}+1}{2}$, the "golden section" ratio.

These proportions can be expressed in other ways also: see Coxeter [2], pp. 247 and 248. We shall see later that our e_1, e_2, e_3, e_4 are proportional to his d, c, b, a, which are the radii of the four vertex-rings of the tricontagonal projection of $\{3,3,5\}$.

Furthermore, if we designate the radius of one of the vertex-rings, say G, by r(G), the proportions of these radii to e_1 are as follows:

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$$\frac{r(A)}{e_1} = 11.503 ; \qquad \frac{r(G)}{e_1} = 7.3648 ;$$

$$\frac{r(B)}{e_1} = 10.946 ; \qquad \frac{r(H)}{e_1} = 6.8185 ;$$

$$\frac{r(C)}{e_1} = 10.358 ; \qquad \frac{r(I)}{e_1} = 5.8283 ;$$

$$\frac{r(D)}{e_1} = 9.9773 ; \qquad \frac{r(J)}{e_1} = 5.1216 ;$$

$$\frac{r(E)}{e_1} = 9.3286 ; \qquad \frac{r(K)}{e_1} = 3.7025 ;$$

$$\frac{r(F)}{e_1} = 8.9036 ; \qquad \frac{r(L)}{e_1} = 1.0886 .$$

(These proportions were obtained by considering the faces and cells, which will be discussed next.)

The faces are of eight types, differing in shape and size. They are illustrated in Figure 2. The numbers appearing at the centers of the sides denote the sets to which the sides belong: for example, a "2" denotes that the side is contained in the set e_2 of edges. In the last four, the numbers in the "corners" denote the interior angles, in degrees.

The first four are regular pentagons; 30 of each of these four types appear in the projection. The rest have only a line of symmetry. It will be noted also that all the interior angles can be expressed as integral multiples of 6^o. There are 150 of each of the last four types. An explanation of the presence of the regular pentagons will follow the present discussion.

It is interesting to note that, in the projection, none of the regular pentagons shares an edge or a vertex with any other regular pentagon. Also, the regular pentagons occur in four "rings", each ring consisting of pentagons of a different size,

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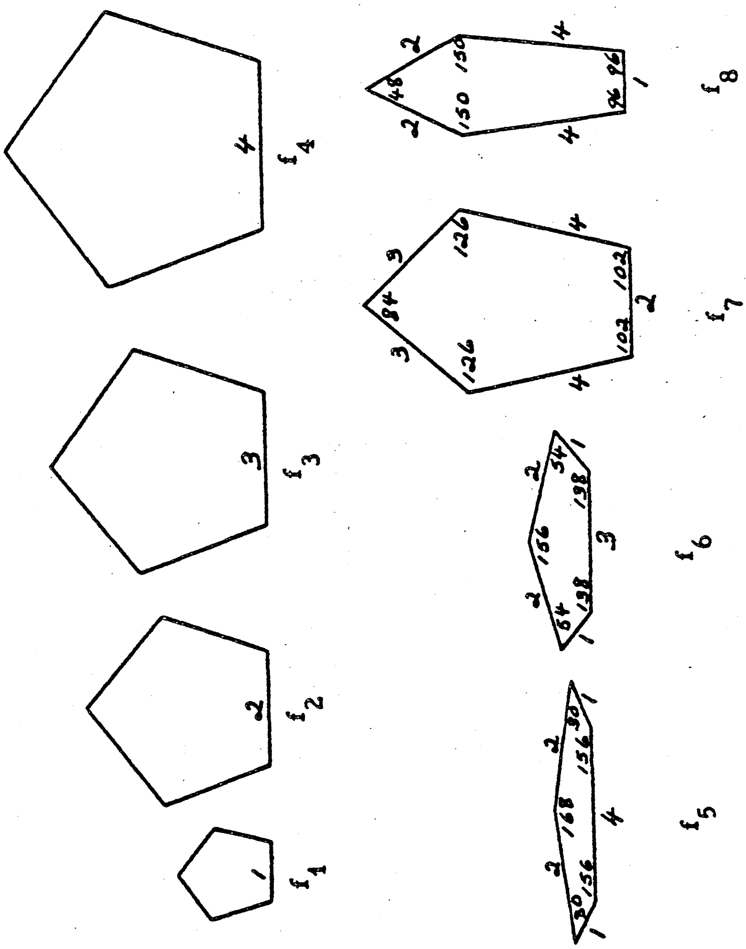

Figure 2

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with the smallest pentagons, f_1, forming the outermost ring; f_2, the next; f_3, the next; and the largest pentagons, f_4, the innermost ring.

From the first property, it follows that every vertex of the projection is a vertex of exactly one regular pentagon. We also see that each set of regular pentagons accounts for three of the vertex-rings; e.g., f_1 has as vertices the members of sets A, B, and D.

Typical regular pentagons are:

$$f_1 : A_0 B_0 D_0 D_{59} B_{59} ;$$

$$f_2 : C_3 E_4 G_4 G_1 E_1 ;$$

$$f_3 : F_1 F_4 H_5 J_3 H_0 ;$$

$$f_4 : I_5 I_{10} K_{14} L_8 K_1 .$$

The more numerous irregular pentagons are scattered throughout the projection: for example, two f_5's are $A_0 A_2 A_4 B_3 B_0$ and $H_5 I_5 K_1 K_{58} J_3$.

There are four types of cells, illustrated in Figure 3. The most prominent difference among the types is the amount of distortion caused by projection. Each of the four sets forms a "ring", with c_1 the outermost, and so on, until we arrive at c_4, the innermost. The more violently compressed cells are thus on the outside, as might be expected.

There are 30 cells of each type, and the centroids of all the cells of a given type are vertices of a regular triacontagon; we thus have four concentric triancontagons (in fact, the 120 points are the vertices of a projection of the reciprocal polytope $\{3, 3, 5\}$; see Coxeter [2]).

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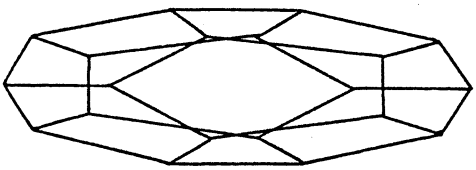
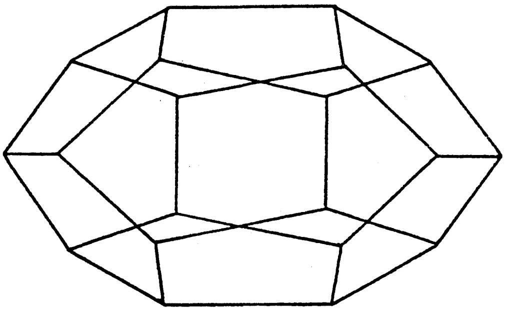
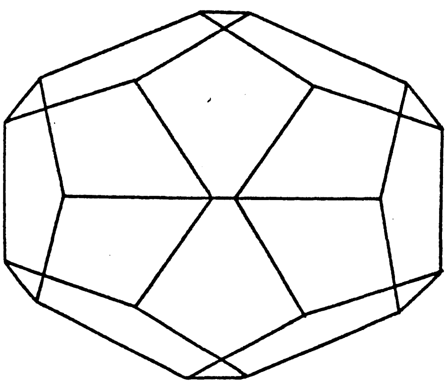


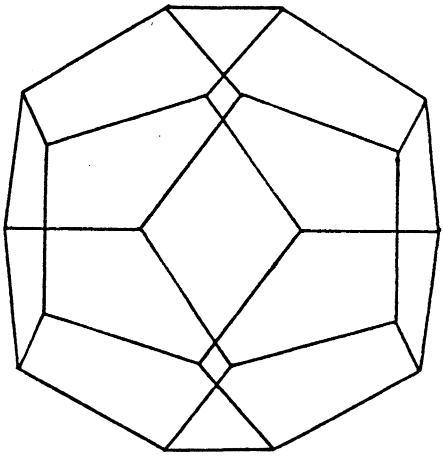
c_1



c_2



c_3



c_4

Figure 3

Each type of cell has among its faces representatives of exactly one of the four types of regular pentagon. The f_1's are faces of the c_1's, and so forth.

Every cell has two perpendicular lines of symmetry, of which one passes through the center of symmetry of the projection.

The triacontagonal projection of the 120-cell was first drawn by W. A. Wythoff, but not in reproducible form. The drawing accompanying the present article also appears in Coxeter [1], p. 403.

3. The regular pentagons. As previously mentioned, 120 of the 720 faces of the 120-cell project into regular pentagons. We shall demonstrate this using two distinct methods. Both involve properties of the triacontagonal projection of the 600-cell (Coxeter [2]).

It is important to note that the centers of the 600 tetrahedra in the latter are the vertices of a triacontagonal projection of the 120-cell. That is to say, when a 600-cell is projected into a regular 30-gon, the inscribed 120-cell will also project into a regular 30-gon, as will any 120-cell in reciprocal position to the 600-cell.

Let us, for convenience, refer to the 120-cell and 600-cell themselves as Π and Π', respectively, and to their projections as P and P'.

One can classify the edges of Π' into 72 sets, such that the edges of each set form a regular decagon lying in a plane passing through the center of the polytope. Thus the edges of Π' comprise 72 of these decagons. (Manning [4], p. 323). Each of the four rings of vertices of P' accounts for three of these regular decagons. For example (using Coxeter's labelling of the vertices of P') one such decagon has as its vertices the points

$$A_0 A_6 A_{12} A_{18} A_{24} A_{30} A_{36} A_{42} A_{48} A_{54}$$

(Coxeter [2], p. 277).

Thus, twelve of the regular decagons of Π' project into regular decagons in P'.

Let us consider the relationship between the projection plane and the planes containing these twelve decagons. Call the former Q; and a typical one of the latter, R. Let Q pass through the center of Π'. Each R-plane contains exactly one decagon. It has at least one point (the center of Π') in common with Q. Furthermore, it has only this point in common with Q; for, since a figure in R projects into a similar figure in Q, the two planes must be isocline (Manning [4], p. 123).

Given a fixed Π', it is possible to construct any number of 120-cells in dual position to this Π', differing only in size. Choose Π so that the midpoints of its faces coincide with the midpoints of the edges of the Π'. With Π and Π' in this relationship. each pentagon of Π is the equatorial pentagon of a pentagonal dipyramid, whose apices are the ends of the edge of Π' which intersects the pentagon. It is possible to construct a semi-regular polytope bounded by 720 such dipyramids. This is the reciprocal of $\left\{ {3 \atop 3,5} \right\}$ in the notation of Coxeter [2], p. 146.

An edge of Π' lying in one of the R-planes will be perpendicular to the Π-pentagon which bisects it. In addition, the line joining the center of Π' to the point at which the edge and pentagon intersect is also perpendicular to the pentagon. This line also lies in R. We therefore have two distinct lines in one plane, both perpendicular to another plane at the same point; the two planes must be completely orthogonal. (See Sommerville [6], p. 31.)

If one of two completely orthogonal planes is isocline to a third, then the other will likewise be isocline to the third. Hence, this particular face of Π is isocline to the projection plane.

This means that this pentagon projects into a regular pentagon. Thus we have shown that P must contain some regular pentagons. In fact, we have their number: it is equal to the number of edges of P' which occur in the regular decagons, namely 120.

This treatment also explains the occurrence of the regular pentagons in four distinct "rings", and the fact that all the pentagons of a single "ring" are the same size.

The second method for showing the existence of the regular pentagons makes use of symmetry, and, again, of the regular decagons of P'.

The cells of Π have central symmetry, so that any face of a cell is parallel to its opposite in the same cell. If we place a second cell on this face, we will have three parallel faces, one common to the two dodecahedra. Making use of the transitivity of parallelism in this way, we can build a "chain" of n dodecahedra, with $n+1$ parallel faces.

When $n = 10$, we find that if we "bend" the chain (by rotation about each of the 9 interior faces) until the other two "free" faces meet, we have a generalized "zone", consisting of 10 dodecahedra joined in pairs by 10 mutually parallel faces, which will fit around the equator of a 120-cell (so that the dodecahedra coincide with cells of the polytope). The 120-cell has 72 of the zones (of which any two have exactly two cells in common). We can thus classify the faces of the 120-cell into 72 sets, such that two faces are in the same set if and only if they are parallel.

Now consider a 600-cell (Π') in dual position. Since the two figures have the same axes of symmetry, any portions of Π' which have pentagonal symmetry (or whose symmetry groups contain as a subgroup the pentagonal group) must lie in planes parallel to those containing faces of Π. Since the equatorial decagons of Π' have such symmetry, they must lie in such planes. In fact, there is a one-to-one correspondence between the 72 decagons and the 72 sets of parallel faces of Π.

12 of these decagons project into regular decagons. All of the faces lying in planes parallel to the planes containing these decagons will project regularly. Again, we see that there will be 120 regular pentagons in Π.

This second method establishes another useful fact: it can be seen that the four edge-lengths $e_1, \ldots, e_4$ of P are in the same ratio as the radii of the four vertex-rings of P' :

$$\frac{e_1}{d} = \frac{e_2}{c} = \frac{e_3}{b} = \frac{e_4}{a} .$$

That is, we have verified the remark made on page 389.

We note also that we are enabled to show that four of the vertex-rings of P are in the same ratio as the four vertex-rings of P': namely, B, F, G, and K.

This is interesting because it is known (Coxeter [2], p. 269) that, among the vertices of the 120-cell, we can find the vertices of a 600-cell (just as we can select from among the vertices of a dodecahedron 8 vertices which will be the vertices of a cube). The 240 vertices of the four vertex-rings B, F, G, K are the vertices of two triacontagonal projections of the 600-cell, one being obtainable from the other by a very small rotation about the center of symmetry.

A comparison of the two methods discussed above shows that for each set of mutually parallel faces of Π, there will be a second set all of whose faces are completely orthogonal to all the faces of the first. If the projections of the faces of one set are f_1's, those of the faces of the other set will be f_4's; there is likewise a correspondence between f_2 and f_3.

Let s represent the length of an edge of Π. Then we have

$$e_1^2 + e_4^2 = e_2^2 + e_3^2 = s .$$

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