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THE STABLE AND UNSTABLE TYPES OF CLASSIFYING SPACES

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ABSTRACT. The main purpose of this paper is to study groups G_1, G_2 such that $H^*(BG_1, \mathbf{Z}/p)$ is isomorphic to $H^*(BG_2, \mathbf{Z}/p)$ in $\mathcal{U}$, the category of unstable modules over the Steenrod algebra $\mathcal{A}$, but not isomorphic as graded algebras over $\mathbf{Z}/p$.

0. **Introduction.** Let G be a finite group. A classification of the stable homotopy type of BG is given by Martino and Priddy's paper [4] in purely algebraic terms. It is known that the stable type of BG does not determine G up to isomorphism; however [4] shows that for each prime p, the local stable type of BG depends on the conjugacy classes of homomorphisms from p-groups Q into G. One application to the classification theorem in [4] is the case G_1, G_2 are finite groups with normal Sylow p-subgroups P_1, P_2. Then BG_1 and BG_2 have the same stable homotopy type, localized at p, if and only if $P_1 \cong P_2$ (say P) and $W_{G_1}(P)$ is pointwise conjugate to $W_{G_2}(P)$ in $\mathrm{Out}(P)$. The paper [4] gives the example of groups G_1, G_2 illustrating this theorem. For these groups $H^*(BG_1, \mathbf{Z}/p)$ and $H^*(BG_2, \mathbf{Z}/p)$ are isomorphic in $\mathcal{U}$, the category of unstable modules over the Steenrod algebra $\mathcal{A}$, but are not isomorphic in $\mathcal{K}$, the category of unstable algebras over $\mathcal{A}$. The goal of this note is to exhibit groups G_1, G_2 such that $H^*(BG_1, \mathbf{Z}/p)$ and $H^*(BG_2, \mathbf{Z}/p)$ are isomorphic in $\mathcal{U}$, but are not even isomorphic even as graded algebras over $\mathbf{Z}/p$. These algebras have the added advantage of a much smaller Krull dimension than those of [4].

Section One gives some information on the classification of the p-local stable homotopy type of BG. This includes the main classification theorem and its application in case of finite groups with normal Sylow p-subgroups. We give an example of two finite groups with stably homotopy equivalent classifying spaces localized at $p > 2$. Then in Section Two, we demonstrate the cohomology of these classifying spaces which are necessarily isomorphic in $\mathcal{U}$, are not isomorphic as graded algebras over $\mathbf{Z}/p$. To show this, we calculate the invariant elements of their cohomology groups in dimension 3 and 6, and then we compare cup products in dimension 6 so that we obtain the result that two cohomology rings have different algebra structures.

1. **A classification of the stable type of** BG**.** Let G be a finite group. We denote BG a classifying space of G, which has a contractible universal principal G bundle EG. With G. Carlsson's solution of the Segal conjecture it has become possible to determine the complete p-local stable decomposition $BG \simeq X_1 \vee X_2 \vee \cdots \vee X_n$. The suspension

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spectrum of BG and its wedge summands have played an important role in homotopy theory. In paper [5], the authors give a characterization of the indecomposable summands of BG in terms of modular representation theory of $\mathrm{Out}(Q)$ modules for $Q < P$ the Sylow p-subgroup of G. This is the characterization which is used to study the stable type of BG in [4]. It is known that the stable type of BG does not determine G up to isomorphism. A simple example [due to N. Minami] is given by $Q_{4p} \times Z/2$ and $D_{2p} \times Z/4$ where p is an odd prime, Q_{4p} is the generalized quaternion group of order $4p$ and D_{2p} is the dihedral group of order $2p$. It is even worse for p-local classifying spaces since BG and $BG/O_{p'}(G)$ have isomorphic mod p homology and hence equivalent stable types. Here $O_{p'}(G)$ is the maximal normal subgroup of G of order prime to p. But there is a good result in this direction by Nishida.

THEOREM 1.1 [6]. *Let G_1, G_2 be finite groups with Sylow p-subgroups P_1, P_2. If BG_1 and BG_2 are stably equivalent localized at p, then $P_1 \cong P_2$.* ∎

However the following classification theorem which is established by J. Martino and S. Priddy gives us a necessary and sufficient condition.

THEOREM 1.2 [4]. *For two finite groups G_1, G_2, the following are equivalent.*

(1) Localized at p, BG_1 and BG_2 are stably equivalent.

(2) For every p-group Q, $F_p \operatorname{Rep}(Q, G_1) \cong F_p \operatorname{Rep}(Q, G_2)$ as $\mathrm{Out}(Q)$ modules. $\operatorname{Rep}(Q, G) = \operatorname{Hom}(Q, G)/G$ with G acting by conjugation.

(3) For every p-group Q, $F_p \operatorname{Inj}(Q, G_1) \cong F_p \operatorname{Inj}(Q, G_2)$ as $\mathrm{Out}(Q)$ modules. $\operatorname{Inj}(Q, G) < \operatorname{Rep}(Q, G)$ consists of conjugacy classes of injective homomorphisms. ∎

This classification simplifies if G has a normal Sylow p-subgroup. Then the stable homotopy type depends on the *Weyl group* of the Sylow p-subgroup.

DEFINITION 1.3. Two subgroups $H, K < G$ are called *pointwise conjugate* in G if there is a bijection of sets $H \xrightarrow{\alpha} K$ such that $\alpha(h) = g_h^{-1} h g_h$ for $g_h \in G$ depending on $h \in H$. ∎

Alternately it is easy to see that an equivalent condition is $|H \cap (g)| = |K \cap (g)|$ for all $g \in G$, where (g) denotes the conjugacy class of g. We assume G has a normal Sylow p-subgroup P. We set $G = P \rtimes H$ for p'-group H by Zassenhaus's theorem, and $G = P \cdot H$, $H \cap P = \{1\}$. Let $W_G(P)$ denote the *Weyl group* of $P < G$ *i.e.* $W_G(P) = N_G(P)/P \cdot C_G(P)$ where $N_G(P)$ is the normalizer and $C_G(P)$ is the centralizer of P in G. Then $W_G(P) \leq \mathrm{Out}(P)$.

THEOREM 1.4 [4]. *Suppose G_1 and G_2 are finite groups with normal Sylow p-subgroups P_1 and P_2. Then BG_1 and BG_2 have the same stable homotopy type, localized at p, if and only if $P_1 \cong P_2$ ($\approx P$ say) and $W_{G_1}(P)$ is pointwise conjugate to $W_{G_2}(P)$ in* $\mathrm{Out}(P)$. ∎

To see the relation between Theorem 1.2 and 1.4 refer to the paper [4].

Let us give G_1, G_2 such that BG_1 is stably equivalent to BG_2 localized at $p > 2$.

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EXAMPLE 1.5. Let p, l be different odd primes such that $p \equiv 1 \pmod{l}$. We set P be an elementary abelian p-group of rank l^2, *i.e.* $P = (\mathbf{Z}/p)^{l^2}$. Then $\operatorname{Out} P = \mathrm{GL}_{l^2}(\mathbf{F}_p)$. Let $H_1' = (\mathbf{Z}/l)^3$ and $H_2' = U_3(\mathbf{F}_l)$ so that H_1' is not isomorphic to H_2' where $U_3(\mathbf{F}_l)$ is 3×3 upper triangular matrices over $\mathbf{F}_l$. Let Q_1, Q_2 be the subgroups of H_1', H_2' given by

$$Q_1 = \langle (1, 0, 0) \rangle,$$
$$Q_2 = \left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Then up to isomorphism $Q_i \cong Q(= \mathbf{Z}/l)$ $(i = 1, 2)$. Thus the inclusion $\rho\colon Q \hookrightarrow \mathrm{GL}_1(\mathbf{F}_p) = \mathbf{F}_p^*$ is a 1-dimensional representation where $\mathbf{F}_p^*$ is a cyclic group of order $p-1$ which has a generator ζ. (In fact this is a primitive $p-1$-th root of unity.) Now $l \mid p-1$, hence we set $l \cdot k = p-1$ for some k. Then $\zeta^{\frac{p-1}{l}} = \zeta^k = \omega$ is a primitive l-th root of unity. We define $\rho(q) = \omega$ where q is the generator of Q. Then ρ induces representations $f_1 = \mathrm{Ind}_{Q_1}^{H_1'}(\rho)\colon H_1' \longrightarrow \mathrm{GL}_{l^2}(\mathbf{F}_p)$ and $f_2 = \mathrm{Ind}_{Q_2}^{H_2'}(\rho)\colon H_2' \longrightarrow \mathrm{GL}_{l^2}(\mathbf{F}_p)$. These induced representations are defined by the following composition maps.

$$(*) \qquad f_i = \mathrm{Ind}_Q^{H_i'}(\rho) : H_i' \xrightarrow{\alpha} Q^{l^2} \rtimes \Sigma_{l^2} \xrightarrow{\rho^{l^2} \times 1} \mathrm{GL}_1(\mathbf{F}_p)^{l^2} \rtimes \Sigma_{l^2} \longrightarrow \mathrm{GL}_{l^2}(\mathbf{F}_p)$$
$$h \xrightarrow{\alpha} (q_1, \ldots, q_{l^2}, \sigma) \xrightarrow{\rho^{l^2} \times 1} \big(\rho(q_1), \ldots, \rho(q_{l^2}), \sigma\big) \longrightarrow \mathbf{T}_{\bar{\sigma}}$$

where for fixed $i = 1, 2$ we define $q_k \in Q$ and $\sigma \in \Sigma_{l^2}$ by choosing coset representatives $\{s_k | k = 1, \ldots, l^2\}$ for H_i'/Q and then setting $hs_k = s_{\sigma(k)} q_k$. $\mathbf{T}_{\bar{\sigma}}$ is the $l^2 \times l^2$ matrix with the $\rho(q_i)'$s replacing the ones of the permutation matrix $\bar{\sigma}$ in $\mathrm{GL}_{l^2}(\mathbf{F}_p)$.

For $h \in H_i'$, $hs_k \in s_j Q$ for some $s_j \in \mathbb{R}_i$ $(1 \leq i \leq 2, 1 \leq j, k \leq l^2)$ where $\mathbb{R}_i$ is a set of coset representatives of H_i'/Q, hence there exists σ such that $\sigma(k) = j$ and $hs_k = s_{\sigma(k)} q_k$ for some $q_k \in Q$. Here $s_{\sigma(k)}$ and q_k are uniquely determined. Thus α is injective. Therefore the induced representations f_i $(i = 1, 2)$ are injective. Now we set $f_1(H_1') = H_1$ and $f_2(H_2') = H_2$. These groups H_1 and H_2 act on P. It follows that $G_i = P \rtimes H_i$ $(i = 1, 2)$ are not isomorphic and satisfy $O_{p'}(G_i) = 1$. This implies $H_i \cap C_{G_i}(P) = \{1\}$. Thus $W_{G_i}(P) = P \cdot H_i / P \cdot C_{G_i}(P) \cong H_i / H_i \cap C_{G_i}(P) = H_i$. Now we need to show that H_1 is pointwise conjugate to H_2 in $\mathrm{GL}_{l^2}(\mathbf{F}_p)$.

If M is an $m_1 \times n_1$ matrix and N is an $m_2 \times n_2$ matrix, then we note that the tensor product of M and N is a matrix of size $m_1 m_2 \times n_1 n_2$. For a given matrix M, we denote ωM by M_ω for some $\omega \in \mathbf{F}_p$.

Let $h_1' = (1, 0, 0)$, $h_2' = (0, 1, 0)$ and $h_3' = (0, 0, 1)$ be the generators of H_1'. Then by the representation map $(*)$, we get the generators $f_1(h_1') = I \otimes I_\omega$, $f_1(h_2') = I \otimes M$, $f_1(h_3') = M \otimes I$, where I is an $l \times l$ identity matrix and M is the $l \times l$ permutation matrix of $(12 \cdots l)$. We set the images of the generators h_1, h_2, h_3. Therefore H_1 is generated by $\langle h_1, h_2, h_3 \rangle$.

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Let

$$\overline{h}_1' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \overline{h}_2' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{h}_3' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

be generators of H_2'. Here $\overline{h}_1' = [\overline{h}_2', \overline{h}_3']$. Then, similarly, we obtain the generators $\overline{h}_1 = I \otimes I_\omega, \overline{h}_2 = D \otimes M, \overline{h}_3 = M \otimes I$, where D is an $l \times l$ diagonal matrix with $\omega, \omega^2, \ldots, \omega^{l-1}, 1$ on the diagonal. We also have $\overline{h}_1 = [\overline{h}_2, \overline{h}_3]$. Thus H_2 is generated by $\langle \overline{h}_1, \overline{h}_2, \overline{h}_3 \rangle$.

We claim H_1 is pointwise conjugate to H_2 in $\mathrm{GL}_{l^2}(\mathbf{F}_p)$. First we notice $h_1 = \overline{h}_1$, $h_3 = \overline{h}_3$. Let J be a subgroup generated by $\langle h_1, h_3 \rangle$ in H_1. Then for any $h \in J$, $(I \otimes I)^{-1} h (I \otimes I) = h \in H_2$. Now we consider the elements in $H_1 - J$ and $H_2 - J$. For the element $h \in H_1 - J$, h is of the form $\omega^k (I \otimes M^i)(M^j \otimes I) = \omega^k (M^j \otimes M^i)$ for some $1 \le i \le l-1$, $1 \le j, k \le l$. Also for the element $\overline{h} \in H_2 - J$, $\overline{h}$ is of the form $\omega^k (D \otimes M)^i (M^j \otimes I) = \omega^k (D^i \otimes M^i)(M^j \otimes I) = \omega^k (D^i M^j \otimes M^i)$ for some $1 \le i \le l-1$, $1 \le j, k \le l$.

We show that $M^j \otimes M^i$ is similar to $D^i M^j \otimes M^i$ for each i, j. First it is enough to show that M^j is similar to $D^i M^j$. Here M^j is also a permutation matrix and $D^i M^j$ is a matrix replacing ones of M^j by $\omega^i, \omega^{2i}, \ldots, \omega^{(l-1)i}, 1$. Then both M^j and $D^i M^j$ have the same characteristic polynomial $f(t) = t^l - 1 = 0$. To see this, let $\lambda \in \mathbf{F}_p$ be an eigenvalue of M^j. Since M^j is a cyclic permutation matrix of order l, $\lambda^l = 1$ and λ is an l th root of unity. (*i.e.* λ is a root of $t^l - 1 = 0$.) Similarly, we can see $(D^i M^j)^l = I_{l \times l}$, since

$$\begin{aligned}
(D^i M^j)^l &= D^i M^j D^i M^j \cdots D^i M^j \\
&= D^i (M^j D^i M^{-j})(M^{2j} D^i M^{-2j}) \cdots (M^{(l-1)j} D^i M^{-(l-1)j}) M^{lj} \\
&= D^i \prod_{k=1}^{l-1} (M^{kj} D^i M^{-kj}) (M^l)^j \\
&= D^i \prod_{k=1}^{l-1} \tau_0^{kj}(D^i) \quad \text{since } M^l = I \\
&= \prod_{k=1}^{l} \tau_0^{kj}(D^i) \\
&= \Big(\prod_{k=1}^{l} \tau_0^{kj}(D) \Big)^i \\
&= I \quad \text{since each diagonal entry is } \prod_{i=1}^{l} \omega^i = 1, \text{ for odd prime } l.
\end{aligned}$$

Hence each eigenvalue of $D^i M^j$ is also a root of $t^l - 1 = 0$. We chose ω as a primitive l th root of unity. Then they have l distinct eigenvalues $\omega, \omega^2, \ldots, \omega^{l-1}, 1$ in $\mathbf{F}_p$, and hence they are diagonalizable. Thus there exist $P, Q \in \mathrm{GL}_l(\mathbf{F}_p)$ such that $P^{-1} M^j P = D$, $Q^{-1} D^i M^j Q = D$, and hence $Q P^{-1} M^j P Q^{-1} = (P Q^{-1})^{-1} M^j (P Q^{-1}) = D^i M^j$. Thus M^j is similar to $D^i M^j$. Now we choose $P Q^{-1} \otimes I \in \mathrm{GL}_{l^2}(\mathbf{F}_p)$ such that

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$(PQ^{-1}\otimes I)^{-1}(M^j\otimes M^i)(PQ^{-1}\otimes I) = (PQ^{-1})^{-1}M^j(PQ^{-1})\otimes I^{-1}M^iI = D^iM^j\otimes M^i$. Therefore $M^j\otimes M^i$ is similar to $D^iM^j\otimes M^i$, $1\le i\le l-1$, $1\le j\le l$. Obviously $\omega^k(M^j\otimes M^i)$ is similar to $\omega^k(D^iM^j\otimes M^i)$ where $1\le k\le l$. This completes our claim. Therefore by Theorem 1.4, BG_1 is stably equivalent to BG_2 at $p>2$. ∎

Thus we conclude $H^*(BG_1,\mathbf{Z}/p)$ is isomorphic to $H^*(BG_2,\mathbf{Z}/p)$ in $\mathcal{U}$, the category of unstable modules over $\mathcal{A}$. Now $H^*(BG_i,\mathbf{Z}/p) = H^*(BP\rtimes H_i,\mathbf{Z}/p) = H^*(BP,\mathbf{Z}/p)^{H_i}$. But we have $H^*(BP,\mathbf{Z}/p) = H^*\big(B(\mathbf{Z}/p)^{l^2},\mathbf{Z}/p\big) = \mathbf{Z}/p[y_1,\ldots,y_{l^2}]\otimes E[x_1,\ldots,x_{l^2}]$ where $|x_i|=1$, $|y_i|=2$, $y_i=\beta x_i$ and β is the Bockstein homomorphism. Thus $H^*(BG_i,\mathbf{Z}/p) = (\mathbf{Z}/p[y_1,\ldots,y_{l^2}:2]\otimes E[x_1,\ldots,x_{l^2}:1])^{H_i}$ $(i=1,2)$.

2. **Unstable homotopy type of** BG. In this section, we demonstrate two groups such that $H^*(BG_1)$ is isomorphic to $H^*(BG_2)$ in $\mathcal{U}$, but not isomorphic as graded algebras over $\mathbf{Z}/p$. From now on we consider the case $l=3$, $p=7$ in Example 1.8. Then $G_1 = P\rtimes H_1$, $G_2 = P\rtimes H_2$ where $P=(\mathbf{Z}/7)^9$, $H_1\cong(\mathbf{Z}/3)^3$, $H_2\cong U_3(\mathbf{F}_3)$ and H_1, $H_2\le \mathrm{GL}_9(\mathbf{F}_7)$. According to the Theorem 1.4, BG_1 is stably homotopy equivalent to BG_2, localized at $p=7$. However, we shall show that $H^*(BG_1,\mathbf{Z}/7)$ is not even isomorphic to $H^*(BG_2,\mathbf{Z}/7)$ as graded algebras over $\mathbf{Z}/7$. Note $H^*(BG_i,\mathbf{Z}/7) = H^*(BP,\mathbf{Z}/7)^{H_i} = (\mathbf{Z}/7[y_1,\ldots,y_9:2]\otimes E[x_1,\ldots,x_9:1])^{H_i}$ for $i=1,2$. By using the representation map $(*)$ constructed in Section 1, we obtain the generators $h_1 = I\otimes 2I, h_2 = I\otimes M, h_3 = M\otimes I$ in H_1 and $\bar h_1 = I\otimes 2I, \bar h_2 = D\otimes M, \bar h_3 = I\otimes M$ in H_2, where I is an 3×3 identity matrix, M is the permutation matrix of (123) and D is an 3×3 diagonal matrix with $2,4,1$ on the diagonal.

First we give the straightforward calculation of the invariants of the action of H_1 and H_2 on $H^*(BP,Z/7)$ in dimension 3 and 6. (Here we give the invariants in dimension 6 relating to cup products.)

(1) Invariants in $H^*(BP,\mathbf{Z}/7)^{H_1}$

(i) dimension **3**

$$a_1 = x_1x_3x_2 + x_4x_6x_5 + x_7x_9x_8$$
$$a_2 = x_1x_7x_4 + x_2x_8x_5 + x_3x_9x_6$$
$$a_3 = x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8$$
$$a_4 = x_1x_8x_6 + x_2x_9x_4 + x_3x_7x_5$$
$$a_5 = x_1x_3x_5 + x_2x_1x_6 + x_3x_2x_4 + x_7x_9x_2 + x_8x_7x_3 + x_9x_8x_1 + x_4x_6x_8 + x_5x_4x_9 + x_6x_5x_7$$
$$a_6 = x_1x_3x_8 + x_2x_1x_9 + x_3x_2x_7 + x_4x_6x_2 + x_5x_4x_3 + x_6x_5x_1 + x_7x_9x_5 + x_8x_7x_6 + x_9x_8x_4$$
$$a_7 = x_1x_3x_4 + x_2x_1x_5 + x_3x_2x_6 + x_7x_9x_1 + x_8x_7x_2 + x_9x_8x_3 + x_4x_6x_7 + x_5x_4x_8 + x_6x_5x_9$$
$$a_8 = x_1x_3x_7 + x_2x_1x_8 + x_3x_2x_9 + x_7x_9x_4 + x_8x_7x_5 + x_9x_8x_6 + x_4x_6x_1 + x_5x_4x_2 + x_6x_5x_3$$

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$$a_9 = x_1x_3x_6 + x_2x_1x_4 + x_3x_2x_5 + x_7x_9x_3 + x_8x_7x_1 + x_9x_8x_2 + x_4x_6x_9 + x_5x_4x_7 + x_6x_5x_8$$

$$a_{10} = x_1x_3x_9 + x_2x_1x_7 + x_3x_2x_8 + x_7x_9x_6 + x_8x_7x_4 + x_9x_8x_5 + x_4x_6x_3 + x_5x_4x_1 + x_6x_5x_2$$

$$a_{11} = x_1x_4x_9 + x_2x_5x_7 + x_3x_6x_8 + x_7x_1x_6 + x_8x_2x_4 + x_9x_3x_5 + x_4x_7x_3 + x_5x_8x_1 + x_6x_9x_2$$

$$a_{12} = x_1x_4x_8 + x_2x_5x_9 + x_3x_6x_7 + x_7x_1x_5 + x_8x_2x_6 + x_9x_3x_4 + x_4x_7x_2 + x_5x_8x_3 + x_6x_9x_1$$

(ii) dimension **6**

$$e_1 = x_1x_2x_3x_4x_5x_9 + x_2x_3x_1x_5x_6x_7 + x_3x_1x_2x_6x_4x_8 + x_4x_5x_6x_7x_8x_3 + x_5x_6x_4x_8x_9x_1 + x_6x_4x_5x_9x_7x_2 + x_7x_8x_9x_1x_2x_6 + x_8x_9x_7x_2x_3x_4 + x_9x_7x_8x_3x_1x_5$$

$$e_2 = x_1x_2x_3x_4x_8x_9 + x_2x_3x_1x_5x_9x_7 + x_3x_1x_2x_6x_7x_8 + x_4x_5x_6x_7x_2x_3 + x_5x_6x_4x_8x_3x_1 + x_6x_4x_5x_9x_1x_2 + x_7x_8x_9x_1x_5x_6 + x_8x_9x_7x_2x_6x_4 + x_9x_7x_8x_3x_4x_5$$

$$e_3 = x_1x_2x_3x_4x_5x_8 + x_2x_3x_1x_5x_6x_9 + x_3x_1x_2x_6x_4x_7 + x_4x_5x_6x_7x_8x_2 + x_5x_6x_4x_8x_9x_3 + x_6x_4x_5x_9x_7x_1 + x_7x_8x_9x_1x_2x_5 + x_8x_9x_7x_2x_3x_6 + x_9x_7x_8x_3x_1x_4$$

$$e_4 = x_1x_2x_3x_5x_7x_8 + x_2x_3x_1x_6x_8x_9 + x_3x_1x_2x_4x_9x_7 + x_4x_5x_6x_8x_1x_2 + x_5x_6x_4x_9x_2x_3 + x_6x_4x_5x_7x_3x_1 + x_7x_8x_9x_2x_4x_5 + x_8x_9x_7x_3x_5x_6 + x_9x_7x_8x_1x_6x_4$$

$$e_5 = x_1x_2x_3x_4x_5x_7 + x_2x_3x_1x_5x_6x_8 + x_3x_1x_2x_6x_4x_9 + x_4x_5x_6x_7x_8x_1 + x_5x_6x_4x_8x_9x_2 + x_6x_4x_5x_9x_7x_3 + x_7x_8x_9x_1x_2x_4 + x_8x_9x_7x_2x_3x_5 + x_9x_7x_8x_3x_1x_6$$

$$e_6 = x_1x_2x_3x_4x_7x_8 + x_2x_3x_1x_5x_8x_9 + x_3x_1x_2x_6x_9x_7 + x_4x_5x_6x_7x_1x_2 + x_5x_6x_4x_8x_2x_3 + x_6x_4x_5x_9x_3x_1 + x_7x_8x_9x_1x_4x_5 + x_8x_9x_7x_2x_5x_6 + x_9x_7x_8x_3x_6x_4$$

$$e_7 = x_1x_2x_4x_6x_7x_9 + x_2x_3x_5x_4x_8x_7 + x_3x_1x_6x_5x_9x_8 + x_4x_5x_7x_9x_1x_3 + x_5x_6x_8x_7x_2x_1 + x_6x_4x_9x_8x_3x_2 + x_7x_8x_1x_3x_4x_6 + x_8x_9x_2x_1x_5x_4 + x_9x_7x_3x_2x_6x_5$$

$$e_8 = x_1x_2x_4x_5x_7x_9 + x_2x_3x_5x_6x_8x_7 + x_3x_1x_6x_4x_9x_8 + x_4x_5x_7x_8x_1x_3 + x_5x_6x_8x_9x_2x_1 + x_6x_4x_9x_7x_3x_2 + x_7x_8x_1x_2x_4x_6 + x_8x_9x_2x_3x_5x_4 + x_9x_7x_3x_1x_6x_5$$

$$e_9 = x_1x_2x_3x_7x_8x_9 + x_4x_5x_6x_1x_2x_3 + x_7x_8x_9x_4x_5x_6$$

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$$e_{10} = x_1x_2x_5x_6x_7x_9 + x_2x_3x_6x_4x_8x_7 + x_3x_1x_4x_5x_9x_8$$
$$e_{11} = x_1x_2x_4x_8x_6x_9 + x_2x_3x_5x_9x_4x_7 + x_3x_1x_6x_7x_5x_8$$
$$e_{12} = x_1x_3x_4x_6x_7x_9 + x_2x_1x_5x_4x_8x_7 + x_3x_2x_6x_5x_9x_8$$

(2) Invariants in $H^*(BP, \mathbf{Z}/7)^{H_2}$

(i) dimension **3**

$$\bar{a}_1 = x_1x_3x_2 + x_4x_6x_5 + x_7x_9x_8$$
$$\bar{a}_2 = x_1x_7x_4 + x_2x_8x_5 + x_3x_9x_6$$
$$\bar{a}_3 = x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8$$
$$\bar{a}_4 = x_1x_8x_6 + x_2x_9x_4 + x_3x_7x_5$$
$$\bar{a}_5 = x_1x_3x_5 + 2x_2x_1x_6 + 4x_3x_2x_4 + x_7x_9x_2 + 2x_8x_7x_3 + 4x_9x_8x_1 + x_4x_6x_8 + 2x_5x_4x_9 + 4x_6x_5x_7$$
$$\bar{a}_6 = x_1x_3x_8 + 4x_2x_1x_9 + 2x_3x_2x_7 + x_7x_9x_5 + 4x_8x_7x_6 + 2x_9x_8x_4 + x_4x_6x_2 + 4x_5x_4x_3 + 2x_6x_5x_1$$
$$\bar{a}_7 = x_1x_3x_4 + 2x_2x_1x_5 + 4x_3x_2x_6 + x_7x_9x_1 + 2x_8x_7x_2 + 4x_9x_8x_3 + x_4x_6x_7 + 2x_5x_4x_8 + 4x_6x_5x_9$$
$$\bar{a}_8 = x_1x_3x_7 + 4x_2x_1x_8 + 2x_3x_2x_9 + x_7x_9x_4 + 4x_8x_7x_5 + 2x_9x_8x_6 + x_4x_6x_1 + 4x_5x_4x_2 + 2x_6x_5x_3$$
$$\bar{a}_9 = x_1x_3x_6 + 2x_2x_1x_4 + 4x_3x_2x_5 + x_7x_9x_3 + 2x_8x_7x_1 + 4x_9x_8x_2 + x_4x_6x_9 + 2x_5x_4x_7 + 4x_6x_5x_8$$
$$\bar{a}_{10} = x_1x_3x_9 + 4x_2x_1x_7 + 2x_3x_2x_8 + x_7x_9x_6 + 4x_8x_7x_4 + 2x_9x_8x_5 + x_4x_6x_3 + 4x_5x_4x_1 + 2x_6x_5x_2$$
$$\bar{a}_{11} = x_1x_4x_9 + x_2x_5x_7 + x_3x_6x_8 + x_7x_1x_6 + x_8x_2x_4 + x_9x_3x_5 + x_4x_7x_3 + x_5x_8x_1 + x_6x_9x_2$$
$$\bar{a}_{12} = x_1x_4x_8 + x_2x_5x_9 + x_3x_6x_7 + x_7x_1x_5 + x_8x_2x_6 + x_9x_3x_4 + x_4x_7x_2 + x_5x_8x_3 + x_6x_9x_1$$

(ii) dimension **6**

$$\bar{e}_1 = x_1x_3x_2x_4x_6x_8 + 2x_2x_1x_3x_5x_4x_9 + 4x_3x_2x_1x_6x_5x_7 + x_4x_6x_5x_7x_9x_2 + 2x_5x_4x_6x_8x_7x_3 + 4x_6x_5x_4x_9x_8x_1 + x_7x_9x_8x_1x_3x_5 + 2x_8x_7x_9x_2x_1x_6 + 4x_9x_8x_7x_3x_2x_4$$
$$\bar{e}_2 = x_1x_3x_2x_5x_7x_9 + 4x_2x_1x_3x_6x_8x_7 + 2x_3x_2x_1x_4x_9x_8 + x_4x_6x_5x_8x_1x_3 + 4x_5x_4x_6x_9x_2x_1 + 2x_6x_5x_4x_7x_3x_2 + x_7x_9x_8x_2x_4x_6 + 4x_8x_7x_9x_3x_5x_4 + 2x_9x_8x_7x_1x_6x_5$$
$$\bar{e}_3 = x_1x_3x_2x_4x_6x_7 + 2x_2x_1x_3x_5x_4x_8 + 4x_3x_2x_1x_6x_5x_9 + x_4x_6x_5x_7x_9x_1$$

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$$
\begin{aligned}
&+2x_5x_4x_6x_8x_7x_2 + 4x_6x_5x_4x_9x_8x_3 + x_7x_9x_8x_1x_3x_4 + 2x_8x_7x_9x_2x_1x_5\\
&+4x_9x_8x_7x_3x_2x_6\\
\overline{e}_4 &= x_1x_3x_2x_4x_7x_9 + 4x_2x_1x_3x_5x_8x_7 + 2x_3x_2x_1x_6x_9x_8 + x_4x_6x_5x_7x_1x_3\\
&+4x_5x_4x_6x_8x_2x_1 + 2x_6x_5x_4x_9x_3x_2 + x_7x_9x_8x_1x_4x_6 + 4x_8x_7x_9x_2x_5x_4\\
&+2x_9x_8x_7x_3x_6x_5\\
\overline{e}_5 &= x_1x_3x_2x_4x_6x_9 + 2x_2x_1x_3x_5x_4x_7 + 4x_3x_2x_1x_6x_5x_8 + x_4x_6x_5x_7x_9x_3\\
&+2x_5x_4x_6x_8x_7x_1 + 4x_6x_5x_4x_9x_8x_2 + x_7x_9x_8x_1x_3x_6 + 2x_8x_7x_9x_2x_1x_4\\
&+4x_9x_8x_7x_3x_2x_5\\
\overline{e}_6 &= x_1x_3x_2x_6x_7x_9 + 4x_2x_1x_3x_4x_8x_7 + 2x_3x_2x_1x_5x_9x_8 + x_4x_6x_5x_9x_1x_3\\
&+4x_5x_4x_6x_7x_2x_1 + 2x_6x_5x_4x_8x_3x_2 + x_7x_9x_8x_3x_4x_6 + 4x_8x_7x_9x_1x_5x_4\\
&+2x_9x_8x_7x_2x_6x_5\\
\overline{e}_7 &= x_1x_2x_4x_6x_7x_9 + x_2x_3x_5x_4x_8x_7 + x_3x_1x_6x_5x_9x_8 + x_4x_5x_7x_9x_1x_3\\
&+x_5x_6x_8x_7x_2x_1 + x_6x_4x_9x_8x_3x_2 + x_7x_8x_1x_3x_4x_6 + x_8x_9x_2x_1x_5x_4\\
&+x_9x_7x_3x_2x_6x_5\\
\overline{e}_8 &= x_1x_2x_4x_5x_7x_9 + x_2x_3x_5x_6x_8x_7 + x_3x_1x_6x_4x_9x_8 + x_4x_5x_7x_8x_1x_3\\
&+x_5x_6x_8x_9x_2x_1 + x_6x_4x_9x_7x_3x_2 + x_7x_8x_1x_2x_4x_6 + x_8x_9x_2x_3x_5x_4\\
&+x_9x_7x_3x_1x_6x_5\\
\overline{e}_9 &= x_1x_2x_3x_7x_8x_9 + x_4x_5x_6x_1x_2x_3 + x_7x_8x_9x_4x_5x_6\\
\overline{e}_{10} &= x_1x_2x_5x_6x_7x_9 + x_2x_3x_6x_4x_8x_7 + x_3x_1x_4x_5x_9x_8\\
\overline{e}_{11} &= x_1x_2x_4x_8x_6x_9 + x_2x_3x_5x_9x_4x_7 + x_3x_1x_6x_7x_5x_8\\
\overline{e}_{12} &= x_1x_3x_4x_6x_7x_9 + x_2x_1x_5x_4x_8x_7 + x_3x_2x_6x_5x_9x_8
\end{aligned}
$$

Next we compute cup products of the generators of $H^*(BG_i, \mathbf{Z}/7)$ in dimension **3**. Table 1 and Table 2 show the cup products in dimension **6**. Each a_j and $\overline{a_j}$ $(j = 1, \ldots, 12)$ is the generator of $H^3(BG_i, \mathbf{Z}/7)$. These cup product structures give the main clue for proving the Proposition 2.1.

With this information, we prove the following proposition.

PROPOSITION 2.1. *$H^*(BP, \mathbf{Z}/7)^{H_1}$ and $H^*(BP, \mathbf{Z}/7)^{H_2}$ are not isomorphic as graded algebras over $\mathbf{Z}/7$.*

PROOF. Suppose $\varphi_* : H^*(BP, \mathbf{Z}/7)^{H_1} \longrightarrow H^*(BP, \mathbf{Z}/7)^{H_2}$ is an isomorphism as graded algebras over $\mathbf{Z}/7$. We consider the following diagram.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ker} f_u & \longrightarrow & H^3(BP, \mathbf{Z}/7)^{H_1} \otimes H^3(BP, \mathbf{Z}/7)^{H_1} & \xrightarrow{f_u} & H^6(BP, \mathbf{Z}/7)^{H_1}\\
 & & \big\downarrow{\scriptstyle \varphi_3\otimes\varphi_3} & & \big\downarrow{\scriptstyle \varphi_3\otimes\varphi_3} & & \big\downarrow{\scriptstyle \varphi_6}\\
0 & \longrightarrow & \operatorname{Ker} g_u & \longrightarrow & H^3(BP, \mathbf{Z}/7)^{H_2} \otimes H^3(BP, \mathbf{Z}/7)^{H_2} & \xrightarrow{g_u} & H^6(BP, \mathbf{Z}/7)^{H_2}
\end{array}
$$

where f_u and g_u are cup product maps and the rows are exact.

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	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_1	0	0	0	0	e_1	e_2	e_3	e_4	e_5	e_6	0	0
a_2	0	0	0	0	0	0	e_3	$6e_4$	$6e_5$	e_6	e_7	e_8
a_3	0	0	0	0	$6e_1$	$6e_2$	e_3	0	0	e_6	$6e_7$	$6e_8$
a_4	0	0	0	0	$6e_1$	$6e_2$	0	e_4	e_5	0	e_7	e_8
a_5	$6e_1$	0	e_1	e_1	0	α_1	0	0	0	0	0	0
a_6	$6e_2$	0	e_2	e_2	$6\alpha_1$	0	0	0	0	0	0	0
a_7	$6e_3$	$6e_3$	$6e_3$	0	0	0	0	0	0	α_2	0	0
a_8	$6e_4$	e_4	0	$6e_4$	0	0	0	0	α_3	0	0	0
a_9	$6e_5$	e_5	0	$6e_5$	0	0	0	$6\alpha_3$	0	0	0	0
a_{10}	$6e_6$	$6e_6$	$6e_6$	0	0	0	$6\alpha_2$	0	0	0	0	0
a_{11}	0	$6e_7$	e_7	$6e_7$	0	0	0	0	0	0	0	α_4
a_{12}	0	$6e_8$	e_8	$6e_8$	0	0	0	0	0	0	$6\alpha_4$	0

$^*\alpha_1 = 3e_9 + 3e_{10} + 3e_{11}$, $\alpha_2 = 3e_9 + 4e_{10} + 3e_{12}$, $\alpha_3 = 4e_9 + 3e_{11} + 3e_{12}$,
$\alpha_4 = 3e_{10} + 4e_{11} + 3e_{12}$

TABLE 1: Cup products in $H^6(BP, \mathbf{Z}/7)^{H_1}$

	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	$\bar{a}_9$	$\bar{a}_{10}$	$\bar{a}_{11}$	$\bar{a}_{12}$
$\bar{a}_1$	0	0	0	0	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$	$\bar{e}_5$	$\bar{e}_6$	0	0
$\bar{a}_2$	0	0	0	0	0	0	$4\bar{e}_3$	$5\bar{e}_4$	$5\bar{e}_5$	$4\bar{e}_6$	$\bar{e}_7$	$\bar{e}_8$
$\bar{a}_3$	0	0	0	0	$5\bar{e}_1$	$5\bar{e}_2$	$\bar{e}_3$	0	0	$\bar{e}_6$	$6\bar{e}_7$	$6\bar{e}_8$
$\bar{a}_4$	0	0	0	0	$3\bar{e}_1$	$3\bar{e}_2$	0	$\bar{e}_4$	$\bar{e}_5$	0	$\bar{e}_7$	$\bar{e}_8$
$\bar{a}_5$	$6\bar{e}_1$	0	$2\bar{e}_1$	$4\bar{e}_1$	0	β_1	$4\bar{e}_4$	$\bar{e}_7$	$6\bar{e}_6$	$3\bar{e}_8$	$5\bar{e}_3$	$5\bar{e}_5$
$\bar{a}_6$	$6\bar{e}_2$	0	$2\bar{e}_2$	$4\bar{e}_2$	$6\beta_1$	0	$3\bar{e}_7$	$6\bar{e}_3$	$\bar{e}_8$	$4\bar{e}_5$	$5\bar{e}_4$	$5\bar{e}_6$
$\bar{a}_7$	$6\bar{e}_3$	$3\bar{e}_3$	$6\bar{e}_3$	0	$3\bar{e}_4$	$4\bar{e}_7$	0	$5\bar{e}_8$	$2\bar{e}_2$	β_2	$3\bar{e}_5$	$3\bar{e}_1$
$\bar{a}_8$	$6\bar{e}_4$	$2\bar{e}_4$	0	$6\bar{e}_4$	$6\bar{e}_7$	$\bar{e}_3$	$2\bar{e}_8$	0	β_3	$5\bar{e}_1$	$6\bar{e}_6$	$6\bar{e}_2$
$\bar{a}_9$	$6\bar{e}_5$	$2\bar{e}_5$	0	$6\bar{e}_5$	$\bar{e}_6$	$6\bar{e}_8$	$5\bar{e}_2$	$6\beta_3$	0	$2\bar{e}_7$	$6\bar{e}_1$	$6\bar{e}_3$
$\bar{a}_{10}$	$6\bar{e}_6$	$3\bar{e}_6$	$6\bar{e}_6$	0	$4\bar{e}_8$	$3\bar{e}_5$	$6\beta_2$	$2\bar{e}_1$	$5\bar{e}_7$	0	$3\bar{e}_2$	$3\bar{e}_4$
$\bar{a}_{11}$	0	$6\bar{e}_7$	$\bar{e}_7$	$6\bar{e}_7$	$2\bar{e}_3$	$2\bar{e}_4$	$4\bar{e}_5$	$\bar{e}_6$	$\bar{e}_1$	$4\bar{e}_2$	0	β_4
$\bar{a}_{12}$	0	$6\bar{e}_8$	$\bar{e}_8$	$6\bar{e}_8$	$2\bar{e}_5$	$2\bar{e}_6$	$4\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$4\bar{e}_4$	$6\beta_4$	0

$^*\beta_1 = 3\bar{e}_9 + 6\bar{e}_{10} + 5\bar{e}_{11}$, $\beta_2 = 6\bar{e}_9 + \bar{e}_{10} + 3\bar{e}_{12}$, $\beta_3 = 2\bar{e}_9 + 5\bar{e}_{11} + 3\bar{e}_{12}$,
$\beta_4 = 3\bar{e}_{10} + 4\bar{e}_{11} + 3\bar{e}_{12}$

TABLE 2: Cup products in $H^6(BP, \mathbf{Z}/7)^{H_2}$

Therefore the diagram commutes, *i.e.* $\varphi_6 \circ f_u = g_u \circ (\varphi_3 \otimes \varphi_3)$. This implies $\varphi_6(a_i a_j) = \varphi_3(a_i)\varphi_3(a_j)$, that is, its algebraic structure is preserved under the map φ_*. Then $\operatorname{Ker} f_u \cong \operatorname{Ker} g_u$. We consider $\operatorname{Ker} f_u = \{\sum n_{ij} a_i \otimes a_j \mid f_u(\sum n_{ij} a_i \otimes a_j) = \sum n_{ij} a_i a_j = 0\}$. We briefly explain how to compute a basis $\bar{X}$ for $\operatorname{Ker} f_u$. By inspection of Table 1, if the cup product is zero, then it is obvious. Otherwise, we consider the elements whose image is a scalar multiple of e_i, $i = 1, \ldots, 9$. For example, in case of e_1, $f_u(n_1 a_1 \otimes a_5 + n_2 a_3 \otimes a_5 + n_3 a_4 \otimes a_5) = n_1 e_1 + 6n_2 e_1 + 6n_3 e_1 = (n_1 + 6n_2 + 6n_3)e_1$. To find basis elements in $\operatorname{Ker} f_u$, we set $(n_1 + 6n_2 + 6n_3)e_1 = 0$. Then $(n_1, n_2, n_3) = (1, 1, 0)$ or $(1, 0, 1)$ over $Z/7$. Therefore we

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can let $a_1 \otimes a_5 + a_3 \otimes a_5$ and $a_1 \otimes a_5 + a_4 \otimes a_6$ belong to $\overline{X}$. Proceeding in a similar manner we determine the following basis.

$$\overline{X} = \{a_1\otimes a_1, a_2\otimes a_2, a_3\otimes a_3, a_4\otimes a_4, a_5\otimes a_5, a_6\otimes a_6, a_7\otimes a_7, a_8\otimes a_8, a_9\otimes a_9, a_{10}\otimes a_{10}, a_{11}\otimes a_{11}, a_{12}\otimes a_{12}, a_1\otimes a_2, a_1\otimes a_3, a_1\otimes a_4, a_1\otimes a_{11}, a_1\otimes a_{12}, a_2\otimes a_3, a_2\otimes a_4, a_2\otimes a_5, a_2\otimes a_6, a_3\otimes a_4, a_3\otimes a_8, a_3\otimes a_9, a_4\otimes a_7, a_4\otimes a_{10}, a_5\otimes a_7, a_5\otimes a_8, a_5\otimes a_9, a_5\otimes a_{10}, a_5\otimes a_{11}, a_5\otimes a_{12}, a_6\otimes a_7, a_6\otimes a_8, a_6\otimes a_9, a_6\otimes a_{10}, a_6\otimes a_{11}, a_6\otimes a_{12}, a_7\otimes a_8, a_7\otimes a_9, a_7\otimes a_{11}, a_7\otimes a_{12}, a_8\otimes a_{10}, a_8\otimes a_{11}, a_8\otimes a_{12}, a_9\otimes a_{10}, a_9\otimes a_{11}, a_9\otimes a_{12}, a_{10}\otimes a_{11}, a_{10}\otimes a_{12}, a_1\otimes a_5+a_3\otimes a_5, a_1\otimes a_5+a_4\otimes a_5, a_1\otimes a_6+a_3\otimes a_6, a_1\otimes a_6+a_4\otimes a_6, a_1\otimes a_7+6(a_3\otimes a_7), a_2\otimes a_7+6(a_3\otimes a_7), a_1\otimes a_3+6(a_4\otimes a_8), a_2\otimes a_8+a_4\otimes a_8, a_1\otimes a_9+6(a_4\otimes a_9), a_2\otimes a_9+a_4\otimes a_9, a_1\otimes a_{10}+6(a_3\otimes a_{10}), a_2\otimes a_{10}+6(a_3\otimes a_{10}), a_2\otimes a_{11}+6(a_4\otimes a_{11}), a_3\otimes a_{11}+a_4\otimes a_{11}, a_2\otimes a_{12}+6(a_4\otimes a_{12}), a_3\otimes a_{12}+a_4\otimes a_{12}\}.$$

Here $|\overline{X}| = 66$. Thus the dimension of $\operatorname{Ker} f_u$ is 66.

Next we consider $\operatorname{Ker} g_u = \{\sum n_{ij}\bar{a}_i \otimes \bar{a}_j \mid g_u(\sum n_{ij}\bar{a}_i \otimes \bar{a}_j) = \sum n_{ij}\bar{a}_i\bar{a}_j = 0\}$. We use the same method as $\overline{X}$ to compute a basis $\overline{Y}$ for $\operatorname{Ker} g_u$. Thus by inspection of Table 2, $\overline{Y}$ consists of the following elements.

$$\overline{Y} = \{\bar{a}_1\otimes\bar{a}_1, \bar{a}_2\otimes\bar{a}_2, \bar{a}_3\otimes\bar{a}_3, \bar{a}_4\otimes\bar{a}_4, \bar{a}_5\otimes\bar{a}_5, \bar{a}_6\otimes\bar{a}_6, \bar{a}_7\otimes\bar{a}_7, \bar{a}_8\otimes\bar{a}_8, \bar{a}_9\otimes\bar{a}_9, \bar{a}_{10}\otimes\bar{a}_{10}, \bar{a}_{11}\otimes\bar{a}_{11}, \bar{a}_{12}\otimes\bar{a}_{12}, \bar{a}_1\otimes\bar{a}_2, \bar{a}_1\otimes\bar{a}_3, \bar{a}_1\otimes\bar{a}_4, \bar{a}_1\otimes\bar{a}_{11}, \bar{a}_1\otimes\bar{a}_{12}, \bar{a}_2\otimes\bar{a}_3, \bar{a}_2\otimes\bar{a}_4, \bar{a}_2\otimes\bar{a}_5, \bar{a}_2\otimes\bar{a}_6, \bar{a}_3\otimes\bar{a}_4, \bar{a}_3\otimes\bar{a}_8, \bar{a}_3\otimes\bar{a}_9, \bar{a}_4\otimes\bar{a}_7, \bar{a}_4\otimes\bar{a}_{10}, \bar{a}_1\otimes\bar{a}_5+\bar{a}_9\otimes\bar{a}_{11}, \bar{a}_3\otimes\bar{a}_5+5(\bar{a}_9\otimes\bar{a}_{11}), \bar{a}_4\otimes\bar{a}_5+3(\bar{a}_9\otimes\bar{a}_{11}), \bar{a}_7\otimes\bar{a}_{12}+3(\bar{a}_9\otimes\bar{a}_{11}), \bar{a}_8\otimes\bar{a}_{10}+5(\bar{a}_9\otimes\bar{a}_{11}), \bar{a}_1\otimes\bar{a}_6+2(\bar{a}_{10}\otimes\bar{a}_{11}), \bar{a}_3\otimes\bar{a}_6+3(\bar{a}_{10}\otimes\bar{a}_{11}), \bar{a}_4\otimes\bar{a}_6+6(\bar{a}_{10}\otimes\bar{a}_{11}), \bar{a}_7\otimes\bar{a}_9+4(\bar{a}_{10}\otimes\bar{a}_{11}), \bar{a}_8\otimes\bar{a}_{12}+5(\bar{a}_{10}\otimes\bar{a}_{11}), \bar{a}_1\otimes\bar{a}_7+\bar{a}_9\otimes\bar{a}_{12}, \bar{a}_2\otimes\bar{a}_7+4(\bar{a}_9\otimes\bar{a}_{12}), \bar{a}_3\otimes\bar{a}_7+\bar{a}_9\otimes\bar{a}_{12}, \bar{a}_5\otimes\bar{a}_{11}+5(\bar{a}_9\otimes\bar{a}_{12}), \bar{a}_6\otimes\bar{a}_8+6(\bar{a}_9\otimes\bar{a}_{12}), \bar{a}_1\otimes\bar{a}_8+2(\bar{a}_{10}\otimes\bar{a}_{12}), \bar{a}_2\otimes\bar{a}_8+3(\bar{a}_{10}\otimes\bar{a}_{12}), \bar{a}_4\otimes\bar{a}_8+2(\bar{a}_{10}\otimes\bar{a}_{12}), \bar{a}_5\otimes\bar{a}_7+\bar{a}_{10}\otimes\bar{a}_{12}, \bar{a}_6\otimes\bar{a}_{11}+3(\bar{a}_{10}\otimes\bar{a}_{12}), \bar{a}_1\otimes\bar{a}_9+2(\bar{a}_7\otimes\bar{a}_{11}), \bar{a}_2\otimes\bar{a}_9+3(\bar{a}_7\otimes\bar{a}_{11}), \bar{a}_4\otimes\bar{a}_9+2(\bar{a}_7\otimes\bar{a}_{11}), \bar{a}_5\otimes\bar{a}_{12}+3(\bar{a}_7\otimes\bar{a}_{11}), \bar{a}_6\otimes\bar{a}_{10}+\bar{a}_7\otimes\bar{a}_{11}, \bar{a}_1\otimes\bar{a}_{10}+\bar{a}_8\otimes\bar{a}_{11}, \bar{a}_2\otimes\bar{a}_{10}+4(\bar{a}_8\otimes\bar{a}_{11}), \bar{a}_3\otimes\bar{a}_{10}+\bar{a}_8\otimes\bar{a}_{11}, \bar{a}_5\otimes\bar{a}_9+6(\bar{a}_8\otimes\bar{a}_{11}), \bar{a}_6\otimes\bar{a}_{12}+5(\bar{a}_8\otimes\bar{a}_{11}), \bar{a}_2\otimes\bar{a}_{11}+3(\bar{a}_9\otimes\bar{a}_{10}), \bar{a}_3\otimes\bar{a}_{11}+4(\bar{a}_9\otimes\bar{a}_{10}), \bar{a}_4\otimes\bar{a}_{11}+3(\bar{a}_9\otimes\bar{a}_{10}), \bar{a}_5\otimes\bar{a}_8+3(\bar{a}_9\otimes\bar{a}_{10}), \bar{a}_6\otimes\bar{a}_7+2(\bar{a}_9\otimes\bar{a}_{10}), \bar{a}_2\otimes\bar{a}_{12}+4(\bar{a}_7\otimes\bar{a}_8), \bar{a}_3\otimes\bar{a}_{12}+3(\bar{a}_7\otimes\bar{a}_8), \bar{a}_4\otimes\bar{a}_{12}+4(\bar{a}_7\otimes\bar{a}_8), \bar{a}_5\otimes\bar{a}_{10}+5(\bar{a}_7\otimes\bar{a}_8), \bar{a}_6\otimes\bar{a}_9+4(\bar{a}_7\otimes\bar{a}_8), 5(\bar{a}_5\otimes\bar{a}_6)+(\bar{a}_7\otimes\bar{a}_{10})+6(\bar{a}_{11}\otimes\bar{a}_{12}), 4(\bar{a}_5\otimes\bar{a}_6)+(\bar{a}_8\otimes\bar{a}_9)+6(\bar{a}_{11}\otimes\bar{a}_{12})\}.$$

Here $|\overline{Y}| = 68$. Thus the dimension of $\operatorname{Ker} g_u$ is 68.

Since $\operatorname{Ker} f_u$ and $\operatorname{Ker} g_u$ have different dimensions, $\operatorname{Ker} f_u$ is not isomorphic to $\operatorname{Ker} g_u$. Thus our assumption leads to a contradiction. Therefore $\varphi_6(a_ia_j) \neq \varphi_3(a_i)\varphi_3(a_j)$. This means the algebraic structure is not preserved under the map φ_*. This completes the proof. ∎

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