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On the boundedness operator

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This paper is a continuation of the study of the boundedness operator δ. By determination of the congruences (that is, collapsings) of the smallest lattice containing δ and closed under application of δ, a new classification of all topological spaces is obtained according to boundedness criteria.

Recently [2, 3], I defined and studied the boundedness operator $\delta : 2^{2^X} \to 2^{2^X}$ where $X \neq \emptyset$ and for each $B \subset 2^X$,

$$\delta B = \{S \subset X : H \subset B,\ H \cup \{S\} \text{ has finite intersection property implies } \cap H \neq \emptyset\} .$$

A subset of a topological space is called bounded [2, 3] if it is contained in some finite union of members of every open cover of the whole space. Let γB be the family of all intersections of (non-empty) families of finite unions of members of B [1, 5], δ^ν is defined inductively by $\delta^\nu = \delta\left(\delta^{\nu-1}\right)$, $\left(\gamma^2 = \gamma\right)$, and multiplication of δ and γ by functional composition.

Then $\delta\gamma B$ is the family of the bounded subsets of the space (X, T) where B is a subbasis for the family T^c of the closed sets of it [3, 2]. The study of δ by analogy to the study of the compactness operator ρ of de Groot, Herrlich, Strecker, Wattel [1, 4, 5] provided [2, 3] the relations

(1) $H \subset B \Rightarrow \delta B \subset \delta H$;

(2) $\delta\gamma = \delta$;

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(3) $H \subset \gamma B \cap \delta B$, H has finite intersection property implies $\cap H \neq \emptyset$;

(4) $\gamma B \wedge \delta^{\nu} B \subset \delta^{\nu} B$ where $H \wedge B = \{H \cap B : H \in H,\ B \in B\}$;

(5) $\delta^{\nu} \wedge \delta^{\nu+1} = \delta^{\nu} \cap \delta^{\nu+1}$;

(6) $\delta B = \delta\gamma\left(B \cup \delta^2 B\right)$

(7) $\delta \subset \delta^3$;

(8) $\delta^2 = \delta^4$;

(9) $\delta^3 \subset \delta \cup \delta^2 \Rightarrow \delta = \delta^3$;

(10) $\gamma\delta = \delta$, $A \subset B \in \delta B \Rightarrow A \in \delta B$.

The last two relations give to δ "nicer" properties than those of ρ . Relations of the above type are interesting since a complete classification of all topological spaces according to boundedness criteria has been obtained [2, 3] by the determination of the congruences of the resulting monoid $\{\gamma, \delta, \delta^2, \delta^3\}$.

Another classification according to boundedness criteria is obtained here by means of the smallest lattice L (with order induced by containment) containing δ and closed under application of δ to the members of L . The ideas and techniques are inspired by [1]. We have

(11) $\delta \cap \delta^2 = \delta^2 \cap \delta^3$;

(12) $\delta \cap \delta^2 = \delta\left(\delta \cup \delta^2\right) = \delta\left(\delta^2 \cup \delta^3\right)$;

(13) $\delta^2 \cup \delta^3 \subset \delta\left(\delta \cap \delta^2\right)$;

(14) $\delta^2\left(\delta \cap \delta^2\right) = \delta \cap \delta^2$.

Proof of (11). By (7), $\delta \cap \delta^2 \subset \delta^2 \cap \delta^3 \subset \delta^2$. Assume that $S \in \delta^2 B \cap \delta^3 B$, $\emptyset \neq H \subset B$, $H \cup \{S\}$ has finite intersection property. Then, by (4), $H^* = H \wedge \{S\} \subset \delta^2 B \cap \delta^3 B = \gamma\left(\delta^2 B\right) \cap \delta\left(\delta^2 B\right)$ and H^* has finite intersection property. It follows by (3) that $\emptyset \neq \cap H$ and thus, $S \in \delta B$; that is, $\delta^2 \cap \delta^3 \subset \delta$.

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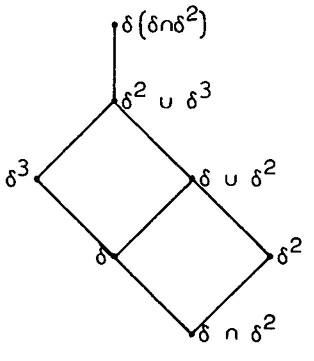
Proof of (12). By (1), $\delta(\delta^2 \cup \delta^3) \subset \delta(\delta \cup \delta^2) \subset \delta^2 \cap \delta^3$. Let $S \in \delta^2 B \cap \delta^3 B$, $\emptyset \neq H \subset \delta^2 B \cup \delta^3 B$, $H \cup \{S\}$ have finite intersection property. Then, as before, $H^* = H \wedge \{S\} \subset \delta^2 B \cap \delta^3 B$ and thus $S \in \delta(\delta^2 B \cup \delta^3 B)$.

Proof of (13). $\delta \supset \delta \cap \delta^2 \subset \delta^2$ implies by (1) that $\delta^2 \subset \delta(\delta \cap \delta^2) \supset \delta^3$.

Proof of (14). By (1), (13), and (12), we have $\delta(\delta(\delta \cap \delta^2)) \subset \delta \cap \delta^2$. Conversely, by (7) and (12), $\delta \cap \delta^2 = \delta(\delta \cup \delta^2) \subset \delta^2(\delta(\delta \cup \delta^2)) = \delta^2(\delta \cap \delta^2)$.

The above method of proving the converse of (14) may serve also to obtain a clearer proof of [1, (15)].

By the above relations we get the following lattice L:



L contains δ and is closed under application of δ. By Example 1 below, the seven elements of the lattice are in general, distinct. In order to determine all the possible congruence relations (that is, collapsings) of the lattice, the following will be useful.

(15) (i) $\delta^2 \subset \delta^3$ or $\delta^3 \subset \delta^2$ if and only if

(ii) $\delta \subset \delta^2$ or $\delta^2 \subset \delta$ if and only if

(iii) $\delta(\delta \cap \delta^2) = \delta^2 \cup \delta^3$.

Proof. (i) $\Rightarrow$ (ii). If $\delta^2 \subset \delta^3$ then, by (11),

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$\delta^2 = \delta^2 \cap \delta^3 = \delta \cap \delta^2 \subset \delta$ and if $\delta^3 \subset \delta^2$ then, by (7), $\delta \subset \delta^2$.

(ii) ⇒ (iii). If $\delta \subset \delta^2$ then, by (1), $\delta^3 \subset \delta^2$ and thus $\delta(\delta \cap \delta^2) = \delta^2 = \delta^2 \cup \delta^3$. If $\delta^2 \subset \delta$ then, by (7), $\delta^2 \subset \delta^3$ and thus $\delta(\delta \cap \delta^2) = \delta^3 = \delta^2 \cup \delta^3$.

(iii) ⇒ (i). If there exist $A \in \delta^2\mathcal{B} - \delta^3\mathcal{B}$ and $B \in \delta^3\mathcal{B} - \delta^2\mathcal{B}$ then, by (10), $\delta^2\mathcal{B} \not\ni A \cup B \notin \delta^3\mathcal{B}$. Nevertheless, by (13), $A \in \delta(\delta \cap \delta^2) \ni B$ and by (10), $A \cup B \in \delta(\delta \cap \delta^2) - \delta^2 \cup \delta^3 \neq \emptyset$.

Now, in order to obtain a complete classification of all topological spaces, let $\mathcal{Q}$ denote the class of spaces for which a certain collapsing of L occurs.

If $\delta \not\subset \delta^2$ and $\delta^2 \not\subset \delta$ then, by (15), (9), the space belongs either to the class $\mathcal{Q}_1$: no collapse, or to the class

$$\mathcal{Q}_2 : \delta(\delta \cap \delta^2) \supset \delta^2 \cup \delta^3 = \delta \cup \delta^2 \supset \delta^3 = \delta \supset \delta \cap \delta^2 \subset \delta^2 .$$

If $\delta \subset \delta^2$ then, by (1), (9), $\delta = \delta^3$ and thus, by (15), the space belongs to $\mathcal{Q}_3 : L = \{\delta\}$ or to

$$\mathcal{Q}_4 : \delta = \delta^3 = \delta \cap \delta^2 \subset \delta^2 = \delta \cup \delta^2 = \delta^2 \cup \delta^3 = \delta(\delta \cap \delta^2) .$$

Finally, if $\delta^2 \subset \delta$ then, by (7), (15), the space belongs to $\mathcal{Q}_3$ or to $\mathcal{Q}_5 : \delta^2 = \delta \cap \delta^2 \subset \delta = \delta \cup \delta^2 = \delta^3 = \delta^2 \cup \delta^3 = \delta(\delta \cap \delta^2)$ or to $\mathcal{Q}_6 : \delta^2 = \delta \cap \delta^2 \subset \delta = \delta \cup \delta^2 \subset \delta^3 = \delta^2 \cup \delta^3 = \delta(\delta \cap \delta^2)$.

THEOREM. *Every topological space belongs to one of the non-empty disjoint classes* $\mathcal{Q}_i$, $1 \leq i \leq 6$.

Proof. By the above mentioned argument, the classes $\mathcal{Q}_i$, $1 \leq i \leq 6$, exhaust all possibilities for spaces and they are by construction disjoint. Finally, they are also non-empty because of the following:

EXAMPLE 1. Let (X_1, T_1) be an infinite topological space such that

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the family of its bounded subsets δT_1^c equals the family of the countable subsets of X_1 (for example, this happens in the space Ω_0 of the countable ordinals with the usual order topology [2, Example 5.1.V]). Let also (X_2, T_2) be an infinite discrete space and $(X_1 \cup X_2, T)$ their disjoint topological union. By Theorem 2.4 [2],

$$\delta T^c = \{S \subset X_1 \cup X_2 : S \cap X_1 \text{ countable and } S \cap X_2 \text{ finite}\} .$$

Then it is proved that $\delta^2 T^c = \{M \subset X_1 \cup X_2 : M \cap X_1 \text{ finite}\}$ and $\delta^3 T^c = \{N \subset X_1 \cup X_2 : N \cap X_2 \text{ finite}\}$. It follows that for the space (X, T) , $\delta \not\subset \delta^2$, and $\delta^2 \not\subset \delta$;

(I) if X_1 is uncountable then (X, T) is a Q_1-space;

(II) if X_1 is countable then (X, T) is a Q_2-space;

(III) every finite space is a Q_3-space;

(IV) every infinite discrete space is a Q_4-space;

(V) every infinite compact space is a Q_5-space;

(VI) Ω_0 is a Q_6-space.

With respect to the S-classification of [2, 3] we get $Q_1 \cup Q_6 = S_5$, $Q_3 = S_1 \cup S_2$, $Q_2 \cup Q_4 \cup Q_5 = S_3 \cup S_4$.

Using the following relation (16) one can obtain another proof of (6) by modification of methods given in [5].

(16) $\delta(H \cup B) = \delta H \cap \delta B \cap \delta(H \wedge B)$.

Proof. By (1), $H \subset H \cup B \supset B$ and $H \wedge B \subset \gamma(H \cup B)$ imply that $\delta(H \cup B) \subset \delta H \cap \delta B \cap \delta(H \wedge B)$. Conversely, let $S \in \delta H \cap \delta B \cap \delta(H \wedge B)$, $\emptyset \neq \mathcal{D} \subset H \cup B$, and $\mathcal{D} \cup \{S\}$ have the finite intersection property. If $\mathcal{D}_1 = \mathcal{D} \cap H = \emptyset$ or $\mathcal{D}_2 = \mathcal{D} \cap B = \emptyset$ then $\cap\mathcal{D} \neq \emptyset$. If $\mathcal{D}_1 \neq \emptyset \neq \mathcal{D}_2$ then $\emptyset \neq \mathcal{D}^* = \mathcal{D}_1 \wedge \mathcal{D}_2 \subset H \wedge B$ and $\mathcal{D}^* \cup \{S\}$

has finite intersection property. Therefore $\emptyset \neq \bigcap\mathcal{D}^* = \bigcap\mathcal{D}$. Since in each case $\emptyset \neq \bigcap\mathcal{D}$, we conclude that $S \in \delta(H \cup B)$.

Proof of (6). By (2), (16),

$$\delta\gamma(B \cup \delta^2 B) = \delta(B \cup \delta^2 B) = \delta B \cap \delta^3 B \cap \delta(B \wedge \delta^2 B) ,$$

and by (7), (1), (4), $\delta B \subset \delta^3 B \subset \delta(B \wedge \delta^2 B)$.

Despite the similarity of the above result to that of [1] there exists no connection between them in that neither constitutes a generalization of the other. However, in the light of (10), some of the following open problems may be easier than the corresponding ones for ρ [1, 5].

(a) Given an operator $f : 2^{2^X} \to 2^{2^X}$ *, under what conditions does it coincide with the boundedness operator* δ *?* (Boundedness axiomatization problem.)

(b) Determine whether or not $\{H \subset 2^X : \delta H = B\}$ *may be empty, as well as conditions to guarantee the existence of a largest* H *(or* H *of maximal families).*

(c) The corresponding problems for δ^2 .

(d) Determine the possible relations between δ *and* ρ *(some are evident).*

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