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PERIODIC ALGORITHMS AND THEIR APPLICATION

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0. Introduction. In two previous papers [**1**, **2**] we investigated the zeros of certain arithmetic functions. Using units of cubic fields, we also succeeded to construct, almost by accident, and as a by-product so to speak, entirely new and comparatively complicated combinatorial identities. In an interesting paper combinatorialist L. Carlitz [**10**] proved those identities in an elementary way. In a/m, we had to prove that the units used were fundamental ones.

Encouraged by these results, we took a closer look at this method that had led to the construction of combinatorial identities. Since the latter are such an important tool in mathematics, we thought it would be "einer Messe wert" to generalize these results and lay the theoretical foundations of a new method for the construction of highly sophisticated combinatorial identities. This indeed was done in a lengthy paper which soon took on the dimensions of a book [**3**]. But this laborious undertaking amply justified itself: not only did we succeed in disclosing many infinite classes of (sometimes quite sophisticated) new combinatorial identities, but we could also disclose the applicability of these results to such fields as transcendental numbers and the solution of Diophantine equations.

In [**3**], we demonstrated this new method of constructing combinatorial identities by narrowing the general theory to a few, though very broad, special cases. The reason for this was rather a pedagogical drive—we thought that this would be the best way to make the reader quickly acquainted with the new technique so that the student could soon manipulate with this delicate tool independently. This gap will be filled out in this paper. For in the present paper we again develop a newer method of constructing combinatorial identities which starts from the "old-new" method, follows its pathway, but later diverges into new directions. This divergence is based on the properties of a new algorithm which will be defined later. So the reader is asked to accept with olympic patience the exposition of the theory of the method, once called the new method, on how to construct interesting combinatorial identities.

1. The old-new method. Let

$$(1.1)\qquad \begin{cases} w \text{ be an } n\text{-th degree algebraic irrational over } \mathbf{Q}, \quad n \geqq 2; \\ e \text{ be an } n\text{-th degree unit in } \mathbf{Q}(w), \text{ not a root of unity.} \end{cases}$$

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The reader will have no difficulty in applying our method when e is a unit of degree k, $k|n$. Let

$$(1.2)\quad \begin{cases} e^n = a_0(1 + a_1 e + \ldots + a_{n-1}e^{n-1}) \text{ be the field equation of } e \text{ in } \mathbf{Q}(w), \\ e^{-n} = a_0 - a_{n-1}e^{-1} - \ldots - a_1 e^{-(n-1)} \text{ be the field equation of } e^{-1} \text{ in } \mathbf{Q}(w), \\ |a_0| = 1, \quad a_i \in \mathbf{Z}. \quad (i = 0, 1, \ldots, n-1.) \end{cases}$$

We now calculate the positive and negative powers of e. Since, by hypothesis, e is an n-th degree irrational in $\mathbf{Q}(w)$, we can use its n powers $e^0, e, e^2, \ldots, e^{n-1}$ as a basis for the maximal order of $\mathbf{Q}(w)$, and set

$$(1.3)\quad \begin{aligned} &e^m = x_{0,m} + x_{1,m}e + x_{2,m}e^2 + \ldots + x_{n-1,m}e^{n-1}, \\ &x_{i,m} \in \mathbf{Z}; \quad i = 0, 1, \ldots, n-1; \quad m = 0, 1, \ldots . \end{aligned}$$

We introduce the notation

$$(1.4)\quad a_0 = a_{01}; \quad a_0 a_i = a_{i,1}, \quad i = 1, \ldots, n-1.$$

From (1.2), (1.3) we obtain, using the notation (1.4)

$$e^{m+1} = \left(\sum_{i=0}^{n-2} x_{i,m}e^{i+1}\right) + x_{n-1,m}\sum_{i=0}^{n-1} a_{i,1}e^i.$$

$$e^{m+1} = \left(\sum_{i=0}^{n-2} x_{i,m}e^{i+1}\right) + \left(x_{n-1,m}\sum_{i=1}^{n-1} a_{i,1}e^i\right) + a_{01}x_{n-1,m}$$

and so

$$(1.5)\quad e^{m+1} = \left(\sum_{i=0}^{n-2} (x_{i,m} + a_{i+1,1}x_{n-1m})e^{i+1}\right) + a_{01}x_{n-1,m}.$$

By definition

$$(1.6)\quad e^{m+1} = \left(\sum_{i=0}^{n-2} x_{i+1,m+1}e^{i+1}\right) + x_{0,m+1}.$$

From (1.5), (1.6) we obtain, by comparison of coefficients of equal powers of e,

$$x_{n-1,m} = a_0 x_{0,m+1} \quad (\text{since } a_{01}^2 = 1),$$

$$x_{i,m} + a_{i+1,1}x_{n-1,m} = x_{i,m} + a_{i+1}x_{0,m+1} = x_{i+1,m+1} \quad (\text{since } a_{i+1,1}a_0 = a_{i+1})$$

hence

$$(1.7)\quad x_{n-1,m} = a_0 x_{0,m+1}; \quad x_{i,m} + a_{i+1}x_{0,m+1} = x_{i+1,m+1} \quad (i = 0, 1, \ldots, n-2).$$

From (1.7) we obtain, for $m > 0$,

$$(1.8)\quad \begin{aligned} x_{1,m} &= x_{0,m-1} + a_1 x_{0,m} \\ x_{2,m} &= x_{1,m-1} + a_2 x_{0,m} = x_{0,m-2} + a_1 x_{0,m-1} + a_2 x_{0,m} \\ x_{k,m} &= a_0 \sum_{i=0}^{k} a_{i,1}x_{0,m+i}, \quad k = 1, \ldots, n-1. \end{aligned}$$

From (1.7), (1.8) we obtain, for $k = n-1$,

$$(1.9)\quad x_{0,m+1} = \sum_{i=0}^{n-1} a_{i,1}x_{0,m-n+1+i},$$

and, substituting in (1.9) $m + n - 1$ for m,

(1.10) $$x_{0,m+n} = \sum_{i=0}^{n-1} a_{i,1}x_{0,m+i}.$$

With (1.8) we have expressed the coefficients of the powers of e in (1.3) as functions of $x_{0,v}$, and (1.10) is the recursion formula by means of which we shall calculate $x_{0,v}$ explicitly. To this end we use Euler's generating functions. We have by definition

(1.11) $$x_{0,0} = 1; \quad x_{0,1} = x_{0,2} = \ldots = x_{0,n-1} = 0.$$

With the initial values (1.11) and the recursion formula (1.10) we obtain, denoting by u an indeterminate (to be subdued to restrictions later) and summing over v,

$$\begin{aligned}\textstyle\sum_0^\infty x_{0,v}u^v &= x_{0,0} + x_{0,1}u + \ldots + x_{0,n-1}u^{n-1} + \textstyle\sum_n^\infty x_{0,v}u^v \\ &= 1 + \textstyle\sum_0^\infty x_{0,v+n}u^{v+n} \\ &= 1 + \textstyle\sum_0^\infty(\sum_{i=0}^{n-1} a_{i,1}x_{0,v+1})\, u^{v+n} =\end{aligned}$$
$$1 + u^n a_0(\textstyle\sum_0^\infty x_{0,v}u^v) + a_{1,1}u^{n-1}(\sum_0^\infty x_{0,v+1}u^{v+1}) +$$
$$a_{2,1}u^{n-2}(\textstyle\sum_0^\infty x_{0,v+2}u^{v+2}) + \ldots + a_{n-1,1}u(\sum_0^\infty x_{0,v+n-1}u^{v+n-1})$$
$$= 1 + a_0u^n\textstyle\sum_0^\infty x_{0,v}u^v + a_{1,1}u^{n-1}[(\sum_0^\infty x_{0,v}u^v) - 1] +$$
$$a_{2,1}u^{n-2}[(\textstyle\sum_0^\infty x_{0,v}u^v) - 1] + \ldots + a_{n-1,1}u[(\sum_0^\infty x_{0,v}u^v) - 1].$$

Carrying over all summands containing $\sum_0^\infty x_{0,v}u^v$ as a factor to the left side, we obtain, on summing over i and v,

(1.12) $$(1 - \textstyle\sum_0^{n-1} a_{i,1}u^{n-i})\sum_0^\infty x_{0,v}u^v = 1 - \sum_1^{n-1} a_{i,1}u^{n-i}, \quad \text{or}$$
$$(1 - \textstyle\sum_0^{n-1} a_{i,1}u^{n-i})\sum_0^\infty x_{0,v}u^v = 1 - \sum_0^{n-1} a_{i,1}u^{n-i} + a_0u^n.$$

Since u is an indeterminate, we can choose $1 - \sum_0^{n-1} a_{i,1}u^{n-1} \neq 0$, and obtain from (1.12)

$$\textstyle\sum_0^\infty x_{0,v}u^v = 1 + a_0u^n/(1 - \sum_0^{n-1} a_{i,1}u^{n-i}),$$
$$1 + x_{0,1}u + x_{0,2}u^2 + \ldots + x_{0,n-1}u^{n-1} + \textstyle\sum_n^\infty x_{0,v}u^v =$$
$$1 + a_0u^n/(1 - \textstyle\sum_0^{n-1} a_{i,1}u^{n-i}),$$

and so

(1.13) $$\textstyle\sum_0^\infty x_{0,v+n}u^v = a_0/(1 - \sum_0^{n-1} a_{i,1}u^{n-i}).$$

If we choose

$$\max\,[|a_{0,1}|, |a_{1,1}|, \ldots, |a_{n-1,1}|] = a \geqq 1,$$
$$|u| \leqq 1/2an < 1/2 < 1,$$

we obtain

$$|\textstyle\sum_0^{n-1} a_{i,1}u^{n-i}| \leqq \sum_0^{n-1}|a_{i,1}|\;|u^{n-i}| < \sum_0^{n-1} a|u| < 1/2,$$

hence

$$\textstyle\sum_0^{n-1} a_{i,1}u^{n-i} \neq 1,$$

as was demanded previously; also we can expand the right side of (1.13) as an infinite convergent power series of u and obtain

$$(1.13a)\quad \sum_0^\infty x_{0,v+n}u^v = a_0\sum_0^\infty(a_{0,1}u^n + a_{1,1}u^{n-1} + \ldots + a_{n-1,1}u)^j$$
$$= \sum_0^\infty a_0^{j+1}u^j(a_{n-1} + a_{n-2}u + \ldots + a_2u^{n-3} + a_1u^{n-2} + u^{n-1})^j.$$

We now want to know, on both sides of (1.13a), the coefficient of u^m. On the left side this is $x_{0,m+n}$. On the right side we shall obtain it from the sum of the coefficients of all x^m which appear in the expression, and only in it;

$$(1.13b)\quad S = \sum_{i=0} a_0^{m-i+1}u^{m-i}(a_{n-1} + a_{n-2}u + \ldots + a_2u^{n-3} + a_1u^{n-2} + u^{n-1})^{m-i}.$$

Using the binomial expansion, we obtain (since $a_0 = \pm 1$)

$$(1.13c)\quad a_0^{m+1}S = \sum_{i=0} a_0^i u^{m-i} \sum_{y_k} \binom{m-i}{y_1, y_2, \ldots, y_n} a_{n-1}^{y_1} \ldots \times a_1^{y_{n-1}}u^{y_2+2y_3+\ldots+(n-1)y_n}$$

the last sum being taken over y; satisfying $y_1 + y_2 + \ldots + y_n = m - i$. Since we want the exponent of u to be m, we obtain from (1.13c),

$$m - i + \sum_{j=0}^{n-1} jy_{j+1} = m, \quad \text{or}$$

$$(1.13d)\quad \sum_0^{n-1} jy_{j+1} = i.$$

Adding (1.13d) with the sigma bounds of (1.13c), viz.

$$\sum_0^{n-1} y_{j+1} = m - i,$$

we obtain

$$\sum_0^{n-1}(j + 1)y_{j+1} = m,$$

or

$$(1.13e)\quad \sum_1^n jy_j = m.$$

To obtain the upper bound for i, we note that $\max y_n = m - i$; hence, from (1.13e), choosing $y_1 = y_2 = \ldots = y_{n=1} = 0$, $ny_n = m$, $n(m - i) = m$,

$$(1.13f)\quad i = [m(n - 1)/n].$$

Taking into account (1.13c), (1.13e), (1.13f), we finally obtain

$$(1.14)\quad x_{0,m+n} = a_0^{m+1} \sum_{i=0} \sum_{y_k} a_0^i\left(\prod_{k=1}^{n-1} a_{n-k}^{y_k}\right)\binom{m-i}{y_1, y_2, \ldots, y_n},$$

the last sum being constrained by the conditions

$$\sum_1^n y_j = m - i \quad \text{and} \quad \sum_1^n jy_j = m, \quad \text{and}$$

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$\gamma = [m(n-1)/n]$. (1.14) supplies the desired explicit expression for $x_{0,v}$, ($v = n, n+1, \ldots$), and with it the explicit formula for the positive powers of e. Formula (1.14) is comparatively simple, especially if $a_0 = +1$. Of course, the two Diophantine equations under the second sigma sign have to be solved simultaneously. We now find an explicit formula, similar to (1.14), for the negative powers of e. We obtain from (1.2), with the notation of (1.4)

$$(1.2a) \qquad e^{-n} = a_0(1 - a_{n-1,1}e^{-1} - a_{n-2,1}e^{2} - \ldots - a_{1,1}e^{-(n-1)}).$$

We further set, as in (1.3), using $1, e^{-1}, \ldots, e^{-(n-1)}$ as a basis,

$$(1.3a) \qquad \begin{cases} e^{-m} = y_{0,m} + y_{1,m}e^{-1} + y_{2,m}e^{-2} + \ldots + y_{n-1,m}e^{-(n-1)}, \\ \qquad y_{i,m} \in \mathbf{Z}; \quad i = 0, 1, \ldots, n-1; \quad m = 0, 1, \ldots . \end{cases}$$

If we compare the first of formulas (1.2) with (1.2a), we see that e^{-n} is obtained from e^n by the substitution

$$(1.2b) \qquad \begin{cases} -a_{n-i,1} \leftrightarrow a_{i,1}, \quad (i = 1, 2, \ldots, n-1) \\ \qquad -a_{n-i} \leftrightarrow a_i, \quad (i = 1, \ldots, n-1), \end{cases}$$

since $a_{i,1} = a_0 a_i$, hence $-a_{n-i}a_0 \leftrightarrow a_i a_0$.

With this substitution, all formulas that hold for e^m also hold for e^{-m}, and we obtain especially, from (1.8)

$$y_{k,m} = y_{0,m-k} - a_0 \sum_1^k a_{n-i,1} y_{0,m-k+i}, \quad \text{hence}$$

$$(1.8a) \qquad y_{k,m} = y_{0,m-k} - \sum_1^k a_{n-i} y_{0,m-k+i};$$

from (1.10)

$$(1.19a) \quad y_{0,m+n} = a_0(y_{0,m} - \sum_1^{n-1} a_{n-i} y_{0,m+i})$$

and finally from (1.14),

$$y_{0,m+n} = a_0^{m+1} \sum_{i=0}^{\gamma} \sum_{t_k} a_0^{i} \left(\prod_{k=1}^{n-1} (-a_k)^{t_k}\right) \binom{m-i}{t_1, t_2, \ldots, t_n},$$

and so

$$(1.14a) \quad y_{0,m+n} = a_0^{m+1} \sum_{i=0}^{\gamma} \sum_{tk} (-1)^{m-i+tn} a_0^{i} \left(\prod_{k=1}^{n-1} a_k^{tn}\right) \binom{m-i}{t_1, t_2, \ldots, t_n},$$

where again $\gamma = [m(n-1)/n]$, and $\sum_{t_k}$ is constrained by $\sum_1^n t_j = m - i$ and $\sum_1^n jt_j = m$.

2. Combinatorial identities from the old-new method. In this chapter we shall outline the technique how to construct infinitely many new combinatorial identities by our previous old-new method. We obtain from (1.3) and (1.3a)

$$(2.1)\qquad \begin{aligned} 1 = e^m \cdot e^{-m} &= (y_{0,m} + y_{1,m}e^{-1} + y_{2,m}e^{-2} + \ldots + y_{n-1,m}e^{-(n-1)})e^m \\ &= y_{0,m}e^m + y_{1,m}e^{m-1} + y_{2,m}e^{m-2} + \ldots + y_{n-1,m}e^{m-n+1} \\ &= y_{0,m}(x_{0,m} + x_{1,m}e + x_{2,m}e^2 + \ldots + x_{n-1,m}e^{n-1}) \\ &\quad + y_{1,m}(x_{0,m+1} + x_{1,m-1}e + x_{2,m-1}e^2 + \ldots + x_{n-1,m-1}e^{n-1}) \\ &\quad + y_{2,m}(x_{0,m-2} + x_{1,m-2}e + x_{2,m-2}e^2 + \ldots + x_{n-1,m-2}e^{n-1}) \\ &\quad + \ldots + y_{n-1,m}(x_{0,m-n+1} + x_{1,m-n+1}e + x_{2,m-n+1}e^2 + \ldots \\ &\qquad\qquad + x_{n-1,m-n+1}e^{n-1}). \end{aligned}$$

Since, by hypothesis, e is an n-th degree irrational, we obtain from (2.1), by comparing coefficients of equal powers of e on both sides,

$$(2.2)\qquad \begin{aligned} &x_{0,m}y_{0,m} + x_{0,m-1}y_{1,m} + x_{0,m-2}y_{2,m} + \ldots + x_{0,m-n+1}y_{n-1,m} = 1, \\ &x_{1,m}y_{0,m} + x_{1,m-1}y_{1,m} + x_{1,m-2}y_{2,m} + \ldots + x_{1,m-n+1}y_{n-1,m} = 0, \\ &x_{2,m}y_{0,m} + x_{2,m-1}y_{1,m} + x_{2,m-2}y_{2,m} + \ldots + x_{2,m-n+1}y_{n-1,m} = 0, \\ &\vdots \\ &x_{n-1,m}y_{0,m} + x_{n-1,m-1}y_{1,m} + x_{n-1,m-2}y_{2,m} + \ldots + x_{n-1,m-n+1}y_{n-1,m} = 0. \end{aligned}$$

(2.2) will be considered as a system of n linear equations in n indeterminates $y_{0,m}, y_{1,m}, \ldots, y_{n-1,m}$. Denote the determinant of this system by Δ_m. For those $x_{s,t}$ with $s \neq 0$, substitute from (1.8) to obtain

$$x_{k,m} = x_{0,m-k} + a_1x_{0,m-k+1} + a_2x_{0,m-k+2} + \ldots + a_kx_{0,m-k}, \quad \text{or}$$

$$(1.8a)\qquad x_{k,m} = x_{0,m-k} + \sum_1^k a_ix_{0,m-k+i}, \quad (k = 1, \ldots, n-1)$$

and Δ_m is

$$(2.3)\qquad \begin{vmatrix} x_{0,m} & x_{0,m-1} & \ldots & x_{0,m-n+1} \\ x_{0,m-1} + a_1x_{0,m} & x_{0,m-2} + a_1x_{0,m-1} & \ldots & x_{0,m-n} + a_1x_{0,m-n+1} \\ x_{0,m-2} + a_1x_{0,m-1} + a_2x_{0,m} & x_{0,m-3} + a_1x_{0,m-2} + a_2x_{0,m-1} & \ldots & x_{0,m-n-1} + a_1x_{0,m-n} + a_2x_{0,m-n+1} \\ \vdots & & & \\ x_{0,m-n+1} + \sum_1^{n-1}a_ix_{0,m-n+1+i} & x_{0,m-n} + \sum_1^{n-1}a_ix_{0,m-n+i} & \ldots & x_{0,m-2n+2} + \sum_1^{n-1}a_ix_{0,m-2n+2+i} \end{vmatrix}$$

By elementary row operations, subtracting from the second row the a_1-multiple of the first and the a_1-multiple of the new second, the reader will have no difficulty in seeing that Δ_m is

$$(2.3a)\qquad \begin{vmatrix} x_{0,m} & x_{0,m-1} & x_{0,m-2} & \ldots & x_{0,m-n+1} \\ x_{0,m-1} & x_{0,m-2} & x_{0,m-3} & \ldots & x_{0,m-n} \\ x_{0,m-2} & x_{0,m-3} & x_{0,m-4} & \ldots & x_{0,m-n-1} \\ \vdots & & & & \\ x_{0,m-n+1} & x_{0,m-n} & x_{0,m-n-1} & \ldots & x_{0,m-2n+2} \end{vmatrix}.$$

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Substituting in the determinant $(2.3a)$ for the elements of the first row the expressions from (1.10), and performing on the first row again elementary row operations which leave only one summand in each element of the first row, we obtain $\Delta_m =$

$$(2.3b)\qquad \begin{vmatrix} x_{0,m-n} & x_{0,m-n-1} & x_{0,m-2-n} & \dots & x_{0,m-2n+1} \\ x_{0,m-1} & x_{0,m-2} & x_{0,m-3} & \dots & x_{0,m-n} \\ x_{0,m-2} & x_{0,m-3} & x_{0,m-4} & \dots & x_{0,m-n-1} \\ \vdots & & & & \\ x_{0,m-n+1} & x_{0,m-n} & x_{0,m-n-1} & \dots & x_{0,m-2n+2} \end{vmatrix}.$$

Bringing in $(2.3b)$ the first row down to the last, we obtain

$$(2.4)\qquad \Delta_m = (-1)^{n-1}\,\Delta_{m-1}.$$

From (2.4) we obtain, applying this formula successively,

$$(2.4a)\qquad \Delta_m = (-1)^{k(n-1)}\,\Delta_{m-k}.$$

From (1.10), (1.11) we recall

$$(2.4b)\qquad x_{0,1} = x_{0,2} = \dots = x_{0,n-1} = 0; \quad x_{0,n} = a_0.$$

Thus

$$(2.4c)\qquad \Delta_m = (-1)^{[m-(2n-1)](n-1)}\Delta_{m-[m-(2n-1)]} = (-1)^{(m-1)(n-1)}\Delta_{2n-1}.$$

By definition Δ_{2n-1} equals

$$\begin{vmatrix} x_{0,2n-1} & x_{0,2n-2} & \dots & x_{0,n} \\ x_{0,2n-2} & x_{0,2n-3} & \dots & x_{0,n-1} \\ \vdots & & & \\ x_{0,n} & x_{0,n-1} & \dots & x_{0,1} \end{vmatrix} =$$

$$\begin{vmatrix} x_{0,2n-1} & x_{0,2n-2} & \dots & & & a_0 \\ x_{0,2n-2} & x_{0,2n-3} & \dots & & a_0 & 0 \\ x_{0,2n-3} & x_{0,2n-4} & \dots & a_0 & 0 & 0 \\ \vdots & & & & & \\ a_0 & 0 & \dots & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)^{(n-1)+(n-2)+\dots+1}\,a_0{}^n = (-1)^{n(n-1)/2}a_0{}^n.$$

From this result and $(2.4c)$, we finally obtain

$$(2.5)\qquad \Delta_m = (-1)^{(n-1)(m-1+n/2)}a_0{}^n.$$

With (2.5) we now obtain from (2.2), by Cramer's rule, and observing that $a_0{}^{-1} = a_0$,

$$(2.5a)\qquad y_{0,m} = (-1)^{(n-1)(m-1+n/2)}a_0{}^n \begin{vmatrix} x_{1,m-1} & x_{1,m-2} & \dots & x_{1,m-n+1} \\ x_{2,m-1} & x_{2,m-2} & \dots & x_{2,m-n+1} \\ x_{3,m-1} & x_{3,m-2} & \dots & x_{3,m-n+1} \\ \vdots & & & \\ x_{n-1,m-1} & x_{n-1,m-2} & \dots & x_{n-1,m-n+1} \end{vmatrix}.$$

Substituting in this identity $m + n$ for m, then the value of $y_{0,m+n}$ from (1.14a) and the values of $x_{s,t}$ from (1.8), and then for $x_{0,v}$ those from (1.14), we obtain

$$(2.6) \qquad \sum_{i=0}^{\gamma} \sum_{t_k} (-1)^{t_n - i} a_0{}^i \prod_1^{n-1} a_k{}^{t_k} =$$

$$(-1)^{mn+n(n-1)/2} a_0{}^{m+n+1} \begin{vmatrix} x_{1,m+n-1} & x_{1,m+n-2} & \dots & x_{1,m+1} \\ x_{2,m+n-1} & x_{2,m+n-2} & \dots & x_{2,m+1} \\ \vdots & & & \\ x_{n-1,m+n-1} & x_{n-1,m+n-2} & \dots & x_{n-1,m+1} \end{vmatrix},$$

where γ and $\sum_{t_k}$ are as in (1.14a). In the same way we obtain from the identity

$$1 = e^m e^{-m} = (x_{0,m} + x_{1,m}e + x_{2,m}e^2 + \dots + x_{n-1,m})e^{-m}$$
$$= m_{0,m}e^{-m} + x_{1,m}e^{-(m-1)} + x_{2,m}e^{-(m-2)} + \dots + x_{n-1,m}e^{-(m-n+1)},$$

substituting here the values for the powers of e from (1.3a), an identity expressing $x_{0,m+n}$ by an $(n - 1) \times (n - 1)$ determinant with entries $y_{0,v}$. By this method many more interesting new combinatorial identities are derived, as was done in [**3**].

To illustrate our results, let us set $n = 3$. We obtain from (2.5a), taking into account (1.8),

$$y_{0,m} = -a_0{}^3 \begin{vmatrix} x_{0,m-2} + a_1 x_{0,m-3} & x_{0,m-3} + a_1 x_{0,m-2} \\ x_{0,m} & x_{0,m-1} \end{vmatrix} =$$

$$-a_0{}^3 \begin{vmatrix} x_{0,m+1} - a_2 x_{0,m} & x_{0,m} - a_2 x_{0,m-1} \\ x_{0,m} & x_{0,m-1} \end{vmatrix} = -a_0{}^3 \begin{vmatrix} x_{0,m+1} & x_{0,m} \\ x_{0,m} & x_{0,m-1} \end{vmatrix}$$

and so

$$(2.7) \qquad y_{0,m} = a_0{}^3(x_{0,m}{}^2 - x_{0,m-1}\, x_{0,m+1}).$$

Formula (2.7) is of a beautiful, simple structure, and, of course, of universal validity for any unit in a cubic algebraic number field. The structure of this combinatorial identity becomes more relevant after substituting the values of $x_{0,m}$, from (1.14a).

Concluding this chapter we want to point out that, if in the field equations (1.2) of e or e^{-1} one or more of the coefficients a_i $(i = 1, \dots, n - 1)$ vanishes, formulas (1.14) and (1.14a) have a slightly different form. These situations must be treated in a case to case way, though they differ unessentially from the general case.

To illustrate the value of our theory in application, we shall illustrate it by an example. Let

$$(2.8) \qquad f(x) = x^3 + x^2 + x - 1; \quad f(e) = e^3 + e^2 + e - 1 = 0;$$
$$0.5 < e < 0.6; \quad f(t) \neq 0, \quad t \in \mathbf{R}; \quad e \text{ a unit in } \mathbf{Q}(e).$$

The field equation of e is $f(e) = 0$, and we have

$$(2.8a) \qquad e^3 = 1 - e - e^2; \quad a_0 = 1; \quad a_1 = a_2 = -1; \quad N(e) = 1.$$

We consider formula (1.14), and investigate the bounds of the sums. We have, with $n = 3$,

$$y_1 + y_2 + y_3 = m - i; \quad y_1 + 2y_2 + 3y_3 = m; \quad y_2 + 2y_3 = i;$$

hence

$$(2.8b) \qquad y_2 = i - 2y_3; \quad y_1 = m - 2i + y_3; \quad 0 \leqq y_3 \leqq i/2 \quad \text{and}$$

$$(2.8c) \qquad \binom{m-i}{y_1, y_2, y_3} = (m-i)!/y_1!y_2!y_3!$$

$$= (m-i)!/(m-2i+y_3)!(i-2y_3)!y_3! = \binom{m-i}{i-y_3}\binom{i-y_3}{y_3}.$$

We denote

$$(2.9) \qquad y_3 = j; \quad 0 \leqq j \leqq [i/2]; \quad \binom{m-i}{y_1, y_2, y_3} = \binom{m-i}{i-j}\binom{i-j}{j}.$$

With (2.9) and (2.8a), formula (1.14) takes the form

$$(2.10) \qquad x_{0,m+3} = \sum_{i=0}^{\gamma_m} \sum_{j=0}^{\beta} (-1)^{m-i-j}\binom{m-i}{i-j}\binom{i-j}{j},$$

where we have defined $\gamma_m = [2m/3]$ and $\beta = [i/2]$ and formula (1.14a) becomes

$$(2.10a) \quad y_{0,m+3} = \sum_{i=0}^{\gamma_m} \sum_{j=0}^{\beta} \binom{m-i}{i-j}\binom{i-j}{j}.$$

(2.7), with $a_0 = 1$, gives the combinatorial identity, writing $m + 3$ for m

$$(2.7a) \qquad y_{0,m+3} = x_{0,m+3}{}^2 - x_{0,m+2}x_{0,m+4}.$$

Substituting in (2.7a) the values of (2.10) and (2.10a), we obtain the sophisticated combinatorial identity

$$(2.11) \qquad \begin{aligned} &\sum_{i=0}^{\gamma_m} \sum_{j=0}^{\beta} \binom{m-i}{i-j}\binom{i-j}{j} \\ &= \left(\sum_{i=0}^{\gamma_m} \sum_{j=0}^{\beta} (-1)^{m-i-j}\binom{m-i}{i-j}\binom{i-j}{j}\right)^2 - \\ &- \left(\sum_{i=0}^{\gamma_{m-1}} \sum_{j=0}^{\beta} (-1)^{m-1-i-j}\binom{m-1-i}{i-j}\binom{i-j}{j}\right) \times \\ &\qquad \left(\sum_{i=0}^{\gamma_{m+1}} \sum_{j=0}^{\beta} (-1)^{m+1-i-j}\binom{m+1-i}{i-j}\binom{i-j}{j}\right). \end{aligned}$$

We shall also investigate a few properties of the arithmetic function $x_{0,v}$ in the case (2.8). We obtain from (1.3), (1.8) and (2.8)

$$e^m = x_{0,m} + x_{1,m}e + x_{2,m}e^2, \quad \text{or}$$

$$(2.12) \qquad e^m = x_{0,m} + (x_{0,m-1} - x_{0,m})e + x_{0,m+1}e^2.$$

We are interested in the zeros of the arithmetic function $x_{0,v}$. To this end, we put $x_{0,m+1} = 0$ in (2.12) and obtain

(2.12*a*) $\alpha = x + ye, \quad \alpha = e^m; \quad x = x_{0,m}; \quad y = x_{0,m-1} - x_{0,m}.$

We shall find the norm equation of α and have, from one side

(2.12*b*) $N(\alpha) = N(e^m) = (N(e))^m = 1; \quad N(e) = 1$ from (2.8).

We further have, by a known method,

(2.12*c*)
$$\begin{aligned} \alpha &= x + ye + 0 \cdot e^2, \\ \alpha e &= 0 + xe + ye^2, \\ \alpha e^2 &= y - ye + (x - y)e^2, \quad \text{since } e^3 = 1 - e - e^2. \end{aligned}$$

From (2.12*c*) we obtain

$$n(\alpha) = 1 = \begin{vmatrix} 0 & y & 0 \\ 0 & x & y \\ y & -y & x - y \end{vmatrix},$$

and, calculating this determinant,

(2.13) $x^3 - x^2y + xy^2 + y^3 = 1.$

The Diophantine equation has the following solutions

(2.13*a*) $x_1 = 1, y_1 = 0; \quad x_2 = 0, y_2 = 1; \quad x_3 = -1, y_3 = 2;$
$x_4 = 56; y_4 = -103.$

Thus, since for these values of x_1, $x_{m+1} = 0$, we obtain for the corresponding binary powers of e,

(2.13*b*) $e^0 = 1; \quad e = 1; \quad e^4 = -1 + 2e; \quad e^{17} = 56 - 103e.$

Thus

(2.13*c*) $x_{0,5} = x_{0,18} = 0,$

and substituting for $x_{0,5}$ and $x_{0,18}$ their values from (2.10), we obtain

(2.14)
$$\sum_{i=0}^{1} \sum_{j=0}^{\beta} (-1)^{2-i-j} \binom{2-i}{i-j} \binom{i-j}{j} = 0,$$
$$\sum_{i=0}^{10} \sum_{j=0}^{\beta} (-1)^{15-i-j} \binom{15-i}{i-j} \binom{i-j}{j} = 0.$$

Now, by the profound theory of Delone and Faddeev, as put forward in their classical book, The Theory of Irrationalities of the Third Degree, (2.13) has exactly four solutions. Thus $x_{0,5}$ and $x_{0,18}$ are the only values of the sequence $x_{0,v}$ $(v = 0, 1, \ldots)$ which vanish. In other words, the arithmetic function $x_{0,m}$ has its only zeros (for $m \geqq 3$) at $m = 5$ and $m = 18$. This is expressed by the identities (2.14) which the reader can easily verify.

3. The new algorithm. This, in the sequel abbreviated as NA, ties very closely to the Jacobi-Perron algorithm which was explored and developed by the author in [**4**], and generalized in [**5**]. Compared with the Jacobi-Perron algorithm the NA has the disadvantage of being applicable to algebraic number fields only, but this shortcoming has a justification. The main property of the NA we are interested in is its periodicity. But also the Jacobi-Perron algorithm yields to periodicity only in algebraic number fields—so it is this domain where the two algorithms have a rendezvous. We shall introduce the NA by means of the following.

Definition 1. Let w be an algebraic irrational of degree $n \geqq 2$, and let $a_1^{(0)}(w), a_2^{(0)}(w), \ldots, a_{n-1}^{(0)}(w)$ be $n-1$ polynomials in w with rational coefficients, each of degree k, $1 \leqq k \leqq n-1$. The vector

$$(3.1) \qquad a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w), \ldots, a_{n-1}^{(0)}(w))$$

will be called the *fixed vector*. The sequence $\langle a^{(v)}\rangle$, $v = 0, 1, \ldots$ will be called the NA of the fixed vector $a^{(0)}$ if the recursion formula (3.2) holds:

$$(3.2) \qquad \begin{aligned} a^{(v+1)} = (a_1^{(v)}(w) - b_1^{(v)})^{-1}(a_2^{(v)}(w) - b_2^{(v)}, \ldots, a_{n-1}^{(v)}(w) \\ - b_{n-1}^{(v)}, 1), \end{aligned}$$

$$a_i^{(v)}(0) = b_i^{(v)}; \quad i = 1, \ldots, n-1; \quad v = 0, 1, \ldots.$$

The following sequences of numbers are defined by means of the $b_i^{(v)}$:

$$(3.3) \qquad \begin{aligned} &A_i^{(j)} = \delta_{i,j} \quad (i, j = 0, \ldots, n = 1), \\ &A_i^{(v+n)} = A_i^{(v)} + \textstyle\sum_1^{n-1} b_j^{(v)} A_i^{(v+j)}, \quad (i = 0, 1, \ldots, n-1; \\ &\qquad\qquad\qquad\qquad\qquad\qquad v = 0, 1, \ldots). \end{aligned}$$

The following formulas are crucial for our further investigations. They can easily be proved by induction.

$$(3.4) \qquad \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \ldots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \ldots & A_1^{(v+n-1)} \\ \vdots & & & \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \ldots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}; \quad v = 0, 1, \ldots,$$

$$(3.4a) \qquad \begin{aligned} a^{(0)} = (\mathrm{A}_i^{(v)} + \textstyle\prod_1^{n-1} a_j^{(v)} A_i^{(v+j)})/(\mathrm{A}_0^{(v)} + \sum_1^{n-1} a_j^{(v)} A_0^{(v+j)}); \\ i = 1, \ldots, n-1; \quad v = 0, 1, \ldots \end{aligned}$$

$$(3.4b) \qquad \textstyle\prod_1^v a_{n-1}^{(i)} = A_0^{(v)} + \sum_1^{n-1} a_j^{(v)} A_0^{(v+j)}, \quad m = 1, 2, \ldots$$

$$(3.4c) \qquad \begin{vmatrix} 1 & A_0^{(v+1)} & \ldots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & \ldots & A_1^{(v+n-1)} \\ \vdots & \vdots & & \vdots \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & & A_{n-1}^{(v+n-1)} \end{vmatrix} \begin{aligned} = (-1)^{v(n-1)}/(A_0^{(v)} \\ + \textstyle\sum_1^{n-1} a_j^{(v)} A_0^{(v+j)}), \\ v = 0, 1, \ldots \end{aligned}$$

Definition 2. The NA of $a^{(0)}$ will be called *periodic* if there are two non-negative minimal integers $L, m \geqq 0$ with $m \geqq 1$, such that

$$(3.4d) \qquad a^{(m+v)} = a^{(v)}, \quad v = L, L = 1, \ldots.$$

The sequences

$$(3.4e)\qquad a^{(0)}, a^{(1)}, \ldots, a^{(L-1)}, \quad \text{and} \quad a^{(L)}, a^{(L+1)}, \ldots, a^{(L+m-1)}$$

are called respectively the *primitive preperiod of length* L and the *primitive period of length* m of the NA; if $L = 0$, the NA is called *purely periodic of length* m. If we define the vector

$$(3.4f)\qquad b^{(v)} = (b_1^{(v)}, b_2^{(v)}, \ldots, b_{n-1}^{(v)}), \quad v = 0, 1, \ldots$$

then the sequences

$$(3.4g)\qquad b^{(0)}, b^{(1)}, \ldots, b^{(L-1)}, \quad \text{and} \quad b^{(L)}, b^{(L+1)}, \ldots, b^{(L+m-1)}$$

are called respectively the *companion vectors* of the primitive preperiod and *primitive period* of the NA of $a^{(0)}$.

The reader should note that any periodic NA of $a^{(0)}$ is a purely periodic NA of $a^{(L)}$.

4. Explicit units. Since our theory is based exclusively on explicit units (not necessarily fundamental ones) of infinitely many algebraic number fields of any degree $n \geqq 3$, it is time to ask whether such units exist. If w is a root of the polynomial

$$(4.1)\qquad F(x) = x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_0, \quad |b_0| = 1, \quad b_i \in \mathbf{Q}; \quad i = 1, \ldots, n-1,$$

and if w is an irrational of degree $\geqq 3$, then the question is almost naive, for then w is such a desired unit. Of course there are (countably) infinitely many algebraic number fields $\mathbf{Q}(w)$ for which $w = e$ is an explicit unit. Demanding that $F(x)$ be irreducible over Q, we, at the same time, satisfy our demand (voiced in (1.1)) that e be an n-th degree irrational. The question still remains whether there are infinitely many such irreducible polynomials $F(x)$ of any degree $n \geqq 3$. Since Eisenstein's irreducibility criterion is not applicable here because $|a_0| = 1$, it is possible to apply the author's criterion, given in [**5**], which states that $F(x)$ from (4.1) is irreducible over $\mathbf{Q}$ if

$$(4.1a)\qquad |b_{n-1}| \geqq 2(1 + |b_1| + |b_2| + \ldots + |b_{n-2}|), \quad n \geqq 3.$$

But there are infinitely many additional algebraic number fields of degree $n \geqq 3$ for which one or more (independent) explicit units are known, the latter being algebraic irrationals of degree n. All of these results are due, in chronological order, to Hasse, Halter-Koch, Stender and the author. Here we give a short resume of results concerning these units.

i) In a joint paper [**8**] Hasse and the author proved:
Let $n, D \in N$; $n \geqq 2$; $d|D$; $d \in \mathbf{Z}$; $w^n = D^n + d$; $d > 1$; $D \geqq d(n-2)$; *then* $e_s = (w^s - D^s)/(w-D)^s$; $1 < s|n$ *is a unit in* $\mathbf{Q}(w)$; *if* $d = 1$, $e_s = w^s - D^s$; $s|n$; $s < n$ *is a unit in* $\mathbf{Q}(w)$. *If* $n = p^v$,

p prime, $w^n = D^n + pd$, then, with $D \geqq pd(n-2)$, the same formulas hold for e_s. Let

$$n, D \in N; \quad n \geqq 2; \quad d|D; \quad d < -1;$$
$$w^n = D^n + d; \quad D \geqq 2|d|(n-1); \quad \textit{then}$$
$$e_s = (w^s - D^s)/(w-D)^s; \quad 1 < s|n \textit{ is a unit in } \mathbf{Q}(w);$$
$$\textit{if } d = -1, \quad e_s = w^s - D^s; \quad s|n, \quad s < n \textit{ is a unit in } Q(w).$$

If $n = p^v$, p prime, $w^n = D^n + pd$, then, with $D \geqq 2p|d|(n-1)$, the same formulas hold for e_s.

ii) In a remarkable paper Halter-Koch and Stender [**12**] generalized the results of *i*) to the following extent:
Let n, $D \in N$; $n \geqq 2$; $k \in \mathbf{Z}$; $k|D^{n-1}$ if n is composite, $k|pD^{n-1}$, if $n = p^v$, p a prime, $v \geqq 1$. Then, for $|k| > 1$ $e_s = (w^s - D^s)/(w-D)^s$, $1 < s|n$ are units in $\mathbf{Q}(w)$, $w^n = D^n + k$. For $|k| = 1$, i) is valid.
In a previous paper Stender [**13**] also proved the fundamentality of e_s from *ii*) for $n = 3, 4, 6$.

iii) In a lengthy paper [**9**] Hasse and the author proved:
Let $F(x) = (x - D_0)(x - D_1)\ldots(x - D_{n-1}) - d$; $D_i \in \mathbf{Z}$, $i = 0, 1, \ldots, n-1$; $1 \leqq d|D_0 - D_i$; $D_0 > D_1 > \ldots > D_{n-1}$; $D_0 - D_{n-1} \geqq 2dn$, $n \geqq 2$; then $F(x)$ is irreducible and has exactly n different real roots. Let w be the largest; then

$$e_{s_i} = (w - D_{s_i})^n/d, \quad i \leqq s_1 < s_2 < \ldots < s_{n-1} \leqq n$$

are $n-1$ independent units in $\mathbf{Q}(w)$.
By Dirichlet, this is the maximal number of independent units in $\mathbf{Q}(w)$.

iv) In a remarkable paper [**11**] Halter-Koch generalized the results of iii), choosing for $F(x)$ the polynomial

$$F(x) = \prod_{i=1}^{m}(x - D_i)\prod_{j=m+1}^{n}(x^2 + a_jx + b_j) - d; \quad n \geqq 2;$$

$a_j, b_j \in \mathbf{Z}$; $x^2 + a_jx + b_j$ irreducible over $\mathbf{Q}$,
$D_0 > D_1 > \ldots > D_m$; $D_i \in Q$; $(i = 1, \ldots, m)$; $1 \leqq d|D_i - D_j$;
Halter proved that, under certain restrictions, the number of independent units in $\mathbf{Q}(w)$ equals $n + m - 1$; $\mathbf{Q}(w) = 0$, $w \in \mathbf{R}$.

v) In [**5**] the author proved:
Let $f(x) = x^n + k_1x^{n-1} + k_2x^{n-2} + \ldots + k_{n-1}x - d$, $k_i, d \in \mathbf{Z}$; $i = 1, \ldots, n-1$; $|k_{n-1}| \geqq 2|d|(1 - |k_1| + \ldots + |k_{n-2}|)$; $d|k_i$. Then $f(x)$ is irreducible over $\mathbf{Q}$ and has one, and only one real root w. A unit in $\mathbf{Q}(w)$ is given by $e = w^n/d$, $|d| > 1$; $e = w$ for $|d| = 1$.

vi) In a recent paper [**6**] the author proved:

Let w be an algebraic irrational of degree $2k + 1$, $(k = 1, 2, \ldots)$ defined by

$$w^{2k+1} = a^{k+1}\left(\sum_{i=0}^{k} \left[\binom{2k - i}{i - 1} + \binom{2k + 1 - i}{i} \right] b^{2k+1-2i} a^{k-i}; \right.$$

then $e = 1 + bw - a^{-1}w^2$ is a unit in $\mathbf{Q}(w)$; $a, b \in \mathbf{Z}$. Let further $\bar{w}$ be an algebraic irrational defined by

$$\bar{w}^{2k} = a^{k}\left(\sum_{i=0}^{k} (-1)^{k} \left[\binom{2k - 1 - i}{i - 1} + \binom{2k - i}{i} \right] (b^2 a)^{k-i}, \right.$$

$k = 1, 2, \ldots; \quad a, b \in \mathbf{Z}.$

Then $e = 1 \pm b\bar{w} + a^{-1}\bar{w}^2$ are units in $\mathbf{Q}(w)$. $\left(\text{Here we define } \binom{m}{-1} = 0.\right)$

Though cases i), ii), iii) were quite a novelty when their authors proved these results, they later turned out to be special cases [**5**]. Let

$$w^n = D^n + k, \quad D \in N, \quad |k|\,|D, \quad |k| > 1; \quad n \geqq 2.$$

We obtain

$$w^n - D^n - k = [(w - D) + D]^n - D^n - k, \quad \text{whence}$$

$$(4.2) \qquad (w - D)^n + \binom{n}{1} D(w - D)^{n-1} + \ldots + \binom{n}{n-1} D^{n-1}(w - D) - k = 0.$$

and since $|k|\,|D$, case iv) is applicable, and $e_n^{-1} = (w - D)^n/k$ is a unit in $\mathbf{Q}(w)$. Similarly we obtain the other units e_s, $s|n$ in $\mathbf{Q}(w)$. From (4.2) we further obtain

$$(-1)^n N(w - D) = -k, \text{ and so}$$

$$(4.2a) \qquad N(w - D) = (-1)^{n-1} k.$$

Thus we obtain

$$(4.3) \quad N(e_n^{-1}) = N((w - D)^n/k) = (N(w - D)/k)^n = ((-1)^{n-1}k/k)^n = 1.$$

Now let, with $e_n = e$,

$$(4.4) \qquad e^n + a_{n-1}e^{n-1} + a_{n-2}e^{n-2} + \ldots + a_1 e + a_0 = 0$$

be the field equation in $\mathbf{Q}(w)$ of e. Then $(-1)^n N(e) = a_0$, so that, with (4.3)

$$(4.5) \qquad a_0 = (-1)^n.$$

Because of reasons which will be explained in the next chapter, we are interested in field equations (4.4) with $a_0 = 1$. So it suffices to choose, for the units from i) or ii) n odd.

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We leave it to the reader to verify that case iii) can also be derived from case v), and that the field equations of the corresponding units can be chosen so that the free element equals -1. Thus, the polynomial $f(x)$ from case v) is the most general one serving to illustrate the theory which will be studied in the next chapters.

5. Periodic NA-s and their application. In the sequel we consider an algebraic number field of degree n over Q, generated by n (real valued) parameters, viz. the field $\mathbf{Q}(w)$, w an algebraic number of degree n, and an explicitly stated unit e in $\mathbf{Q}(w)$ which is not a root of unity. Let e be an algebraic irrational of degree n, and let the free element of its field equation be -1. In the previous chapter we have shown that infinitely many choices of such patterns are possible. In a next chapter we shall consider the situation when the free element of the field equation of e may also be $+1$ which is a highly complicated case. Thus let

(5.1) w be an algebraic irrational of degree n, generating $Q(w)$; e be a unit in $Q(e)$ which is not a root of unity, $h(e) = e^n + a_1e^{n-1} + a_2e^{n-2} + , \ldots + a_{n-1}e - 1 = 0$ be its field equation, $h(e)$ be irreducible over Q.

We obtain from (5.1)

$$e^{-1} = e^{n-1} + a_1e^{n-2} + a_2e^{n-3} + \ldots + a_{n-2}e + a_{n-1}. \tag{5.1a}$$

We construct the fixed vector

$$a^{(0)} = (e + a_1, e^2 + a_1e + a_2, e^3 + a_1e^2 + a_2e + a_3, \ldots, e^{n-1} + a_1e^{n-2} + \ldots + a_{n-2}e + a_{n-1}). \tag{5.2}$$

We now carry out the NA of $a^{(0)}$, and obtain from (5.2), following (3.2), the companion vector

$$b^{(0)} = (a_1, a_2, \ldots, a_{n-2}, a_{n-1}). \tag{5.2a}$$

From (5.2), (5.2a) we obtain

$$\begin{aligned} a^{(0)} - b^{(0)} &= (a_1^{(0)} - b_1^{(0)}, a_2^{(0)} - b_2^{(0)}, \ldots, a_{n-1}^{(0)} - b_{n-1}^{(0)}) \\ &= (e, e^2 + a_1e, e^3 + a_1e^2 + a_2e, \ldots, e^{n-1} + a_1e^{n-2} + \ldots + a_{n-2}e). \end{aligned} \tag{5.2b}$$

We obtain from (5.2b), in view of (3.2), and with (5.1a)

$$a^{(1)} = (e + a_1, e^2 + a_1 \ + a_2, \ldots, e^{n-2} + a_1e^{n-3} + \ldots + a_{n-2}, e^{n-1} + a_1e^{n-2} + \ldots + a_{n-2}e + a_{n-1}),$$

and hence

$$a^{(1)} = a^{(0)}. \tag{5.3}$$

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By (5.3), the NA is purely periodic, and the length of the primitive period $m = 1$; hence, with the definition (3.3), we obtain, since $a^{(v)} = a^{(0)}$, $v^{(v)} = b^{(0)}$, $v = 1, 2, \ldots$

$$(5.4)\qquad A_0^{(v+n)} = A_0^{(v)} + \textstyle\sum_1^{n-1} a_j A_0^{(v+j)};\quad v = 0, 1, \ldots.$$

We shall now calculate $A_0^{(v+n)}$ explicitly. The calculations are completely analogous to those of $x_{0,v}$ in Section 1. With u denoting a variable, we have, summing over v,

$$\begin{aligned}(5.4a)\qquad \textstyle\sum_0^\infty A_0^{(v)}u^v &= 1 + \textstyle\sum_n^\infty A_0^{(v)}u^v = 1 + \textstyle\sum_0^\infty A_0^{(v+n)}u^{v+n}\\ &= 1 + u^n \textstyle\sum_0^\infty A_0^{(v+n)}u^v = 1 + u^n \textstyle\sum_0^\infty (A_0^{(v)} + \textstyle\sum_{j=1}^{n-1} a_j A_0^{(v+j)})\, u^v\\ &= 1 + u^n \textstyle\sum_0^\infty A_0^{(v)}e^v + u^{n-1}a_1 \textstyle\sum_0^\infty A_0^{(v+1)}u^{v+1}\\ &\quad + u^{n-2}a_2 \textstyle\sum_0^\infty A_0^{(v+2)}u^{v+2}\\ &\quad + \ldots + u^2 a_{n-2} \textstyle\sum_0 A_0^{(v+n-2)}u^{(v+n-2)} + u a_{n-1} \textstyle\sum_0 A_0^{(v+n-1)}.\end{aligned}$$

With the initial values $A_0^{(1)} = A_0^{(2)} = \ldots = A_0^{(n-1)} = 0$, (5.4$a$) takes the form

$$(5.5)\qquad \textstyle\sum_0^\infty A_0^{(v)}u^v = 1 + u^n \textstyle\sum_0^\infty A_0^{(v)}u^v + \textstyle\sum_{i=1}^{n-1} a_i u^{n-i} \times (\textstyle\sum_0^\infty A_0^{(v)}u^v - 1).$$

Carrying over all the sums from the right to the left side, we obtain from (5.5)

$$(5.5a)\qquad (1 - u^n - \textstyle\sum_1^{n-1} a_i u^{n-1}) \textstyle\sum_0^\infty A_0^{(v)}u^v = 1 - \textstyle\sum_1^{n-1} a_i u^{n-i}.$$

Choosing $1 - u^n - \sum_1^{n-1} a_i u^{n-1} \neq 0$, we obtain from (5.5$a$)

$$\begin{aligned}\textstyle\sum_0^\infty A_0^{(v)}u^v &= (1 - \textstyle\sum_1^{n-1} a_1 u^{n-1})/(1 - u^n - \textstyle\sum_1^{n-1} a_i u^{n-1}),\quad \text{so}\\ \textstyle\sum_0^\infty A_0^{(v)}u^v &= 1 + \textstyle\sum_0^\infty A_0^{(v+n)}u^{v+n} = 1 + u^n/(1 - u^n - \textstyle\sum_1^{n-1} a_i u^{n-i})\end{aligned}$$

and hence

$$(5.5b)\qquad \textstyle\sum_0^\infty A_0^{(v+n)}u^v = (1 - u(u^{n-1} + \textstyle\sum_1^{n-1} a_i u^{n-1-i}))^{-1}.$$

Let max $[|a_1|, |a_2|, \ldots, |a_{n-1}|] = a$, $a \geqq 1$; we choose $|u| \leqq 1/2na < 1/2 < 1$; then

$$|u(u^{n-1} + \textstyle\sum_1^{n-1} a_i u^{n-1-i})| < (1 + (n-1)a)/2na \leqq na/2na = 1/2 < 1,$$

and we obtain from (5.5b)

$$(5.6)\qquad \textstyle\sum_{v=0}^\infty A_0^{(v+n)}u^v = \textstyle\sum_{i=0}^\infty u^i(u^{n-1} + a_1u^{n-2} + a_2u^{n-3} + \ldots + a_{n-2}u + a_{n-1})^i.$$

Formula (5.6) is the same as (1.13a), where, in the latter, $a_0 = 1$, and $x_{0,v+n} = A_0^{(v+n)}$. Hence we obtain, in complete analogy with formula (1.14)

$$(5.7)\qquad A_0^{(m+n)} = \textstyle\sum_{i=0}^{\gamma} \sum_{y_k} \binom{m-i}{y_1, y_2, \ldots, y_n} \prod_{k=1}^{n-1} a_{n-k}^{\ y_k},$$

where again $\gamma = [m(n-1)/n]$ and $\sum_{y_k}$ is constrained by $\sum_1^n y_j = m - i$ and $\sum_1^n j\, y_i = m$.

Formula (5.7) is a most important explicit representation of the $A_0^{(v)}$ which will be used repeatedly in this paper. Because of reasons of approximation of the irrational number w by rationals, the $A_i^{(v)}$ are called the *i-th convergent of order v*. We shall now construct a second periodic NA, starting with the field equation of e^{-1}. We obtain for $h(e^{-1})$ from (5.1)

$$(5.8) \qquad e^{-n} - a_{n-1}e^{-(n-1)} - a_{n-2}e^{-(n-2)} - \ldots - a_2e^{-2} - a_1e^{-1} - 1 = 0.$$

We construct the fixed vector

$$(5.9) \qquad \bar{a}^{(0)} = (e^{-1} - a_{n-1}, e^{-2} - a_{n-1}e^{-1} - a_{n-2}, \ldots, \\ e^{-(n-1)} - a_{n-1}e^{-(n-2)} - \ldots - a_2e^{-1} - a_1).$$

Carrying out the necessary operations of the NA of $\bar{a}^{(0)}$, we obtain, for the companion vector

$$(5.9a) \qquad \bar{b}^{(0)} = (-a_{n-1}, -a_{n-2}, \ldots, -a_2, -a_1).$$

We further obtain that the NA of $\bar{a}^{(0)}$ is purely periodic, with length of the period $m = 1$, so that

$$(5.9b) \qquad \bar{a}^{(v)} = \bar{a}^{(0)}, \quad \bar{b}^{(v)} = \bar{b}^{(0)} \quad \text{and} \\ \bar{b}^{(v)} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-2}, \bar{a}_{n-1}); \quad \bar{a}_i = -a_{n-i}; \quad i = 1, 2, \ldots, n-1.$$

Denoting the convergents of the NA of $\bar{a}^{(0)}$ by $B_0^{(v)}$, we obtain, by definition, and in view of (5.9*b*)

$$(5.9c) \qquad B_i^{(j)} = \delta_{ij} \quad (i, j = 0, 1, \ldots, n-1), \quad \text{and} \\ B_i^{(v+n)} = B_i^{(v)} + \textstyle\sum_1^{n-1} \bar{a}_j B_i^{(v+j)}; \quad v = 0, 1, \ldots; \\ i = 0, 1, \ldots, n-1.$$

Since the sequence $\langle B_0^{(v)} \rangle$ has exactly the same n initial values, and the same recursion pattern as $\langle A_0^{(v)} \rangle$, as is shown by comparison of (5.9*c*) and (5.4), we obtain for $B_0^{(v+n)}$ the same explicit representation (5.7), viz.

$$(5.10) \qquad B_0^{(m+n)} = \textstyle\sum_{i=0}^{\gamma} \sum_{y_k} (-1)^{m-i-y_n} \binom{m-i}{y_1, \ldots, y_n} \prod_1^{n-1} a_k^{y_k},$$

where γ and $\sum_{y_k}$ are as in 5.7. The powers of -1 under the second sigma sign result from the substitution $a_{n-k} \to -a_k$ in (5.7); the exponent of -1 is $y_1 + y_2 + \ldots + y_{n-1} = m - i - y_n$.

6. Linear forms of the convergents. In this chapter we shall show how the $A_i^{(v)}$, $(i = 1, \ldots, n-1; \ v = n, n+1, \ldots)$ can be expressed linearly by the $A_0^{(v)}$, and similarly the $B_i^{(v)}$ by the $B_0^{(v)}$. This representation plays a predominant role in our new method of constructing combinatorial identities.

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We return to the fixed vector (5.2) with $a^{(0)} = a^{(v)}$, $v = 1, 2, \ldots$, and the companion vectors of the NA of $a^{(0)}$ with $b^{(0)} = b^{(v)}$. According to formula (3.4a), we obtain, for $i = 1$, with $s_j^v = a_j$,

$$a_1^{(0)} = \frac{A_1^{(v)} + \sum_1^{n-1} a_j A_1^{(v+j)}}{A_0^{(v)} + \sum_1^{n-1} a_j A_0^{(v+j)}}, \quad \text{and so}$$

$$(6.1) \qquad a_1^{(0)} = \frac{A_1^{(v)} + \sum_1^{n-1} (e^j + a_1 e^{j-1} + \ldots + a_{j-1}e + a_j) A_1^{(v+j)}}{A_0^{(v)} + \sum_1^{n-1} (e^j + a_1 e^{j-1} + \ldots + a_{j-1}e + a_j) A_0^{(v+j)}} = e + a_1.$$

We multiply the second equation of (6.1) by the denominator of the midterm. We remember that e is an n-th degree algebraic irrational. We compare the coefficients of e^0 on both sides of the newly obtained equation and recall that $e^n = 1 + P(e)$ a polynomial in e of degree $n - 1$ without a free element. Then we obtain

$$(6.2) \qquad A_1^{(v)} + \sum_1^{n-1} a_j A_1^{(v+j)} = a_1 (A_0^{(v)} + \sum_1^{n-1} a_j A_0^{(v+j)}) + A_0^{(v+n-1)}.$$

But $A_i^{(v)} + \sum_1^{n-1} a_j A_i^{(v+j)} = A_i^{(v+n)}$, $i = 0, 1, \ldots, n - 1$; $v = 0, 1, \ldots$. Hence we obtain from (6.2)

$$(6.3) \qquad A_1^{(v+n)} = a_1 A_0^{(v+n)} + A_0^{(v+n-1)}.$$

In the same way we obtain from (3.4a) with the periodic NA of the fixed vector $a^{(0)}$ from (5.2)

$$a_2^{(0)} = e^2 + a_1 e + a_2$$
$$= (A_2^{(v)} + \sum_1^{n-1} a_j^{(v)} A_2^{(v+j)})/(A_0^{(v)} + \sum_1^{n-1} a_j^{(v)} A_0^{(v+j)}), \text{ whence}$$

$$(6.4) \qquad (e^2 + a_1 e + a_2)(A_0^{(v)} + \sum_1^{n-1} (e^j + a_1 e^{j-1} + \ldots + a_j (A_0^{(v+j)}))$$
$$= A_2^{(v)} + \sum_1^{n-1} (e^j + a_1 e^{j-1} + \ldots + a_j) A_2^{(v+j)}.$$

We compare coefficients of equal powers of e^0 on both sides of (6.4). Here we must take into consideration that the power of e^{n+1} appears once, the pówer of e^n appears thrice. We know already that $e^n = 1 + P(e)$, as before. We further have from (5.1)

$$(6.4a) \qquad e^{n+1} = - a_1 e^n - a_2 e^{n-1} - \ldots - a_{n-1} e^2 + e$$
$$= - a_1 (1 + P(e)) + M(e),$$

where $P(e)$ and $M(e)$ are polynomials in e of degree $\leqq n - 1$, without a free element. If $H(e)$ denotes a polynomial in e with the same properties as $M(e)$, we obtain

$$(6.4b) \qquad e^{n+1} = - a_1 + H(e).$$

With (6.4b) we now obtain from (6.4),

$$-a_1A_0^{(v+n-1)} + a_1A_0^{(v+n-1)} + A_0^{(v+n-2)} + a_1A_0^{(v+n-1)}$$
$$+ a_2(A_0^{(v)} + \sum_1^{n-1} a_jA_0^{(v+j)}) = A_2^{(v)} + \sum_1^{n-1} a_jA_2^{(v+j)}, \quad \text{or}$$

$$(6.5) \qquad A_2^{(v+n)} = a_2A_0^{(v+n)} + a_1A_0^{(v+n-1)} + A_0^{(v+n-2)}.$$

By the same reasoning, the reader will verify easily the formula

$$(6.6) \qquad A_j^{(v+v)} = \sum_{k=0}^{j} a_{j-k}A_0^{(v+n-k)}; \quad a_0 = 1; \quad j = 1, 2, \ldots, n-1.$$

We have proved (6.6) for $j = 1, 2$ in (6.4), (6.5). Formula (6.6) is best proved by induction, whereby the formula $a_{k+1}^{(0)} = e\, a_k^{(0)} + a_{k+1}$, $(k = 0, 1, \ldots, n-2;\ a_0^{(0)} = 1)$ is useful. In the same way we obtain, since the $B_i^{(0)}$ and $A_i^{(v)}$ are identical with $\bar{a}_i \to -a_{n-i}$, $(i = 1, \ldots, n-1)$,

$$B_j^{(v+n)} = B_0^{(v+n-j)} + \sum_{k=0}^{j-1} \bar{a}_{j-k}B_0^{(v+n-k)},$$

$$(6.7) \qquad B_j^{(v+n)} = B_0^{(v+n-j)} - \sum_0^{j-1} a_{k+n-j}B_0^{(v+n-k)}; \quad j = 1, 2, \ldots, n-1.$$

7. Periodic NA-s, the plus case. We are now turning to the case when the free element in the field equation of e (and hence of e^{-1}) is $+1$, viz.

$$(7.1) \qquad e^n + c_1e^{n-1} + c_2e^{n-2} + \ldots + c_{n-2}e^2 + c_{n-1}e + 1 = 0,$$
$$c_i \in \mathbf{Z}, \quad i = 1, 2, \ldots, n-1.$$

We carry out the NA of the initial vector

$$(7.2) \qquad a^{(0)} = (e + c_1, e^2 + c_1e + c_2, \ldots, e^{n-2} + c_1e^{n-3} + \ldots + c_{n-2},$$
$$e^{n-1} + c_1e^{n-2} + \ldots + c_{n-2}e + c_{n-1}).$$

We have by definition

$$(7.2a) \qquad b^{(0)} = (c_1, c_2, \ldots, c_{n-2}, c_{n-1}) \quad \text{and } a^{(0)} - b^{(0)}$$
$$= (e, e^2 + c_1e, e^3 + c_1e^2 + c_2e, \ldots, e^{n-2} + c_1e^{n-3} + c_{n-3}e,$$
$$e^{n-1} + c_1e^{n-2} + \ldots + c_{n-2}e).$$

We have from (7.1)

$$(7.2b) \qquad e^{-1} = -(e^{n-1} + c_1e^{n-2} + c_2e^{n-3} + \ldots + c_{n-2}e + c_{n-1}),$$

and obtain from (7.2a) and (7.2b), by definition

$$a^{(1)} = (e + c_1, e^2 + c_1e + c_2, \ldots, e^{n-2} + c_1e^{n-3} + \ldots + c_{n-2},$$
$$-(e^{n-4} + c_1e^{n-2} + \ldots, + c_{n-2}e + c_{n-1})), \quad \text{whence}$$

$$(7.3) \qquad a^{(1)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-2}^{(0)}, -a_{n-1}^{(0)})$$
$$a_k^{(0)} = e^k + \sum_{i=1}^{k} c_ie^{k-i}, \quad k = 1, \ldots, n-1.$$

From (7.3) we obtain by definition

$$(7.3a) \qquad b^{(1)} = (c_1, c_2, \ldots, c_{n-2}, -c_{n-1}).$$

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Carrying out the NA, now of $a^{(1)}$, we obtain, pursuing this procedure from one vector to the following one, using induction

$$(7.3b)\qquad a^{(k)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-k-1}^{(0)}, -a_{n-k}^{(0)}, -a_{n-k+1}^{(0)}, \ldots, a_{n-1}^{(0)});$$
$$k = 1, 2, \ldots, n-1; \quad \text{and}$$
$$b^{(k)} = (c_1, c_2, \ldots, c_{n-k+1}, -c_{n-k}, -c_{n-k+1}, \ldots, -c_{n-1}^{(0)});$$
$$k = 1, 2, \ldots, n-1.$$

We now obtain from (7.3b), for k $= n - 1$,

$$(7.3c)\qquad \begin{aligned} a^{(n-1)} - b^{(n-1)} &= (-a_1, -a_2, \ldots, -a_{n-2}, -a_{n-1}) \\ &= (-e, -(e^2 + c_1e), -(e^3 + c_1e^2 + c_2e), \ldots, \\ &\qquad - (e^{n-1} + c_1e^{n-2} + \ldots + c_{n-3}e^2 + c_{n-2}e)) \end{aligned}$$

and from (7.3c), taking into account (7.2b),

$$\begin{aligned} a^{(n)} &= (e + c_1, e^2 + c_1e + c_2, \ldots, e^{n-2} + c_1e^{n-3} + \ldots + c_{n-2}, \\ &\qquad e^{n-1} + c_1e^{n-2} + \ldots + c_{n-1}) = a^{(0)}, \end{aligned}$$

and so

$$(7.4)\qquad a^{(n)} = a^{(0)}.$$

Thus, the NA of $a^{(0)}$, in the case of the $+1$ free element in the field equation of e, is purely periodic, and the length of the period $m = n$. We repeat the companion vectors of this NA, viz.

$$(7.4b)\qquad \begin{aligned} b^{(0)} &= (c_1, c_2, \ldots, c_{n-2}, c_{n-1}) \\ b^{(1)} &= (c_1, c_2, \ldots, c_{n-2}, -c_{n-1}) \\ b^{(2)} &= (c_1, c_2, \ldots, c_{n-3}, -c_{n-2}, -c_{n-1}) \\ &\vdots \\ b^{(n-1)} &= (-c_1, -c_2, \ldots, -c_{n-2}, -c_{n-1}) \\ b^{(n)} &= b^{(0)}. \end{aligned}$$

The difficulty in calculating explictly the $A_0^{(v)}$ rests with the fact that, instead of one recurrency relation for $A_0^{(v)}$, as was the case with (5.4), we have, according to (7.4a) n recursion relations, viz.

$$(7.5)\qquad \begin{aligned} A_0^{((s+1)n)} &= A_0^{(sn)} + \textstyle\sum_1^{n-1} c_iA_0^{(sn+i)}, \\ A_0^{((s+1)n+1)} &= A_0^{(sn+1)} + \textstyle\sum_1^{n-2} c_iA_0^{(sn+1+i)} - c_{n-1}A_0^{(s+1)n}, \\ A_0^{((s+1)n+2)} &= A_0^{(sn+2)} + \textstyle\sum_1^{n-3} c_iA_0^{(sn+2+1)} - c_{n-2}A_0^{((sn+1)n)} \\ &\qquad - c_{n-1}A_0^{(s+1)n+1}, \\ A_0^{((s+1)n+3)} &= A_0^{(sn+3)} + \textstyle\sum_1^{n-4} c_iA_0^{(sn+3+i)} \\ &\qquad - \textstyle\sum_{n-3}^{n-1} c_iA_0^{(sn+3+i)}, \\ A_0^{((s+1)n+k)} &= A_0^{(sn+k)} + \textstyle\sum_1^{n-1-k} c_iA_0^{(sn+k-i)} \\ &\qquad - \textstyle\sum_{n-k}^{n-1} c_iA_0^{(sn+k-i)}, \quad (k = 4, \ldots, n-2) \\ A_0^{((s+1)n+n-1)} &= A_0^{(sn+n-1)} - \textstyle\sum_1^{n-1} c_iA_0^{(sn+n-1+i)}. \end{aligned}$$

The task of calculating the $A_0^{(v)}$ from the recurrency relations (7.5), is for-

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midable. These n relations, though, have something in common: the summands of the $A_0^{((s+1)n+k)}$, $(k = 0, 1, \ldots, n-1)$ are, up to the signs, the same, and also the alteration of the signs follows an exact pattern. The author has solved this problem in a previous paper [7] where a similar problem was investigated. Therefore, the explicit structure of the $A_0^{(v)}$ is given here explicitly. The corresponding proof would occupy some fifty pages, and the reader is asked to accept the result on trust. His time and energy will only profit from skipping these lengthy, laborious readings.

The formula for $A_0^{(m+n)}$ in its explicit form is completely analogous to formula (5.7) of the -1 free element case of the field equation of e, with the difference that here there appears, under the summation, a power of -1, viz.

$$(7.6)\qquad A_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_k} (-1)^{m-i-[m/n]} \binom{m-i}{y_1, y_2, \ldots, y_n} \prod_1^{n-1} c_{n-k}^{y_k}, \quad m = 0, 1, \ldots,$$

with γ and $\sum_{y_k}$ as before. Similarly we treat the $+1$ free element case of the field equation of e^{-1} which, following (7.1) has now the form

$$(7.1a)\qquad e^{-n} + c_{n-1}e^{-(n-1)} + c_{n-2}e^{-(n-2)} + \ldots + c_2e^{-2} + c_1e^{-1} + 1 = 0.$$

Thus the field equation of e^{-1}, in the plus 1 case, is obtained from that of the minus 1 case by the substitution $c_k \to c_{n-k}$ $(k = 1, \ldots, n-1)$. In complete analogy with formulas (5.10) and (7.6), we obtain here for the corresponding convergents $B_0^{(v)}$:

$$(7.7)\qquad B_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_k} (-1)^{m-i-[m/n]} \binom{m-i}{y_1, y_2, \ldots, y_n} \prod_1^{n-1} c_k^{\,y_k}, \quad m = 0, 1, \ldots$$

8. Combinatorial identities of convergents. We are now sufficiently equipped with the necessary tools to develop our new method of constructing sophisticated combinatorial identities. We also believe that our new approach is superior to our previous new-old one, but in final judgment probably each stands on its own method. We first handle the minus one free element case. Formula (3.4b), viz.

$$\prod_1^v a_{n-1}^{(i)} = A_0^{(v)} + \sum_1^{n-1} a_j^{(v)} A_0^{(v+j)}$$

will be used repeatedly. In the minus one case it takes the form, in view of (5.2), (5.3) and (5.4), and taking into account (5.1a),

$$(8.1)\qquad \prod_1^v a_{n-1}^{(i)} = \prod_1^v e^{-1} = A_0^{(v)} + \sum_1^{n-1}(e^j + a_1e^{j-1} + \ldots + a_{j-1}e + a_j)A_0^{(v+j)},$$

and from (8.1) we obtain

$$(8.1a)\qquad e^{-v} = A_0^{(v)} + \sum_{j=1}^{n-1} a_jA_0^{(v+j)} + \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1-k} a_jA_0^{(v+k+j)}\right)e^k,$$

$$a_0 = 1,\ v = 1, 2, \ldots.$$

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But $A_0^{(v)} + \sum_1^{n-1} a_j A_0^{(v+j)} = A_0^{(v+n)}$; hence, from (8.1$a$),

(8.2) $$e^{-v} = A_0^{(v+n)} + \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1-k} a_j A_0^{(v+k+j)}\right) e^k;\ a_0 = 1;\ v = 1, 2, \ldots.$$

Carrying out the NA of the starting vector $\bar{a}^{(0)}$ from (5.9), we have to substitute in (8.2)

$$a_j \to -a_{n-j},\quad j = 0, 1, \ldots, n-1;\quad a_n = -1$$

and obtain, substituting also $e \to e^{-1}$

(8.3) $$e^v = B_0^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1-k} a_{n-j} B_0^{(v+k+j)}\right) e^{-k};\quad a_n = -1;\ v = 1, 2, \ldots$$

We further return to the important formula (3.4c) which, in the case of the NA of $a^{(0)}$, from (5.2), and with (8.1), takes the form

(8.4) $$\Delta_v = \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \ldots A_0^{(v+n-1)} \\ e + a_1 & A_1^{(v+1)} & A_0^{(v+2)} & \ldots A_1^{(v+n-1)} \\ e^2 + a_1 e + a_2 & A_2^{(v+1)} & A_2^{(v+2)} & \ldots A_2^{(v+n-1)} \\ \vdots & \vdots & & \vdots\ \ \vdots \\ e^{n-1} + a_1 e^{n-2} + \ldots + a_{n-1} & A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)} e^v.$$

Substituting for $A_j^{(v+k)}$, $(j = 1, \ldots, n-1;\ k = 1, \ldots, n-1)$ their values as linear forms of $A_0^{(v+k)}$ from (6.6), we obtain that Δ_v is the determinant of the matrix with entries Δ_{ij} given by

(8.5) $$\begin{aligned} &\Delta_{11} = 1,\ \Delta_{12} = A_0^{(v+1)}, \ldots, \Delta_{1n} = A_0^{(v+n-1)} \\ &\Delta_{21} = e + a_1,\ \Delta_{22} = a_1 A_0^{(v+1)} + A_0^{(v)} \ldots \\ &\qquad\qquad \Delta_{2n} = a_1 A_0^{(v+n-1)} + A_0^{(v+n-2)} \\ &\Delta_{31} = e^2 + a_1 e + a_2,\ \Delta_{32} = a_2 A_0^{(v+1)} + a_i A_0^{(v)} + A_0^{(v-1)} \ldots. \\ &\qquad\qquad \Delta_{3n} = a_2 A_0^{(v+n-1)} + a_1 A_0^{(v+n-1)} + A_0^{(v+n-2)} \\ &\Delta_{n1} = e^{n-1} + a_1 e^{n-2} + \ldots + a_{n-1},\ \Delta_{n2} = a_{n-1} A_0^{(v+1)} + a_{n-2} A_0^{(v)} \\ &\qquad + \ldots + a_1 A_0^{(v-n+3)} + A_0^{(v-n+2)}, \ldots, \Delta_{nn} = a_{n-1} A_0^{(v+n-1)} \\ &\qquad\qquad + a_{n-2} A_0^{(v+n-2)} + \ldots + A_0^{(v)}. \end{aligned}$$

In the determinant (8.5) we carry out the following elementary row operations: $R_2 - a_1 R_1$, $R_3 - a_2 R_1 - a_1 R_2, \ldots, R_j - a_{j-1} R_1 - a_{j-2} R_2 - \ldots - a_1 R_{j-1}$, $j = 2, \ldots, n$; after that Δ_v takes the form which, together with (8.4), yields

(8.6) $$\begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & A_0^{(v+n-1)} \\ e & A_0^{(v)} & A_0^{(v+1)} & A_0^{(v+n-2)} \\ e^2 & A_0^{(v-1)} & A_0^{(v)} & A_0^{(v+n-3)} \\ \vdots & \vdots & \vdots & \vdots \\ e^{n-1} & A_0^{(v-n+2)} & A_0^{(v-n+3)} & A_0^{(v)} \end{vmatrix} = (-1)^{v(n-1)} e^v$$

$$= (-1)^{v(n-1)} \left[B_0^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1-k} a_{n-j} B_0^{(v+k+j)}\right) e^{-k}\right];\quad a_n = -1, v = 0, 1, \ldots$$

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the last equation resulting from (8.3). We have from (8.2)

$$(8.6a)\qquad e^{-k} = A_0^{(k+n)} + \sum_{s=1}^{n-1} \left(\sum_{j=0}^{n-1-s} a_j A_0^{(k+s+j)}\right) e^s;\ a_0 = 1;\ k = 0, 1, \ldots .$$

Substituting the value of e^{-k} from (8.6a) into (8.6), we obtain

$$(8.6b)\qquad (-1)^{v(n-1)} \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \ldots & A_0^{(v+n-1)} \\ e & A_0^{(v)} & A_0^{(v+1)} & \ldots & A_0^{(v+n-2)} \\ e^2 & A^{(v-1)} & A^{(v)} & \ldots & A^{(v+n-3)} \\ \vdots & \vdots & \vdots & & \vdots \\ e^{n-1} & A_0^{(v-n+2)} & A_0^{(v-n+3)} & \ldots & A_0^{(v)} \end{vmatrix} =$$

$$B_0^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{t=0}^{n-1-k} a_{n-t} B_0^{(v+k+t)}\right) \times \left(A_0^{(k+n)} + \sum_{s=1}^{n-1} \left(\sum_{j=0}^{n-1-s} a_j A_0^{k+s+j}\right) e^s\right).$$

(8.6b) is a highly significant formula, for if we compare coefficients of equal powers of e on both sides of (8.6b), we obtain n combinatorial identities. To state only one of them, we obtain from (8.6b), comparing the coefficients of e^0 on both sides of (8.6b),

$$(8.7)\qquad (-1)^{v(n-1)} \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \ldots & A_0^{(v+n-2)} \\ A_0^{(v-1)} & A_0^{(v)} & \ldots & A_0^{(v+n-3)} \\ A_0^{(v-2)} & A_0^{(v-1)} & \ldots & A_0^{(v+n-4)} \\ \vdots & \vdots & & \vdots \\ A_0^{(v-n+2)} & A_0^{(v-n+3)} & \ldots & A_0^{(v)} \end{vmatrix} =$$

$$B_0^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{t=0}^{n-1-k} a_{n-t} B_0^{(v+k+t)}\right) A_0^{(k+n)};\ a_n = -1;\ v = 0, 1, \ldots .$$

(8.7) is indeed an amazing combinatorial identity, and its structure, harmonic and esthetic as it may appear to the eye, is highly sophisticated, especially if we substitute for the $A_0^{(v)}$ and $B_0^{(v)}$ their explicit values. This would horrify any commercial publisher.

In exactly the same way we obtain the combinatorial identity from the NA of $\bar{a}^{(0)}$ which, in a certain sense, can be regarded as the inverse of the identity (8.7);

$$(8.7a)\qquad (-1)^{v(n-1)} \begin{vmatrix} B_0^{(v)} & B_0^{(v+1)} & \ldots & B_0^{(v+n-2)} \\ B_0^{(v-1)} & B_0^{(v)} & \ldots & B_0^{(v+n-3)} \\ B_0^{(v-2)} & B_0^{(v-1)} & \ldots & B_0^{(v+n-4)} \\ \vdots & \vdots & & \vdots \\ B_0^{(v-n+2)} & B_0^{(v-n+3)} & \ldots & B_0^{(v)} \end{vmatrix} =$$

$$A_0^{(v+n)} + \sum_{k=1}^{n-1} \left(\sum_{t=0}^{n-1-k} a_t A_0^{(v+k+t)}\right) B_0^{(k+n)};\ a_0 = 1;\ v = 0, 1, \ldots .$$

We shall now simplify the right sides of (8.7) and (8.7a). It is surprising indeed that these complicated structures assume such a simple form, as we shall show in the sequel. It is this simple pattern of the right sides of the

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above mentioned formulas which will supply the combinatorial identities, so harmonic in their beautiful structure. We obtain, with $a_n = -1$,

$$\begin{aligned}
&B_0{}^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{t=0}^{n-1-k} a_{n-t} B_0{}^{(v+k+t)}\right) A_0{}^{(k+n)} \\
&= B_0{}^{(v)} - a_{n-1}B_0{}^{(v+1)} - a_{n-2}B_0{}^{(v+2)} - \ldots - a_2 B_0{}^{(v+n-2)} - \\
&\quad a_1 B_0{}^{(v+n-1)} + (B_0{}^{(v+1)} - a_{n-1}B_0{}^{(v+2)} - a_{n-2}B_0{}^{(v+3)} - \ldots - \\
&\quad a_3 B_0{}^{(v+n-2)} - a_2 B_0{}^{(v+n-1)})A_0{}^{(n+1)} + (B_0{}^{(v+2)} - a_{n-1}B_0{}^{(v+3)} - \\
&\qquad a_{n-2}B_0{}^{(v+4)} - \ldots - a_4 B_0{}^{(v+n-2)} - a_3 B_0{}^{(v+n-1)})\, A_0{}^{(n+2)} \\
&\quad + \ldots + (B_0{}^{(v+n-2)} - a_{n-1}B_0{}^{(v+n-1)})\, A_0{}^{(n+n-2)} \\
&\qquad\qquad + B_0{}^{(v+n-1)}\, A_0{}^{(n+n-1)} \\
&= B_0{}^{(v)} + B_0{}^{(v+1)}(A_0{}^{(n+1)} - a_{n-1}) + B_0{}^{(v+2)}(A_0{}^{(n+2)} - a_{n-2} \\
&\quad - a_{n-1}A_0{}^{(n+1)}) + B_0{}^{(v+3)}(A_0{}^{(n+3)} - a_{n-3} - a_{n-2}A_0{}^{(n+1)} \\
&\quad - a_{n-1}A_0{}^{(n+2)}) + \ldots. + B_0{}^{(v+k)}(A_0{}^{(n+k)} - a_{n-k} - a_{n-k+1}A_0{}^{(n+1)} \\
&\quad - a_{n-k+2}A_0{}^{(n+2)} - \ldots - a_{n-1}A_0{}^{(n+k-1)}) + \ldots. \\
&\quad + B_0{}^{(v+n-2)}(A_0{}^{(n+n-2)} - a_2 - a_3 A_0{}^{(n+1)} - \ldots - a_{n-1}A_0{}^{(n+n-3)}) \\
&\quad + B_0{}^{(v+n-1)}(A_0{}^{(n+n-1)} - a_1 - a_2 A_0{}^{(n+1)} - \ldots - a_{n-1}A_0{}^{(n+n-2)}).
\end{aligned}$$

Now

$$A_0{}^{(n+k)} = a_{n-k} + a_{n-k+1}A_0{}^{(n+1)} + a_{n-k+2}A_0{}^{(n+2)} + \ldots + a_{n-1}A_0{}^{(n+k-1)},$$

hence

$$\begin{aligned}
&A_0{}^{(n+k)} + a_{n-k} - a_{n-k+1}A_0{}^{(n+1)} - a_{n-k+2}A_0{}^{(n+2)} - \ldots \\
&\qquad\qquad - a_{n-1}A_0{}^{(n+k-1)} = 0, \quad k = 1, \ldots, n-1,
\end{aligned}$$

and we obtain the important formula

$$(8.8)\qquad B_0{}^{(v+n)} - \sum_{k=1}^{n-1} \left(\sum_{t=0}^{n-1-k} a_{n-t} B_0{}^{(v+k+t)}\right) A_0{}^{(k+n)} = A_0{}^{(v)}.$$

In the same way we shall simplify, for the general case, the right side of formula (8.7a) which in certain cases turned out to be $A_0{}^{(v)}$, and will hold so generally. We have

$$\begin{aligned}
B_0{}^{(n+1)} &= -a_1, \\
B_0{}^{(n+2)} &= -a_2 - a_1 B_0{}^{(n+1)},
\end{aligned}$$

and generally, by induction

$$(8.9)\qquad B_0{}^{(n+k)} = -\sum_0^{k-1} a_{k-i} B_0{}^{(n+i)}, \quad k = 1, \ldots, n-1.$$

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We now obtain for the right side of (8.7a)

$$
\begin{aligned}
&A_0^{(v+n)} + \textstyle\sum_{k=1}^{n-1}(A_0^{(v+k)} + a_1A_0^{(v+k-1)} + \dots \\
&\quad + a_{n-k-1}A_0^{(v+n-1)})B_0^{(k+n)} = A_0^{(v)} + a_1A_0^{(v+1)} + a_2A_0^{(v+2)} + \dots \\
&\quad + a_{n-1}A_0^{(v+n-1)} + (A_0^{(v+1)} + a_1A_0^{(v+2)} + \dots \\
&\quad + a_{n-2}A_0^{(v+n-1)})B_0^{(n+1)} + (A_0^{(v+2)} + a_1A_0^{(v+3)}) + \dots \\
&\quad + a_{n-3}A_0^{(v+n-1)})B_0^{(n+2)} + \dots + (A_0^{(v+n+2)} \\
&\quad + a_1A_0^{(v+n-1)})B_0^{(n+n-2)} + B_0^{(n+n-1)}A_0^{(v+n-1)} = A_0^{(v)} \\
&\quad + A_0^{(v+1)}(B_0^{(n+1)} + a_1) + A_0^{(v+2)}(B_0^{(n+2)} + a_1B_0^{(n+1)} + a_2) \\
&\quad + A_0^{(v+3)}(B_0^{(n+3)} + a_1B_0^{(n+2)} + a_2B_0^{(n+1)} + a_3) + \dots \\
&\quad + A_0^{(v+n-2)}(B_0^{(n+n-2)} + a_1B_0^{(n+n-3)} + \dots + a_{n-2}) \\
&\qquad + A_0^{(v+n-1)}(B_0^{(n+n-1)} + a_1B_0^{(n+n-2)} + \dots + a_{n-1}) = A_0^{(v)},
\end{aligned}
$$

in view of (8.9). This we wished to prove.

Formulas (8.7) and (8.7a) now take the final form

$$
(8.10)\quad
\begin{vmatrix}
A_0^{(v)} & A_0^{(v+1)} & \dots & A_0^{(v+n-2)} \\
A_0^{(v-1)} & A_0^{(v)} & \dots & A_0^{(v+n-3)} \\
\vdots & \vdots & & \vdots \\
A_0^{(v-v+2)} & A_0^{(v-n+3)} & \dots & A_0^{(v)}
\end{vmatrix}
= (-1)^{v(n-1)}\, B_0^{(v)}; \quad \text{and}
$$

$$
\begin{vmatrix}
B_0^{(v)} & B_0^{(v+1)} & \dots & B_0^{(v+n-2)} \\
B_0^{(v+1)} & B_0^{(v)} & \dots & B_0^{(v+n-3)} \\
\vdots & \vdots & & \vdots \\
B_0^{(v-n+2)} & B_0^{(v-n+3)} & \dots & B_0^{(v)}
\end{vmatrix}
= (-1)^{v(n-1)}\, A_0^{(v)};
$$

$$
A_0^{(m+n)},\ B_0^{(m+n)} \quad (m = 0, 1, \dots) \text{ from (5.7), (5.10).}
$$

(8.10) supplies the two basic combinatorial identities in the special case of the minus one free element in the field equations of the units e and e^{-1}, both algebraic numbers of degree n.

The blessings and the fertility of formula (8.10) are unlimited. First of all, it supplies two sophisticated combinatorial identities which could never be proved by so-called elementary methods. Secondly, an infinite number of new, and still more complicated identities can be conjured up from them. Thus we obtain, for instance,

$$
(8.11)\quad A_0^{(v)}B_0^{(v)} =
\begin{vmatrix}
A_0^{(v)} & A_0^{(v+1)} & \dots & A_0^{(v+n-2)} \\
A_0^{(v-1)} & A_0^{(v)} & \dots & A_0^{(v+n-3)} \\
\vdots & \vdots & & \vdots \\
A_0^{(v-n+2)} & A_0^{(v-n+3)} & \dots & A_0^{(v)}
\end{vmatrix}
\times
\begin{vmatrix}
B_0^{(v)} & B_0^{(v+1)} & \dots & B_0^{(v+n-2)} \\
B_0^{(v-1)} & B_0^{(v)} & \dots & B_0^{(v+n-3)} \\
\vdots & \vdots & & \vdots \\
B_0^{(v-n+2)} & B_0^{(v-n+3)} & \dots & B_0^{(v)}
\end{vmatrix},
$$

and in the same way we produce $A_0^{(v_1)}A_0^{(v_2)} \ldots A_0^{(v_k)}$, $k = 2, 3, \ldots$, even $A_0^{(v)B_0^{(v)}}$, and much more.

Further important identities can be obtained from formulas (7.2), (8.3). Since $e^{-v}\cdot e^v = 1$, we obtain n combinatorial identities by comparing coefficients of equal powers of e. Of course, to this end we have to express e^{-1}, $e^{-2}, \ldots, e^{-n}$ through polynomials in $e, e^2, \ldots, e^{n-1}$, which can be easily found from the corresponding field equations of e and e^{-1}. In the same way we obtain numerous combinatorial identities from $e^{-(s+t)}\, e^s e^t = 1$, $(s, t \in \mathbf{Z})$ $s^{-(s+t+v)} \times e^s e^t e^v = 1$, etc., as was done in [**3**].

9. Special cases. In this chapter we shall illustrate our theory by a few special cases which are of a certain significance in various mathematical problems. Let

$$(9.1)\qquad f(x) = x^n + ax - 1;\quad a \in \mathbf{Z};\quad f(x) \text{ irreducible over } \mathbf{Q};$$
$$f(w) = 0;\quad w \text{ a unit in } \mathbf{Q}(w);\quad w = e.$$

The field equation of e is

$$(9.1a)\qquad e^n + aw - 1 = 0,$$

and, compared with the general field equation of the unit e from (5.1), we have, in this case,

$$a_1 = a_2 = \ldots = a_{n-2} = 0;\quad s_{n-1} = a.$$

Since here

$$(9.1b)\qquad y_2 = y_3 = \ldots = y_{n-1} = 0,$$

formula (5.7) takes the form

$$(9.1c)\qquad A_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_1, y_n} \binom{m-i}{y_1,\, y_n} a^{y_1},$$

where $\gamma = [m(n-1)/n]$ and $y_1 + y_n = m - i$ and $y_1 + ny_n = m$. We obtain from (9.1c)

$$i = (n-1)y_n;\quad y_1 = m - i - y_n = m - ny_n;$$
$$\binom{m-i}{y_1,\, y_n} = \binom{m-i}{y_n,\, m-i-y_n} = \binom{m-i}{y_n};$$

since

$$\binom{m-i}{y_n} = \binom{m-(n-1)y_n}{y_n},$$
$$y_n \leqq m - (n-1)y_n,\quad \text{hence}\quad ny_n \leqq m.$$

Writing

$$y_n = j;\quad 0 \leqq j \leqq [m/n],$$

(9.1c) takes the form

$$(9.2)\qquad A_0^{(m+n)} = \sum_0^{[m/n]} \binom{m-(n-1)j}{j} a^{m-nj}.$$

(9.2) is a beautiful formula which will serve us well in the sequel.

Further, in our special case (9.1), formula (5.10) takes the form, with $y_1 = y_2 = \ldots = y_{n-2} = 0$,

$$(9.3)\qquad B_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_{n-1},y_n} (-1)^{m-i-y_n} \binom{m-i}{y_{n-1},\, y_n} a^{y_{n-1}},$$

where $\gamma = [m(n-1)/n]$ and $y_{n-1} + y_n = m - i$ and $ny_n + (n+1)y_{n-1} = m$.

We shall now put (9.3) in better shape, which is not a simple tailoring process. We obtain from the two Diophantine equations under the second summation

$$(n-2)y_{n-1} + (n-1)y_n = i,$$
$$(n-2)y_{n-1} + (n-1)\,[m-i-y_{n-1}] = i,$$
$$-y_{n-1} + (n-1)m - (n-1)i = i,\quad \text{whence}$$
$$(9.3a)\qquad y_{n-1} = (n-1)m - ni.$$
$$y_n = m - i - (n-1)m + ni,\quad \text{and}$$
$$(9.3b)\qquad y_n = (n-1)i - (n-2)m.$$

Since $y_{n-1}, y_n > 0$, we obtain

$$(n-2)m/(n-1) \leqq i \leqq (n-1)m/n,\quad \text{so}$$
$$(9.3c)\qquad [(n-2)m/(n-1)] \leqq i \leqq [(n-1)m/n].$$

Since $y_{n-1} + y_n = m - i$, the multinomial coefficient in (9.3) becomes binomial, and the formula takes the form

$$(9.4)\qquad B_0^{(m+n)} = \sum_{\beta}^{\gamma} (-1)^{(n-1)m-ni} \binom{m-i}{(n-1)m-ni} a^{(n-1)m-ni},$$

where $\gamma = [(n-1)m/n]$ and $\beta = [(n-2)m/(n-1)]$.

Though no objection can be raised against (9.4) from an esthetic viewpoint, it can be given a very simple form for a special value of m. We choose

$$(9.4a)\qquad m \to m(n-1),$$

and obtain from (9.4)

$$B_0^{((n-1)m+n)} = \sum_{\beta}^{\gamma} (-1)^{(n-1)^2m-ni} \binom{(n-1)m-i}{(n-1)^2m-ni} a^{(n-1)^2m-ni},$$

with $\beta = (n-2)m$ and $\gamma = [(n-1)^2m/n]$.

We now lower the index of the bounds of sigma in (9.4b) by $(n-2)m$, and obtain

$$B_0^{((n-1)m+n)} = \sum_{i=0}^{\gamma} (-1)^{(n-1)^2 m - n(i+(n-2)m)} \times \binom{(n-1)m-(i+(n-2)m)}{(n-1)i} a^{m-ni},$$

with $\gamma = [(n-1)^2 m/n - (n-2)m]$

(9.5) $$B_0^{((n-1)m+n)} = \sum_0^{[m/n]} (-1)^{m-ni} \binom{m-i}{(n-1)i} a^{m-ni}.$$

We shall now investigate a single case, viz.

(9.6) $n = 3;\ a = -1;\quad x^3 + x - 1 = 0$ from (9.1); $\quad v = 2m + 3$ in (8.10).

With the values of (9.6), we obtain from (8.10)

(9.6a) $$B_0^{(2m+3)} = (A_0^{(2m+3)})^2 - A_0^{(2m+2)} A_0^{(2m+4)}.$$

From (9.5) we obtain

(9.6b) $$B_0^{(2m+3)} = \sum_0^{[m/3]} \binom{m-i}{2i}.$$

From (9.2) we obtain

(9.6c) $$A_0^{(2m+3)} = \sum_0^{[2m/3]} (-1)^j \binom{2m-2j}{j};$$

$$A^{(2m+2)} = \sum_{j=0}^{[(2m-1)/3]} (-1)^{1+j} \binom{2m-1-2j}{j}$$

$$A_0^{(2m+4)} = \sum_{j=0}^{[(2m+1)/3]} (-1)^{1+j} \binom{2m+1-2j}{j}.$$

Substituting the values of (9.6b), (9.6c) we obtain a beautiful identity which was proved in [**1**]:

(9.6d) $$\sum_0^{\gamma_1} \binom{m-i}{2i} = \left(\sum_0^{\gamma_2} (-1)^j \binom{2m-2j}{j}\right)^2 - \sum_0^{\gamma_3} (-1)^{1+j} \times \binom{2m-1-2j}{j} \sum_0^{\gamma_4} (-1)^{1+j} \binom{2m+1-2j}{j}$$

with $\gamma_1 = [m/3]$, $\gamma_2 = [2m/3]$, $\gamma_3 = [(2m-1)/3]$ and $\gamma_4 = [(2m+1)/3]$.

For $m = 6$, we obtain from (9.6d), $12 = 0^2 - (-3)4$. Thus $A_0^{(15)} = 0$; also $A_0^{(6)} = 0$, as can be easily verified from (9.6c). As the author has proved in [**1**], these are the only zeros of the arithmetic function $A_0^{(v)}$, $(v = 3, 4, \ldots)$.

For $n = 2$, $a = -1$ in (9.1), the generating polynomial is $f(x) = x^2 - x$

-1, the Fibonacci generating polynomial. (9.5) takes the form

(9.7) $$B_0^{(m+2)} = \sum_0^{[m/2]} \binom{m-i}{i}$$

which are the Fibonacci numbers with $B_0^{(m+2)} = F_{m+1}$, $m = 0, 1, \ldots$. Since $B_0^{(0)} = 1$, $B_0^{(1)} = 0$, it would be correct to introduce the notation $F_0 = 0$, $F_{-1} = 1$, $F_{m+2} = F_m + F_{m+1}$, $m = 1, 0, 1, \ldots$.

As a second special case we investigate the generating polynomial

(9.8) $$x^n = D^n + 1, \quad D \in N; \quad D^n + 1 \quad n\text{-powerfree}$$

which yields the algebraic number field and its unit

(9.8a) $$Q(w); \quad w^n = D^n + 1; \quad e = w - D \quad \text{a unit in } \mathbf{Q}(w).$$

The unit (9.8a) was mentioned in Section 4, i). We shall first construct the field equation of e, and obtain, denoting $D^n + 1 = m$,

$$\begin{aligned}
e &= -D + w + 0 \cdot W + \ldots + 0 \cdot w^{n-1} \\
we &= 0 - Dw + w^2 + \ldots + 0 \cdot w^{n-1} \\
&\vdots \\
w^{n-2}e &= 0 + \ldots + 0 \cdot w^{n-3} - Dw^{n-2} + w^{n-1} \\
w^{n-1}e &= m + 0 \cdot w + \ldots + 0 \cdot w^{n-2} - Dw^{n-1}
\end{aligned}$$

(9.9) $$\begin{vmatrix}
(-D-e) & 1 & 0 & 0 \ldots 0 & 0 & 0 \\
0 & (-D-e) & 1 & 0 \ldots 0 & 0 & 0 \\
0 & 0 & (-D-e) & 0 \ldots 0 & 0 & 0 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 \ldots 0 & (-D-e) & 1 \\
m & 0 & 0 & 0 \ldots 0 & 0 & (-D-e)
\end{vmatrix} = 0.$$

Expanding the determinant (9.9), we obtain

$$(-D - e)^n + (-1)^{n-1} m = 0$$

$$(D + e)^n - m = 0,$$

$$e^n + \sum_1^{n-1} \binom{n}{k} D^k e^{n-k} + D^n - (D^n + 1) = 0 \quad \text{and}$$

(9.10) $$e^n + \binom{n}{1} De^{n-1} + \binom{n}{2} D^2 e^{n-2} + \ldots + \binom{n}{n-2} D^{n-2} e^2 + \binom{n}{n-1} D^{n-1} e - 1 = 0.$$

Comparing (9.10) with the general field equation (5.1), we have

(9.10a) $$a_k = \binom{n}{k} D^k, \quad k = 1, 2, \ldots, n-1.$$

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Substituting the value of a_k from (9.10a) into formula (5.7), we obtain

$$A_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_k} \binom{m-i}{y_1, \ldots, y_n} \prod_1^{n-1} \left[\binom{n}{k} D^{n+k}\right]^{y_k},$$

with $\gamma = [(n-1)m/n]$ and $\sum_1^n y_j = m - i$ and $\sum_i^n jy_j = n$.

We obtain

$$\prod_1^{n-1} \left[\binom{n}{k} D^{n-k}\right]^{y_k} = \left[\prod_1^{n-1} \binom{n}{k}^{y_k}\right] D \sum_1^{n-1} (n-k)y_k.$$

Now

$$\sum_1^{n-1} (n-k)y_n = n \sum_1^{n-1} y_k - \sum_1^{n-1} ky_k$$
$$= n(m - i - y_n) - (m - ny_n) = (n-1)m - ni.$$

Thus

$$(9.11) \qquad A_0^{(m+n)} = \sum_{i=0}^{\gamma} D^{(n-1)m-ni} \sum_{y_k} \binom{m-i}{y_1, \ldots, y_n} \prod_1^{n-1} \binom{n}{k}^{y_k},$$

with $\gamma = [(n-1)m/n]$ and $\sum y_k$ constrained by $\sum_1^n y_j = m - i$ and $\sum_1^n jy_j = n$.

In the same way we obtain from (9.10a) and (5.10)

$$(9.11a) \qquad B_0^{(m+n)} = \sum_{i=0}^{\gamma} \sum_{y_k} (-1)^{m-i-y_n} \binom{m-i}{y_1, \ldots, y_n} D^{m-ny_n} \prod_1^{n-1} \binom{n}{k}^{y_k},$$

with $\sum_{y_k}$ constrained by $\sum_1^n y_j = m - i$ and $\sum_1^n jy_j = m$.

From (9.9) we obtain, for $n = 3$,

$$(9.12) \qquad B_0^{(v)} = (A_0^{(v)})^2 - A_0^{(v-1)} A_0^{(v+1)}, \quad (v = 1, 2, \ldots)$$

We further obtain from $y_1 + y_2 + y_3 = m - i$, $\quad y_1 + 2y_2 + 3y_3 = m$,

$$(9.12a) \qquad y_1 = i - 2y_3, \quad y_2 = m - 2i + y_3,$$
$$\binom{m-i}{y_1, y_2, y_3} = \binom{m-i}{i-y_2} = \binom{i-y_3}{y_3};$$
$$y_3 = j; \quad 0 \leqq j \leqq [i/2].$$

With (9.12a), (9.11) takes the form:

$$(9.13) \qquad A_0^{(m+3)} = \sum_{i=0}^{[2m/3]} D^{2m-3i} \sum_{j=0}^{[i/2]} \binom{m-i}{i-j} \binom{i-j}{j} 2^{m-i-j},$$

$$B_0^{(m+3)} = \sum_{i=0}^{[2m/3]} \sum_{j=0}^{[i/2]} (-1)^{m-i-j} \binom{m-i}{i-j} \binom{i-j}{j} 3^{m-i-j} D^{m-3j}.$$

Substituting the values of $A_0^{(v)}$, $B_0^{(v)}$ from (9.13) into (9.12), we obtain a very extensive, but most harmonic combinatorial identity.

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