Automorphic vector bundles on the stack of $G$-zips

Naoki Imai\textsuperscript{1} and Jean-Stefan Koskivirta\textsuperscript{2}

\textsuperscript{1}Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan; E-mail: naoki@ms.u-tokyo.ac.jp.
\textsuperscript{2}Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama City, Saitama 338-8570, Japan; E-mail: jeanstefan.koskivirta@gmail.com.

Received: 9 September 2020; Accepted: 1 March 2021

2020 Mathematics Subject Classification: Primary – 14G35; Secondary – 20G40

Abstract

For a connected reductive group $G$ over a finite field, we study automorphic vector bundles on the stack of $G$-zips. In particular, we give a formula in the general case for the space of global sections of an automorphic vector bundle in terms of the Brylinski-Kostant filtration. Moreover, we give an equivalence of categories between the category of automorphic vector bundles on the stack of $G$-zips and a category of admissible modules with actions of a 0-dimensional algebraic subgroup a Levi subgroup and monodromy operators.

1. Introduction

The stack of $G$-zips was introduced by Pink-Wedhorn-Ziegler [PWZ11] and [PWZ15] based on the notion of F-zip defined in the work of Moonen-Wedhorn ([MW04]). In this article, we investigate vector bundles on the stack of $G$-zips. Let $G$ be a connected reductive group over a finite field $\mathbb{F}_q$ and let $k$ denote an algebraic closure of $\mathbb{F}_q$. For a cocharacter $\mu: \mathbb{G}_a \to G_k$, Pink-Wedhorn-Ziegler have defined a smooth finite stack $G$-$\text{Zip}^\mu$ over $k$, called the stack of $G$-zips of type $\mu$. Many authors have shown that it is a useful tool to study the geometry of Shimura varieties in characteristic $p$. For example, let $\text{Sh}(G, X)_K$ be a Shimura variety of Hodge type over a number field $E$ with good reduction at a prime $p$. Kisin [Kis10] and Vasiu [Vas99] have constructed an integral model $S_K$ over $\mathcal{O}_E$, at all places $v | p$ in $E$. Denote by $S_K$ the geometric special fibre of $S_K$ and by $G$ the special fibre over $\mathbb{F}_p$ of $G$ (in the context of Shimura varieties, we take $q = p$). Let $\mu$ be the cocharacter attached naturally to $X$. Then Zhang [Zha18] has shown that there exists a smooth morphism of stacks $\zeta: S_K \to G$-$\text{Zip}^\mu$, which is also surjective. The second author and Wedhorn have used the stack $G$-$\text{Zip}^\mu$ to construct $\mu$-ordinary Hasse invariants in [KW18], and this result was later generalised to all Ekedahl-Oort strata with Goldring [GK19a].

In [Kos19], the second author studied the space of global sections of the family of vector bundles $(V_\ell(\lambda))_{\lambda \in X^*(T)}$. To explain what these vector bundles are, first recall that the cocharacter $\mu$ yields a parabolic subgroup $P \subset G$ as well as a Levi subgroup $L \subset P$, which is equal to the centraliser of $\mu$ (see Subsection 2.2.2 for details). Then for any algebraic $P$-representation $(V, \rho)$ over $k$, there is a naturally attached vector bundle $\mathcal{V}(\rho)$ of rank $\dim(V)$ on $G$-$\text{Zip}^\mu$ modelled on $(V, \rho)$ (see Subsection 2.4). We call $\mathcal{V}(\rho)$ an automorphic vector bundle on $G$-$\text{Zip}^\mu$ (cf. [Mil90, Chapter III, §2]).

The vector bundle $V_\ell(\lambda)$ (for $\lambda \in X^*(T)$ a character of a maximal torus $T \subset G$) is by definition the vector bundle attached to the $P$-representation $V_\ell(\lambda) = \text{Ind}_B^P(\lambda)$, where $B \subset P$ is a Borel subgroup (containing $T$ and appropriately chosen), $\text{Ind}$ denotes induction and $I$ denotes the set of simple roots.
of \( L \). For a \( k \)-algebraic group \( H \), we write \( \text{Rep}(H) \) for the category of finite-dimensional algebraic representations of \( H \) over \( k \). The natural projection \( P \to L \) modulo the unipotent radical induces a fully faithful functor \( \text{Rep}(L) \to \text{Rep}(P) \). In particular, all representations of the form \( V_I(\lambda) \) lie in the full subcategory \( \text{Rep}(L) \). In the case when \( G \) is split over \( \mathbb{F}_p \), we showed in a previous work [Kos19, Theorem 1] that \( H^0(G-\text{Zip}^\mu, V_I(\lambda)) \) can be expressed as

\[
H^0(G-\text{Zip}^\mu, V_I(\lambda)) = V_I(\lambda)^{L(\mathbb{F}_p)} \cap V_I(\lambda)_{\leq 0},
\]

where \( V_I(\lambda)^{L(\mathbb{F}_p)} \) denotes the \( L(\mathbb{F}_p) \)-invariant subspace of \( V_I(\lambda) \) and \( V_I(\lambda)_{\leq 0} \subset V_I(\lambda) \) is defined as follows: It is the direct sum of the \( T \)-weight spaces \( V_I(\lambda)_\nu \) for the weights \( \nu \) satisfying \( \langle \nu, \alpha^\vee \rangle \leq 0 \) for any simple root \( \alpha \) outside of \( L \).

In this article, we vastly generalise the formula (1.0.1) to the most general case. We do not assume that \( G \) is split over \( \mathbb{F}_q \) and, more important, we consider arbitrary representations in the larger category \( \text{Rep}(P) \) as opposed to the subcategory \( \text{Rep}(L) \). In the context of Shimura varieties, there are many interesting vector bundles other than the family \( (V_I(\lambda))_I \), which may not always arise from representations in \( \text{Rep}(L) \). For example, in [Urb14], nearly holomorphic modular forms of weight \( k \) and order \( \leq r \) are defined as sections of the vector bundle \( \omega^{(k-r)} \otimes \text{Sym}^r(\mathcal{T}^1_{\text{dr}}) \) on the modular curve \( X(N) \) for some level \( N \geq 1 \). Here, \( \mathcal{T}^1_{\text{dr}} \) is the sheaf of relative de Rham cohomology of the universal elliptic curve \( \mathcal{E} \to X(N) \), and \( 0 < \omega < \mathcal{T}^1_{\text{dr}} \) is the usual Hodge filtration. In this context, the group \( G \) is \( GL_2 \), \( P = B \) is a Borel subgroup of \( G \). The vector bundle \( \mathcal{T}^1_{\text{dr}} \) is attached to the dual of the standard representation of \( GL_2 \) (viewed by restriction as a representation of \( P \)). Similarly, \( \text{Sym}^r(\mathcal{T}^1_{\text{dr}}) \) is attached to the \( r \)th symmetric power of that representation. More generally, on the Siegel-type Shimura variety \( \mathcal{A}_g \) (which parametrises principally polarised abelian varieties of rank \( g \)), the universal abelian scheme yields a rank \( 2g \) vector bundle \( \mathcal{T}^1_{\text{dr}} \) on \( \mathcal{A}_g \). One can extend the definition of \( \mathcal{T}^1_{\text{dr}} \) to Hodge-type Shimura varieties after choosing a Siegel embedding. Furthermore, it extends to a vector bundle on the integral model \( \mathcal{S}_K \) of Kisin and Vasiu. This example shows that it is desirable to also understand vector bundles that arise from general representations of \( P \). In this article, we determine the space \( H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) \) for any cocharacter datum \( (G, \mu) \) (for the definition of cocharacter datum, see Subsection 2.2.2) and for any representation \( (V, \rho) \in \text{Rep}(P) \). By Zhang’s smooth surjective map \( \xi : S_K \to G-\text{Zip}^\mu \), this determines a natural Hecke-equivariant subspace

\[
H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) \xrightarrow{\xi} H^0(S_K, \mathcal{V}(\rho)).
\]

In particular, we obtain Hecke-equivariant sections of \( \mathcal{V}(\rho) \) on \( S_K \). Furthermore, we can potentially study sections on Ekedahl-Oort strata by the same method, as demonstrated in [GK19a]. Another motivation for describing sections on \( G-\text{Zip}^\mu \) is that we would like to determine which weights \( \lambda \) admit nonzero automorphic forms. Specifically, let \( C_K \) denote the set of \( \lambda \in X^*(T) \) such that \( H^0(S_K, \mathcal{V}_I(\lambda)) \neq 0 \). Similarly, let \( C_{\text{zip}} \) be the set of \( \lambda \) such that \( H^0(G-\text{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0 \) (one can show that they are cones in \( X^*(T) \)). The inclusion (1.0.2) shows that \( C_{\text{zip}} \subset C_K \). Denote by \( (\cdot)_{\geq 0} \) the generated \( \mathbb{Q}_{\geq 0} \)-cones. Then one can see [Kos19, Corollary 1.5.3] that \( C_K(\mathbb{Q}_{\geq 0}) \) is independent of \( K \), and we conjecture [GK18, Conjecture 2.1.6] that it coincides with \( C_{\text{zip}}(\mathbb{Q}_{\geq 0}) \). Goldring and the second author proved this conjecture in some case in [GK18, Theorem D].

We show that the space \( H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) \) is given by the intersection of the \( L_{\varphi} \)-invariants of \( V \) with a generalised Brylinski-Kostant filtration (where \( L_{\varphi} \subset L \) is a certain 0-dimensional group, see (3.2.1)). For the general statement, see Theorem 3.4.1. For the sake of brevity, we give a simplified statement in this introduction. Assume here that \( P \) is defined over \( \mathbb{F}_q \) (in this case, \( L_{\varphi} = L(\mathbb{F}_q) \)). Let \( \varphi : X^*(T)_{\mathbb{R}} \to X^*(T)_{\mathbb{R}} \) be the map induced by the Lang torsor \( \varphi : T \to T ; \; g \mapsto g\varphi(g)^{-1} \), where \( \varphi : G \to G \) denotes the \( q \)th power Frobenius homomorphism. Let \( V = \bigoplus V_\nu \) be the weight decomposition of \( V \). For \( \chi \in X^*(T)_{\mathbb{R}} \), let \( \text{Fil}_\chi^p V_\nu \) be the Brylinski-Kostant filtration of \( V_\nu \) (see (3.4.2)).
Theorem 1 (Corollary 3.4.2). Assume that \( P \) is defined over \( \mathbb{F}_q \). For any \((V, \rho) \in \text{Rep}(P)\), we have
\[
H^0(G, \text{Zip}^\mu, \mathbb{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^+(T)} \text{Fil}^P_{\varphi^r(\nu)} V_\nu.
\]

In the more simple case of [Kos19], the space \( V^r(\lambda \leq 0) \) appearing in equation (1.0.1) is a sum of weight spaces of \( V \). In the general case, \( H^0(G, \text{Zip}^\mu, \mathbb{V}(\rho)) \) cannot be written as an intersection of \( V^{L(\mathbb{F}_q)} \) with a sum of weight spaces of \( V \) (see Examples 4.3.2 for a counterexample). We include examples of concrete computations of the space \( H^0(G, \text{Zip}^\mu, \mathbb{V}(\rho)) \) in Section 6.

Our second result concerns the category \( \mathcal{V}(G, \text{Zip}^\mu) \) of vector bundles on \( G, \text{Zip}^\mu \). As explained above, there is a natural functor \( \mathcal{V} : \text{Rep}(P) \to \mathcal{V}(G, \text{Zip}^\mu) \). Denote by \( \mathbb{V}_P(G, \text{Zip}^\mu) \) the full subcategory that is equal to the essential image of \( \mathcal{V} \). We give an explicit description of the category \( \mathbb{V}_P(G, \text{Zip}^\mu) \) of automorphic vector bundles. We define the category of \( L_\varphi \)-modules with \( \Delta^P \)-monodromy (see Definition 5.2.2). Its objects are \( L_\varphi \)-modules \( W \) endowed with a set of monodromy operators indexed by \( \Delta^P \) (where \( \Delta^P \) denotes the set of simple roots outside the parabolic \( P \)). There is a natural functor \( F_{MN} : \text{Rep}(P) \to L_\varphi^* \text{-MN}_{\Delta^P} \) (see (5.2.1)). An \( L_\varphi \)-module with \( \Delta^P \)-monodromy is called admissible if it lies in the essential image of \( F_{MN} \). The category of admissible \( L_\varphi \)-modules \( \Delta^P \)-monodromy is denoted by \( L_\varphi^* \text{-MN}_{\Delta^P} \).

Theorem 2 (Theorem 5.1.5). The functor \( \mathcal{V} : \text{Rep}(P) \to \mathcal{V}(G, \text{Zip}^\mu) \) factors through the functor \( F_{MN} : \text{Rep}(P) \to \text{MN}_{\Delta^P} \) and induces an equivalence of categories
\[
L_\varphi^* \text{-MN}_{\Delta^P} \to \mathcal{V}(G, \text{Zip}^\mu).
\]

In particular, we deduce the following. Let \( S_K \) denote again the good reduction special fibre of a Hodge-type Shimura variety. Similarly, there is a natural functor \( \text{Rep}(P) \to \mathcal{V}(S_K) \), where \( \mathcal{V}(S_K) \) denotes the category of vector bundles on \( S_K \). Write again \( \mathcal{V}_P(S_K) \) for the essential image of \( \text{Rep}(P) \). In this context, we have the following.

Corollary 3 (Corollary 5.1.6). The functor \( \mathcal{V} : \text{Rep}(P) \to \mathcal{V}_P(S_K) \) factors as
\[
\begin{array}{ccc}
\text{Rep}(P) & \xrightarrow{F_{MN}} & L_\varphi^* \text{-MN}_{\Delta^P} \\
& \xrightarrow{\zeta^*} & \mathcal{V}_P(S_K).
\end{array}
\]

The results of this article will be used in the follow-up articles [IK21] and [GIK21], where we study partial Hasse invariants for Shimura varieties of Hodge type.

2. Vector bundles on the stack of \( G \)-zips

2.1. Notation

Throughout the article, \( p \) is a prime number, \( q \) is a power of \( p \) and \( \mathbb{F}_q \) is the finite field with \( q \) elements. We write \( k = \overline{\mathbb{F}_q} \) for an algebraic closure of \( \mathbb{F}_q \). Write \( \sigma \in \text{Gal}(k/\mathbb{F}_q) \) for the \( q \)th power Frobenius. For a \( k \)-scheme \( X \) and \( m \in \mathbb{Z} \), we write \( X^{(q^m)} \) for the base change of \( X \) by \( \sigma^m \) and \( \varphi : X^{(q^m)} \to X^{(q^{m+1})} \) for the relative \( q \)th power Frobenius morphism. For an algebraic representation \( (V, \rho) \) of an algebraic group \( H \) over \( k \), let \( (V^{(q)}, \rho^{(q)}) \) denote the representation \( \rho \circ \varphi : H^{(q^{-1})} \to H \to \text{GL}(V) \).

The notation \( G \) will denote a connected reductive group over \( \overline{\mathbb{F}_q} \). We will always write \( (B, \rho) \) for a Borel pair defined over \( \overline{\mathbb{F}_q} \); that is, \( T \subset B \subset G_k \) are a maximal torus and a Borel subgroup defined over \( \overline{\mathbb{F}_q} \). Let \( B^+ \) be the Borel subgroup of \( G_k \) opposite to \( B \) with respect to \( T \) (i.e., the unique Borel subgroup of \( G \) such that \( B^+ \cap B = T \)). We will use the following notations:

- As usual, \( X^*(T) \) (respectively \( X_*(T) \)) denotes the group of characters (respectively cocharacters) of \( T \). The group \( \text{Gal}(k/\overline{\mathbb{F}_q}) \) acts naturally on these groups. Let \( W = W(G_k, T) \) be the Weyl group of

Downloaded from https://www.cambridge.org/core; IP address: 54.70.40.11, on 08 Aug 2021 at 15:38:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/fms.2021.32
\[ G_k. \] Similarly, \( \text{Gal}(k/\mathbb{F}_q) \) acts on \( W \). Furthermore, the actions of \( \text{Gal}(k/\mathbb{F}_q) \) and \( W \) on \( X^*(T) \) and \( X^+_\nu(T) \) are compatible in a natural sense.

- \( \Phi \subset X^*(T) \) the set of \( T \)-roots of \( G \).
- \( \Phi_+ \subset \Phi \) the system of positive roots with respect to \( B^+ \) (i.e., \( \alpha \in \Phi_+ \) when the \( \alpha \)-root group \( U_\alpha \) is contained in \( B^+ \)). This convention may differ from other authors. We use it to match the conventions of [Jan03, Chapter II, §1.8] and previous publications [GK19a], [Kos19].
- \( \Delta \subset \Phi_+ \) the set of simple roots.
- For \( \alpha \in \Phi \), let \( s_\alpha \in W \) be the corresponding reflection. The system \((W, \{s_\alpha\}_{\alpha \in \Delta})\) is a Coxeter system. Write \( \ell : W \to \mathbb{N} \) for the length function. Hence, \( \ell(s_\alpha) = 1 \) for all \( \alpha \in \Phi \). Let \( w_0 \) denote the longest element of \( W \).
- For a subset \( K \subset \Delta \), let \( W_K \) denote the subgroup of \( W \) generated by \( \{s_\alpha\}_{\alpha \in K} \). Write \( w_{0,K} \) for the longest element in \( W_K \).
- Let \( \vec{K} \) denote the subset of elements \( w \in W \) that have minimal length in the coset \( W_K w \). Then \( \vec{K} \) is a set of representatives of \( W_K \backslash W \). The longest element in the set \( \vec{K} \) is \( w_{0,K}w_0 \).
- \( X^*_\nu(T) \) denotes the set of dominant characters – that is, characters \( \lambda \in X^*(T) \) – such that \( \langle \lambda, \alpha^\vee \rangle \geq 0 \) for all \( \alpha \in \Delta \).
- For a subset \( I \subset \Delta \), let \( X^*_\nu,I(T) \) denote the set of characters \( \lambda \in X^*(T) \) such that \( \langle \lambda, \alpha^\vee \rangle \geq 0 \) for all \( \alpha \in I \). We call them \( I \)-dominant characters.

**Definition 2.1.1.** Let \( P \subset G_k \) be a parabolic subgroup containing \( B \) and let \( L \subset P \) be the unique Levi subgroup of \( P \) containing \( T \). Then we define a subset \( I_P \subset \Delta \) as the unique subset such that \( W(L,T) = W_{I_P} \). For an arbitrary parabolic subgroup \( P \subset G_k \) containing \( T \), we put \( I_P = I_{P'} \subset \Delta \), where \( P' \) is the unique conjugate of \( P \) containing \( B \).

- For a parabolic \( P \subset G_k \), we put \( \Delta^P = \Delta \setminus I_P \).

### 2.2. The stack of \( G \)-zips

In this section, we recall some facts about the stack of \( G \)-zips of Pink-Wedhorn-Ziegler.

#### 2.2.1. Zip datum

Let \( G \) be a connected reductive group over \( \mathbb{F}_q \). In this article, a zip datum is a tuple \( \mathcal{Z} = (G, P, L, Q, M, \varphi) \) consisting of the following objects:

(i) \( P \subset G_k \) and \( Q \subset G_k \) are parabolic subgroups of \( G_k \).

(ii) \( L \subset P \) and \( M \subset Q \) are Levi subgroups such that \( L^{(q)} = M \). In particular, the \( q \)-power Frobenius isogeny induces an isogeny \( \varphi : L \to M \).

If \( H \) is an algebraic group, denote by \( R_u(H) \) the unipotent radical of \( H \). For \( x \in P \), we can write uniquely \( x = \bar{x}u \) with \( \bar{x} \in L \) and \( u \in R_u(P) \). This defines a projection map \( \theta^P_L : P \to L; x \mapsto \bar{x} \). Similarly, we have a projection map \( \theta^Q_M : Q \to M \). The zip group is the subgroup of \( P \times Q \) defined by

\[
E := \{(x, y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}.
\]  

(2.2.1)

In other words, \( E \) is the subgroup of \( P \times Q \) generated by \( R_u(P) \times R_u(Q) \) and elements of the form \((a, \varphi(a))\) with \( a \in L \). Let \( G \times G \) act on \( G_k \) by \((a, b) \cdot g := agb^{-1}\), and let \( E \) act on \( G \) by restricting this action to \( E \). The stack of \( G \)-zips of type \( \mathcal{Z} \) can be defined as the quotient stack

\[
\mathcal{Z} - \text{Zip}^G = [E \backslash G_k].
\]

Although the above definition of \( \mathcal{Z} - \text{Zip}^G \) may be the most concise one, there is a more useful, equivalent definition in terms of torsors: By [PWZ15, §3C and 3D], the stack \( \mathcal{Z} - \text{Zip}^G \) is the stack over \( k \) such that for all \( k \)-scheme \( S \), the groupoid \( G \times \text{Zip}^G(S) \) is the category of tuples \((J, j, p, j_Q, i)\), where \( J \)
is a $G_k$-torsor over $S$, $\mathcal{I}_P \subset \mathcal{I}$ and $\mathcal{I}_Q \subset \mathcal{I}$ are a $P$-subtorus and a $Q$-subtorus of $\mathcal{I}$ respectively and $\iota: (\mathcal{I}_P/R_u(P))^{(p)} \to \mathcal{I}_Q/R_u(Q)$ is an isomorphism of $M$-torsors.

2.2.2. Cocharacter datum

A convenient way to give a zip datum is using cocharacters. A cocharacter datum is a pair $(\mathcal{I}, \mu)$ where $\mathcal{I}$ is a reductive connected group over $\mathbb{F}_q$ and $\mu: \mathbb{G}_{m,k} \to G_k$ is a cocharacter. There is a natural way to attach to $(\mathcal{I}, \mu)$ a zip datum $\mathcal{Z}_\mu$, defined as follows. First, denote by $P_+(\mu)$ (respectively $P_-(\mu)$) the unique parabolic subgroup of $G_k$ such that $P_+(\mu)(k)$ (respectively $P_-(\mu)(k)$) consists of the elements $g \in G(k)$ satisfying that the map

$$\mathbb{G}_{m,k} \to G_k; \; t \mapsto \mu(t)g\mu(t)^{-1}$$

(respectively $t \mapsto \mu(t)^{-1}g\mu(t)$)

extends to a morphism of varieties $\mathbb{A}^1_k \to G_k$. This construction yields a pair of parabolics $(P_+(\mu), P_-(\mu))$ in $G_k$ such that the intersection $P_+(\mu) \cap P_-(\mu) = L(\mu)$ is the centraliser of $\mu$. It is a common Levi subgroup of $P_+(\mu)$ and $P_-(\mu)$. Set $P = P_+(\mu), \; Q = (P_+(\mu))^{(q)}, \; L = L(\mu)$ and $M = (L(\mu))^{(q)}$. Then the tuple $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$ is a zip datum, which we call the zip datum attached to the cocharacter datum $(\mathcal{I}, \mu)$. We write simply $G$$\text{-}\text{Zip}^\mu$ for $G$$\text{-}\text{Zip}^{\mathcal{Z}_\mu}$. For simplicity, we will always consider zip data arising in this way from a cocharacter datum.

2.2.3. Frames

In this article, given a zip datum $\mathcal{Z} = (G, P, L, Q, M, \varphi)$, a frame for $\mathcal{Z}$ is a triple $(B, T, z)$ where $(B, T)$ is a Borel pair of $G_k$ defined over $\mathbb{F}_q$ satisfying the following conditions:

(i) One has the inclusion $B \subset P$.
(ii) $z \in W$ is an element satisfying the conditions

$$5B \subset Q \quad \text{and} \quad B \cap M = 5B \cap M.$$

Remark 2.2.1. Let $(B, T)$ be a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. Then we can find $z \in W$ such that $(B, T, z)$ is a frame. This follows from the proof of [PWZ11, Proposition 3.7].

A frame may not always exist. However, if $(\mathcal{I}, \mu)$ is a cocharacter datum and $\mathcal{Z}_\mu$ is the associated zip datum (Subsection 2.2.2), then we can find a $G(k)$-conjugate $\mu' = \text{ad}(g)\circ \mu$ (with $g \in G(k)$) such that $\mathcal{Z}_{\mu'}$ admits a frame. This follows easily from Remark 2.2.1 and the fact that $G$ is quasi-split over $\mathbb{F}_q$. Hence, it is harmless to assume that a frame exists, and we will only consider a zip datum that admits a frame.

Remark 2.2.2. If the cocharacter $\mu$ is defined over $\mathbb{F}_q$, then so are $P$ and $Q$. In particular, we have in this case $L = M$ and $P, Q$ are opposite parabolic subgroups with common Levi subgroup $L$.

For a zip datum $(G, P, L, Q, M, \varphi)$, we put $I = I_P \subset \Delta$. Note that $\Delta^P = \Delta \setminus I$.

Lemma 2.2.3 ([GK19b, Lemma 2.3.4]). Let $\mu: \mathbb{G}_{m,k} \to G_k$ be a cocharacter, and let $\mathcal{Z}_\mu$ be the attached zip datum. Assume that $(B, T)$ is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. We put $z = \sigma(w_{0,1})w_0$. Then $(B, T, z)$ is a frame for $\mathcal{Z}_\mu$.

2.2.4. Parametrisation of $E$-orbits

Recall that the group $E$ from (2.2.1) acts on $G_k$. We review below the parametrisation of $E$-orbits following [PWZ11].

Assume that $\mathcal{Z}$ has a frame $(B, T, z)$. For $w \in W$, fix a representative $\tilde{w} \in N_G(T)$, such that $(w_1w_2)' = \tilde{w}_1\tilde{w}_2$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, [ABD+66, Exposé XXIII, §6]). For $w \in W$, define $G_w$ as the $E$-orbit of $\tilde{w}z^{-1}$. We note that $G_w$ is independent of the choices of $\tilde{w}$ and a frame by [PWZ11, Proposition 5.8]. If no confusion occurs, we write $w$ instead of $\tilde{w}$. Define a twisted order on $IW$ as follows. For $w, w' \in IW$, write $w' \leq w$ if there exists $w_1 \in W_L$ such that $w' \leq w_1w\sigma(w_1)^{-1}$. This defines a partial order on $IW$ [PWZ11, Corollary 6.3].
Theorem 2.2.4 ([PWZ11, Theorem 6.2, Theorem 7.5]). The map \( w \mapsto G_w \) restricts to a bijection

\[
\mathcal{I}W \to \{E\text{-orbits in } G_k\}. \tag{2.2.2}
\]

For \( w \in \mathcal{I}W \), one has \( \dim(G_w) = \ell(w) + \dim(P) \). Furthermore, for \( w \in \mathcal{I}W \), the Zariski closure of \( G_w \) is

\[
\overline{G}_w = \bigcup_{w' \in \mathcal{I}W, \, w' \leq w} G_{w'}.
\]

Each \( E \)-orbit is locally closed in \( G_k \). Because \( E \) is smooth over \( k \), all \( E \)-orbits are also smooth over \( k \). However, the Zariski closure \( \overline{G}_w \) of \( G_w \) may have highly complicated singularities; see [Kos18] for a description of the normalisation of \( \overline{G}_w \). The closure of an \( E \)-orbit is a union of \( E \)-orbits; hence we obtain a stratification of \( G \).

In particular, there is a unique open \( E \)-orbit \( U_\mathcal{I} \subset G_k \) corresponding to the longest element \( w_0^Iw_0 \in \mathcal{I}W \) via (2.2.2). For an \( E \)-orbit \( G_w \) (with \( w \in \mathcal{I}W \)), we write \( \mathcal{X}_w := [E \setminus G_w] \) for the corresponding locally closed substack of \( G\text{-Zip}^\mathcal{I} = [E \setminus G_k] \).

If \( \mathcal{I} \) arises from a cocharacter datum (Subsection 2.2.2), we write \( U_{\mu} \) for \( U_{\mathcal{I}_{\mu}} \). Using the terminology pertaining to the theory of Shimura varieties, we call \( U_{\mu} \) the \( \mu \)-ordinary stratum of \( G\text{-Zip}^{\mu} \). The corresponding substack \( \mathcal{U}_\mu := [E \setminus U_{\mu}] \) is called the \( \mu \)-ordinary locus. It corresponds to the \( \mu \)-ordinary locus in the good reduction of Shimura varieties, studied, for example, in [Wor13], [Moo04]. For more details about Shimura varieties, we refer to Subsection 2.5.

### 2.3. Reminders about representation theory

If \( H \) is an algebraic group over a field \( K \), denote by \( \text{Rep}(H) \) the category of algebraic representations of \( H \) on finite-dimensional \( K \)-vector spaces. We will denote such a representation by \( (V, \rho) \) or sometimes simply \( \rho \) or \( V \).

Let \( H \) be a split connected reductive \( K \)-group and choose a Borel pair \( (B_H, T) \) defined over \( K \). Irreducible representations of \( H \) are in one-to-one correspondence with dominant characters \( X^+_\ast(T) \). This bijection is given by the highest weight of a representation. For \( \lambda \in X^+_\ast(T) \), let \( \mathcal{L}_\lambda \) be the line bundle attached to \( \lambda \) on the flag variety \( H/B_H \) by the usual associated sheaf construction [Jan03, §5.8]. Define an \( H \)-representation \( V_H(\lambda) \) by

\[
V_H(\lambda) := H^0(H/B_H, \mathcal{L}_\lambda). \tag{2.3.1}
\]

In other words, \( V_H(\lambda) \) is the induced representation \( \text{Ind}^H_{B_H} \lambda \). Then \( V_H(\lambda) \) is a representation of highest weight \( \lambda \). We view elements of \( V_H(\lambda) \) as functions \( f : H \to \mathbb{A}^1 \) satisfying the relation

\[
f(hb) = \lambda(b^{-1})f(h), \quad \forall h \in H, \forall b \in B_H. \tag{2.3.2}
\]

For dominant characters \( \lambda, \lambda' \), there is a natural surjective map

\[
V_H(\lambda) \otimes V_H(\lambda') \to V_H(\lambda + \lambda'). \tag{2.3.3}
\]

In the description given by (2.3.2), this map is simply given by mapping \( f \otimes f' \) (where \( f \in V_H(\lambda) \), \( f' \in V_H(\lambda') \)) to the function \( f f' \in V_H(\lambda + \lambda') \).

Denote by \( W_H := W(H, T) \) the Weyl group and \( w_{0,I} \in W_H \) the longest element. Then \( V_H(\lambda) \) has a unique \( B_H \)-stable line, which is a weight space for the weight \( w_{0,I} \lambda \).
2.4. Vector bundles on the stack of $G$-zips

2.4.1. General theory
For an algebraic stack $\mathcal{X}$, write $\mathfrak{BB}(\mathcal{X})$ for the category of vector bundles on $\mathcal{X}$. Let $X$ be a $k$-scheme and $H$ an affine $k$-group scheme acting on $X$. If $\rho: H \to \text{GL}(V)$ is a finite-dimensional algebraic representation of $H$, it gives rise to a vector bundle $\mathcal{V}_{H,X}(\rho)$ on the stack $[H\backslash X]$. This vector bundle can be defined geometrically as $[H\backslash (X \times_k V)]$, where $H$ acts diagonally on $X \times_k V$. We obtain a functor

$$\mathcal{V}_{H,X}: \text{Rep}(H) \to \mathfrak{BB}([H\backslash X]).$$

In particular, similar to the usual associated sheaf construction [Jan03, Chapter I, §5.8, Equation (1)], the space of global sections $H^0([H\backslash X], \mathcal{V}_{H,X}(\rho))$ is identified with

$$H^0([H\backslash X], \mathcal{V}_{H,X}(\rho)) = \{ f: X \to V \mid f(h \cdot x) = \rho(h)f(x), \quad \forall h \in H, \forall x \in X \}. \quad (2.4.1)$$

2.4.2. Automorphic vector bundles on $G$-Zip

Fix a zip datum

$$Z = (G, P, L, Q, M, \varphi)$$

and a frame $(B, T, z)$ as usual. By the previous paragraph, we obtain a functor $\mathcal{V}_{E,G}: \text{Rep}(E) \to \mathfrak{BB}(G\text{-Zip}^Z)$, which we simply denote by $\mathcal{V}$. For $(V, \rho) \in \text{Rep}(E)$, the space of global sections of $\mathcal{V}(\rho)$ is

$$H^0(G\text{-Zip}^Z, \mathcal{V}(\rho)) = \{ f: G_k \to V \mid f(\epsilon \cdot g) = \rho(\epsilon)f(g), \quad \forall \epsilon \in E, \forall g \in G_k \}.\$$

One has the following easy lemma, which follows from the fact that $G_k$ admits an open dense $E$-orbit (see discussion below Theorem 2.2.4).

Lemma 2.4.1 ([Kos19, Lemma 1.2.1]). Let $(V, \rho)$ be an $E$-representation. Then we have $\dim H^0(G\text{-Zip}^Z, \mathcal{V}(\rho)) \leq \dim(V)$.

The first projection $p_1: E \to P$ induces a functor $p_1^*: \text{Rep}(P) \to \text{Rep}(E)$. If $(V, \rho) \in \text{Rep}(P)$, we write again $\mathcal{V}$ for $\mathcal{V}(p_1^*(\rho))$. Let $\mathfrak{BB}_P(G\text{-Zip}^Z)$ be the essential image of $\mathcal{V}: \text{Rep}(P) \to \mathfrak{BB}(G\text{-Zip}^Z)$. We call $\mathfrak{BB}_P(G\text{-Zip}^Z)$ the category of automorphic vector bundles (cf. [Mil90, Chapter III, Remark 2.3]). The goal of this article is to study the vector bundles $\mathcal{V}(\rho)$ on $G\text{-Zip}^Z$ and determine their properties for $\rho \in \text{Rep}(P)$. In particular, we seek to understand the properties of $\mathcal{V}(\rho)$ in terms of the representation $(V, \rho)$ defining it.

2.4.3. $L$-representations

Let $\theta_P^P: P \to L$ denote again the natural projection modulo the unipotent radical $R_u(P)$, as in Subsection 2.2.1. It induces by composition a functor

$$(\theta_P^P)_*: \text{Rep}(L) \to \text{Rep}(P).$$

It is easy to see that $(\theta_P^P)_*$ is a fully faithful functor, and its image is the full subcategory of $\text{Rep}(P)$ of $P$-representations that are trivial on $R_u(P)$. Hence, we view $\text{Rep}(L)$ as a full subcategory of $\text{Rep}(P)$. If $(V, \rho) \in \text{Rep}(L)$, we write again $\mathcal{V}(\rho)$ as $\mathcal{V}((\theta_P^P)_*(\rho))$. For $x \in X^*_+, T$, write $B_L := B \cap L$ and define an $L$-representation as

$$\mathcal{V}_L(\lambda) = \text{Ind}^L_{B_L}(\lambda).$$

This is the representation defined in (2.3.1) for $H = L$ and $B_H = B_L$. Denote by $\mathcal{V}_L(\lambda)$ the vector bundle on $G\text{-Zip}^Z$ attached to $\mathcal{V}_L(\lambda)$. We call $\mathcal{V}_L(\lambda)$ the automorphic vector bundle associated to the weight $\lambda$ on $G\text{-Zip}^Z$. This terminology stems from Shimura varieties (see Subsection 2.5 for further details). Note that if $\lambda \in X^+(T)$ is not $L$-dominant, then $\mathcal{V}_L(\lambda) = 0$ and hence $\mathcal{V}_L(\lambda) = 0$. In [Kos19], the
second author studied the vector bundles \( \mathcal{V}_I(\lambda) \) on \( G-.\text{Zip}^\mathbb{Z} \). In particular, he investigated the question of determining the set \( C_{\text{zip}} \) of characters \( \lambda \in X^*_e,(T) \) such that the space \( H^0(G-.\text{Zip}^\mathbb{Z}, \mathcal{V}_I(\lambda)) \) is nonzero. In a work in progress [GIK21] with Goldring, we completely determine \( C_{\text{zip}} \) under the condition that \( P \) is defined over \( \mathbb{F}_q \) and the Frobenius \( \sigma \) acts on \( I \) by \( -w_{0,I} \).

### 2.5. Shimura varieties

In this subsection, we explain the link between the stack of \( G \)-zips and Shimura varieties. Let \((G,X)\) be a Shimura datum [Del79, §2.1.1]. In particular, \( G \) is a connected reductive group over \( \mathbb{Q} \). Furthermore, \( X \) provides a well-defined \( G(\mathbb{Q}) \)-conjugacy class \( \{\mu\} \) of cocharacters of \( G^- \). Write \( E = E(G,X) \) for the reflex field of \((G,X)\) (i.e., the field of definition of \( \{\mu\} \)) and \( \mathcal{O}_E \) for its ring of integers. Given an open compact subgroup \( K \subset G(A_f) \), write \( \text{Sh}(G,X)_K \) for the canonical model at level \( K \) over \( E \) (cf. [Del79, §2.2]). For \( K \) small enough in \( G(A_f) \), \( \text{Sh}(G,X)_K \) is a smooth, quasi-projective scheme over \( E \). For a small enough \( K \), every inclusion \( K' \subset K \) induces a finite étale projection \( \pi_{K'/K} : \text{Sh}(G,X)_{K'} \rightarrow \text{Sh}(G,X)_K \).

Let \( g \geq 1 \) and let \((V,\psi)\) be a \( 2g \)-dimensional, nondegenerate symplectic space over \( \mathbb{Q} \). Write \( \text{GSp}(2g) = \text{GSp}(V,\psi) \) for the group of symplectic similitudes of \((V,\psi)\). Write \( X_g \) for the double Siegel half-space [Del79, §1.3.1]. The pair \((\text{GSp}(2g),X_g)\) is called the Siegel-Shimura datum and has reflex field \( \mathbb{Q} \). Recall that \((G,X)\) is of Hodge type if there exists an embedding of Shimura data \( \iota : (G,X) \rightarrow (\text{GSp}(2g),X_g) \) for some \( g \geq 1 \). Henceforth, assume that \((G,X)\) is of Hodge type.

Fix a prime number \( p \) and assume that the level \( K \) is of the form \( K = K_pK^p \) where \( K_p \subset G(\mathbb{Q}_p) \) is a hyperspecial subgroup and \( K^p \subset G(A_f) \) is an open compact subgroup. Recall that a hyperspecial subgroup of \( G(\mathbb{Q}_p) \) exists if and only if \( G_{\mathbb{Q}_p} \) is unramified and is of the form \( K_p = \mathcal{G}(\mathbb{Z}_p) \) where \( \mathcal{G} \) is a reductive group over \( \mathbb{Z}_p \) such that \( \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq G_{\mathbb{Q}_p} \) and \( \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \) is connected.

We assume that \( p > 2 \). For any place \( v \) above \( p \) in \( E \), Kisin [Kis10] and Vasiu [Vas99] constructed a family of smooth \( \mathcal{O}_{E_v} \)-schemes \( \mathcal{S} = (\mathcal{S}_K)_{K'} \) where \( K = K_pK^p \) and \( K^p \) is a small enough compact open subgroup of \( G(A_f^p) \). For \( K^p \subset K_p \), one has again a finite étale projection \( \pi_{K'/K} : \mathcal{S}_{K_pK^p} \rightarrow \mathcal{S}_{K_pK^p} \), where \( K = K_pK^p \) and \( K' = K_pK'^p \) and the tower \( \mathcal{S} = (\mathcal{S}_K)_{K'} \) is an \( \mathcal{O}_{E_v} \)-model of the tower \((\text{Sh}(G,X)_{K})_{K'}\). We write \( \mathcal{S}_K \) for the geometric special fibre of \( \mathcal{S}_K \).

We take a representative \( \mu \in \{\mu\} \) defined over \( E_v \) by [Kot84, (1.1.3) Lemma (a)]. We can also assume that \( \mu \) extends to \( \mu : \mathbb{C}_m,\mathcal{O}_{E_v} \rightarrow \mathcal{S}_{\mathbb{O}_{E_v}} \) [Kim18, Corollary 3.3.11]. Denote by \( \mathcal{L} \subset \mathcal{G}_{E_v} \) the centraliser of the cocharacter \( \mu \). We take a parabolic subgroups \( \mathcal{P} \) of \( \mathcal{G}_{E_v} \), which has \( \mathcal{L} \) as a Levi subgroup. Since \( G_{\mathbb{Q}_p} \) is unramified, it is quasi-split, hence we can choose a Borel subgroup \( \mathcal{B} \subset \mathcal{G}_{\mathbb{Q}_p} \) and a maximal torus \( \mathcal{T} \subset \mathcal{B} \). There is \( g \in \mathcal{G}(E_v) \) such that \( \mathcal{B}_{E_v} \subset g\mathcal{P}g^{-1} \). Write \( g = g_0g_1 \) with \( b \in B(E_v) \) and \( g_0 \in \mathcal{G}(E_v) \) by the Iwasawa decomposition. Then replacing \( \mu \) by its conjugate by \( g_0 \), we may assume that \( \mathcal{B}_{E_v} \subset \mathcal{P} \).

By properness of the scheme of parabolic subgroups of \( \mathcal{S} \) [ABD+66, Exposé XXVI, Corollaire 3.5], the subgroups \( \mathcal{B} \) and \( \mathcal{P} \) extend uniquely to subgroups \( \mathcal{B} \subset \mathcal{G} \subset \mathbb{Z}_p \) and \( \mathcal{P} \subset \mathcal{G}_{\mathbb{O}_{E_v}} \) over \( \mathbb{O}_{E_v} \) respectively. Let \( \mathcal{L} \subset \mathcal{P} \) be the centraliser of \( \mu : \mathbb{C}_m,\mathcal{O}_{E_v} \rightarrow \mathcal{S}_{\mathbb{O}_{E_v}} \). We take a Borel subgroup \( \mathcal{B}^p \) of \( \mathcal{G}_{\mathbb{Q}_p} \) such that \( \mathcal{B} = \mathcal{B} \cap \mathcal{B}^p \). The subgroup \( \mathcal{B}^p \) extends uniquely to a subgroup \( \mathcal{B}^p \subset \mathcal{G} \) over \( \mathbb{Z}_p \). We put \( \mathcal{L} = \mathcal{B} \cap \mathcal{B}^p \). Set \( G = \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \) and denote by \( B, T, P, L \) the geometric special fibre of \( \mathcal{B}, \mathcal{L}, \mathcal{P} \) respectively. By slight abuse of notation, we denote again by \( \mu \) its mod \( p \) reduction \( \mu : \mathbb{C}_m,k \rightarrow G_k \). Then \((G,\mu)\) is a cocharacter datum, and it yields a zip datum \((G, P, L, Q, M, \varphi)\) as in Subsection 2.2.2 (because \( G \) is defined over \( \mathbb{F}_p \), in the context of Shimura varieties, we always take \( q = p \); hence, \( \varphi \) is the \( p \)th power Frobenius).

By a result of Zhang [Zha18, 4.1], there exists a natural smooth morphism

\[
\zeta : S_K \rightarrow G-.\text{Zip}^\mu
\]

This map is also surjective by [SYZ19, Corollary 3.5.3(1)]. The map \( \zeta \) amounts to the existence of a universal \( G \)-zip \( J = (J,P,J_Q,i) \) over \( S_K \), using the description of \( G-.\text{Zip}^\mu \) provided at the end of Subsection 2.2.1. In the construction of Zhang, the \( G_k \)-torsor \( J \) and the \( P \)-torsor \( J_P \) over \( S_K \) are actually the reduction of a \( \mathcal{G} \)-torsor and a \( \mathcal{P} \)-torsor over \( \mathcal{S}_K \), which we denote by \( J \) and \( J_p \) respectively.
Example 2.5.1. We explain the example of the Siegel-type Shimura variety. In this case, one has $G = \text{GSp}(V, \psi)$ for a symplectic space $(V, \psi)$ of dimension $2g$ ($g \geq 1$) over $\mathbb{Q}$. The $\mathbb{Z}_p$-model $\mathcal{G} = \text{GSp}(\Lambda, \psi)$ is given by a self-dual $\mathbb{Z}_p$-lattice $\Lambda \subset V_{\mathbb{Q}_p}$; that is, a lattice satisfying $\Lambda^\vee = \Lambda$, where $\Lambda^\vee := \{ x \in V_{\mathbb{Q}_p} \mid \forall y \in \Lambda, \, \psi(x, y) \in \mathbb{Z}_p \}$. The cocharacter $\mu: \mathbb{G}_m, \mathbb{Z}_p \to \mathcal{G}_{\mathbb{Z}_p}$ induces a decomposition $\Lambda = \Lambda_0 \oplus \Lambda_1$, where $\Lambda_0, \Lambda_1$ are free $\mathbb{Z}_p$-modules of rank $g$. Here $z \in \mathbb{G}_m$ acts via $\mu$ on $\Lambda_i$ by the character $z \mapsto z^i$ for $i \in \{0, 1\}$. Define two filtrations

$$
\Fil_0(\Lambda): \ 0 \subset \Lambda_0 \subset \Lambda \quad \text{and} \quad \Fil_1(\Lambda): \ 0 \subset \Lambda_1 \subset \Lambda.
$$

Then $\mathcal{P}$ can be defined as the parabolic subgroup of $\mathcal{G}$ stabilising $\Fil_0(\Lambda)$. The scheme $\delta_K$ (with $K = K_p K^p$ and $K_p = \mathcal{G}(\mathbb{Z}_p)$ as above) is a moduli space classifying triples $(A, \xi, \eta K^p)$ where $A$ is an abelian variety of rank $g$ endowed with a principal polarisation $\xi$ and a $K^p$-level structure $\eta K^p$. Here $\eta$ is a symplectic isomorphism $H^1(A, \mathbb{A}^p) \simeq V \otimes \mathbb{A}^p$ and $\eta K^p$ is its $K^p$-coset in the set of such isomorphisms.

Let $\mathcal{A} \to \delta_K$ denote the universal abelian scheme. Then

$$\mathcal{H} := H^1_{\text{dR}}(\mathcal{A}/\delta_K)$$

is a rank $2g$ vector bundle on $\delta_K$, and the principal polarisation $\xi$ induces on $\mathcal{H}$ a perfect, symplectic pairing, which we denote by $\psi_\xi$. The vector bundle $\mathcal{H}$ also carries a natural Hodge filtration (which we denote by $\Fil_{\text{Hdg}}$):

$$0 \subset \Omega_{\mathcal{A}/\delta_K} \subset \mathcal{H},$$

where $\Omega_{\mathcal{A}/\delta_K}$ is the push-forward of the sheaf of relative Kähler differentials $\Omega^1_{\mathcal{A}/\delta_K}$ by the structural morphism $f: \mathcal{A} \to \delta_K$. It is a rank $g$-subbundle of $\mathcal{H}$. We obtain a $\mathcal{G}$-torsor $\mathcal{I}$ and a $\mathcal{P}$-torsor $\mathcal{I}_\mathcal{P}$ over $\delta_K$ as follows: For an $\delta_K$-scheme $S$, we define $\mathcal{I}(S)$ by

$$\text{Isom}_{\Omega_S} \left( (\Lambda \otimes \Omega_S, \psi), (\mathcal{H} \otimes_{\Omega_{\delta_K}} \Omega_S, \psi_\xi) \right)$$

and $\mathcal{I}_\mathcal{P}(S)$ by

$$\text{Isom}_{\Omega_S} \left( (\Lambda \otimes \Omega_S, \psi, \Fil_0(\Lambda) \otimes \Omega_S), (\mathcal{H} \otimes_{\Omega_{\delta_K}} \Omega_S, \psi_\xi, \Fil_{\text{Hdg}} \otimes_{\Omega_{\delta_K}} \Omega_S) \right).$$

This defines two fppf sheaves on $\delta_K$. Furthermore, $\mathcal{G}$ acts naturally on $\mathcal{I}$ via its action on $\Lambda$. Furthermore, because the parabolic group $\mathcal{P} \subset \mathcal{G}$ stabilises $\Fil_0(\Lambda)$, the group $\mathcal{P}$ acts naturally on $\mathcal{I}_\mathcal{P}$. This defines respectively a $\mathcal{G}$-torsor and a $\mathcal{P}$-torsor on $\delta_K$.

Over $S_K = \delta_K \otimes \mathbb{F}_p$, the G-zip $\mathfrak{z} = (\mathfrak{J}, \mathfrak{J}_p, \mathfrak{J}_Q, \iota)$ is defined as follows. First define $\mathfrak{J}$ and $\mathfrak{J}_p$ to be the base change to $S_K$ of $\mathcal{J}$ and $\mathcal{J}_p$. To define the $\mathcal{P}$-torsor $\mathfrak{J}_Q$, recall that $H := H^1_{\text{dR}}(A/S_K)$ admits a conjugate filtration $\Fil_{\text{conj}} \subset H$: Let $f: A \to S_K$ denote the universal abelian scheme (with $A := \mathcal{A} \otimes_{\delta_K} S_K$), then there is a conjugate spectral sequence $E_2^{ab} = R^a f_* (\mathcal{H}^b(\Omega^*_A/S_K)) \Rightarrow H^{a+b}(A/S_K)$. For abelian varieties, this spectral sequence degenerates and gives the filtration $\Fil_{\text{conj}}$ on $H^1_{\text{dR}}(A/S_K)$. Note that the conjugate filtration only exists on the special fibre of $\delta_K$, contrary to the Hodge filtration. For an $\delta_K$-scheme $S$, we put

$$\mathfrak{J}_Q(S) = \text{Isom}_{\Omega_S} \left( (\Lambda \otimes \Omega_S, \psi, \Fil_1(\Lambda) \otimes \Omega_S), (H \otimes_{\Omega_{\delta_K}} \Omega_S, \psi_\xi, \Fil_{\text{conj}} \otimes_{\Omega_{\delta_K}} \Omega_S) \right).$$

Because $Q$ stabilises the filtration $\Fil_1(\Lambda) \otimes \mathbb{F}_p$, it acts naturally on $\mathfrak{J}_Q$, and again we obtain a $\mathcal{P}$-torsor on $S_K$. Finally, the isomorphism $i: (\mathfrak{J}_p/R_p(\mathfrak{p}))^{(p)} \to \mathfrak{J}_Q/R_p(Q)$ is naturally induced by the Frobenius and Verschiebung homomorphisms (or, more generally, the Cartier isomorphism; see [MW04, 7.3]).
For each $L$-dominant character $\lambda \in X^*(T)$, we have the unique irreducible representation $V_I(\lambda)$ of $P$ over $\overline{\mathbb{Q}}_p$ of highest weight $\lambda$. Because we are in characteristic zero, $V_I(\lambda)$ coincides with $H^0(P/B, \mathcal{L}_\lambda)$, as defined in (2.3.1) in Subsection 2.3. It admits a natural model over $\mathbb{Z}_p$, namely,

$$V_I(\lambda)_{\mathbb{Z}_p} := H^0(\mathcal{P}/B, \mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda$ is the line bundle attached to $\lambda$ viewed as a character of $\mathcal{T}$. Its reduction modulo $p$ is the $P$-representation $V_I(\lambda) = H^0(P/B, \mathcal{L}_\lambda)$ over $k = \overline{\mathbb{F}}_p$. Because $\mathcal{S}_K$ is endowed naturally with a $\mathcal{P}$-torsor $\mathcal{I}_{\mathcal{P}}$, we obtain a vector bundle $\mathcal{V}_I(\lambda)$ on $\mathcal{S}_K$ by applying the $\mathcal{P}$-representation $V_I(\lambda)_{\mathbb{Z}_p}$ to $\mathcal{I}_{\mathcal{P}}$. The vector bundle $\mathcal{V}_I(\lambda)$ for $\lambda \in X^*(T)_+, I$ is called the automorphic vector bundle associated to the weight $\lambda$.

Furthermore, the map $\zeta: S_K \to G\text{-}\text{Zip}^{\mu}$ induces a factorization of the lower horizontal arrow of the above diagram as

$$\text{Rep}_{\overline{\mathbb{Q}}_p}(P) \xrightarrow{\mathcal{V}} \mathfrak{B}(\mathcal{S}_K) \xrightarrow{\mathcal{V}^*} \mathfrak{B}(S_K). \tag{2.5.1}$$

Note also that for any $P$-representation $(V, \rho)$, the map $\zeta: S_K \to G\text{-}\text{Zip}^{\mu}$ induces by pullback a natural injective morphism

$$H^0(G\text{-}\text{Zip}^{\mu}, \mathcal{V}(\rho)) \to H^0(S_K, \mathcal{V}(\rho)).$$

In Section 3, we determine the space $H^0(G\text{-}\text{Zip}^{\mu}, \mathcal{V}(\rho))$ in all generality (i.e., even for cocharacter data $(G, \mu)$ that are not attached to Shimura varieties). For general pairs $(G, \mu)$ with $\mu$ minuscule (but not necessarily attached to Shimura varieties), one has the following remark.

**Remark 2.5.2.** Let $F$ be a local field with ring of integers $\mathcal{O}$ and residue field $\mathcal{F}_q$. Let $G$ be an unramified reductive group over $\mathcal{O}$. Let $(B, \mathcal{T})$ be a Borel pair of $G$, and let $\mu$ be a dominant cocharacter of $G$. Then Xiao-Zhu define the moduli of local shtukas $\text{Sh}_{\mu}^{\text{loc}}$ classifying modifications bounded by $\mu$ of a $G$-torsor and its Frobenius twist (see [XZ17, Definition 5.2.1]). Similarly, there is a moduli $\text{Sh}_{\mu}^{\text{loc}(m,n)}$ of restricted local shtuka [XZ17, §5.3], with a natural projection $\text{Sh}_{\mu}^{\text{loc}} \to \text{Sh}_{\mu}^{\text{loc}(m,n)}$. In the case when $\mu$ is minuscule, Xiao-Zhu show in [XZ17, Lemma 5.3.6] that there exists a natural perfectly smooth morphism $\text{Sh}_{\mu}^{\text{loc}(2,1)} \to G\text{-}\text{Zip}^{\mu}$, where $\text{pf}$ denotes the perfection and the special fibre of $G$ is again denoted by $G$ (see Subsection 3.5 for further details).

### 3. The space of global sections $H^0(G\text{-}\text{Zip}^{\mu}, \mathcal{V}(\rho))$

#### 3.1. Adapted morphisms

To determine the space $H^0(G\text{-}\text{Zip}^{\mu}, \mathcal{V}(\rho))$ (for $(V, \rho)$ a $P$-representation), we use a similar method as in [Kos19, §3.2], where we studied representations of the type $V_I(\lambda)$. We review some of the notions introduced in [Kos19, §3.2].


Let $X$ be an irreducible normal $k$-variety and let $U \subset X$ be an open subset such that $S = X \setminus U$ is irreducible of codimension 1. For $f \in H^0(U, \mathcal{O}_X)$, denote by $Z_U(f) \subset U$ the vanishing locus of $f$ in $U$ and let $\overline{Z_U(f)}$ be its Zariski closure in $X$. We endow all locally closed subsets of schemes with the reduced structure. Let $Y$ be an irreducible $k$-variety and $\psi : Y \to X$ be a $k$-morphism.

**Definition 3.1.1.** We say that $\psi$ is adapted to $f$ (with respect to $U$) if

(i) $\psi(Y) \cap U \neq \emptyset$ and
(ii) $\psi(Y) \cap S$ is not contained in $\overline{Z_U(f)}$.

**Lemma 3.1.2.** If $\psi(Y)$ intersects $U$ and $\psi(Y) \cap S$ is dense in $S$, then $\psi$ is adapted to any nonzero section $f \in H^0(U, \mathcal{O}_X)$.

**Proof.** We need to show that condition (ii) is satisfied. We may assume that $Z_U(f) \neq \emptyset$. Then, the closed subset $\overline{Z_U(f)}$ has codimension 1 in $X$ and intersects $U$; hence, $Z_U(f) \cap S$ has codimension $\geq 1$ in $S$, so it cannot contain $\psi(Y) \cap S$.

**Lemma 3.1.3 ([Kos19, Lemma 3.2.2]).** Let $\psi : Y \to X$ be a morphism adapted to $f \in H^0(U, \mathcal{O}_X)$. Then $f$ extends to $X$ if and only if $\psi^*(f) \in H^0(\psi^{-1}(U), \mathcal{O}_Y)$ extends to $Y$. In this case, $f$ vanishes along $S$ if and only if $\psi^*(f)$ vanishes along $\psi^{-1}(S)$.

We apply the above notions to the following situation. From now on, let $(G, \mu)$ be a cocharacter datum, with attached zip datum $Z = (G, P, L, Q, M, \varphi)$ as in Subsection 2.2.2. Assume that $(B, T)$ is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. We take a frame $(B, T, z)$ as in Lemma 2.2.3. Consider the variety $G_k$ and the open subset $U_\mu \subset G_k$ (the $\mu$-ordinary stratum, defined after Theorem 2.2.4). The complement of $U_\mu$ in $G_k$ is not irreducible in general, so in order to apply the previous results, we slightly modify the problem. Recall the parametrisation of $E$-orbits in $G_k$ (2.2.2). Using Theorem 2.2.4, we have

$$G_k \setminus U_\mu = \bigcup_{\alpha \in \Delta^P} Z_\alpha, \quad Z_\alpha = \overline{E \cdot s_\alpha},$$

where $E \cdot s_\alpha$ denotes the $E$-orbit of $s_\alpha$ and the bar denotes the Zariski closure. Indeed, by (2.2.2), the $E$-orbits of codimension 1 in $G_k$ are the $E$-orbits of $w z^{-1}$ where $w \in W$ is an element of length $\ell(w_0, t)w_0) - 1$. These elements are of the form $w_0, t s_\alpha w_0$ for $\alpha \in \Delta^P$. Because $z = z_\sigma(w_0, t)w_0$, the element $w z^{-1}$ has the form $w_0, t s_\alpha z_\sigma(w_0, t)$. Because $(w_0, t, z_\sigma(w_0, t)) \in E$, this element generates the same $E$-orbit as $s_\alpha$. This proves the decomposition (3.1.1) above. For any $\alpha \in \Delta^P$, define an open subset

$$X_\alpha := G_k \setminus \bigcup_{\beta \in \Delta^P, \beta \neq \alpha} Z_\beta.$$

Clearly, $U_\mu \subset X_\alpha$ and one has $X_\alpha \setminus U_\mu = E \cdot s_\alpha$. In particular, $X_\alpha \setminus U_\mu$ is irreducible. We define a morphism that satisfies the conditions of Definition 3.1.1 for the pair $(X_\alpha, U_\mu)$.

We take an isomorphism $u_\alpha : \mathcal{O}_\alpha \to U_\alpha$ for $\alpha \in \Phi$ so that $(u_\alpha)_{\alpha \in \Phi}$ is a realisation in the sense of [Spr98, §8.1.4]. In particular, we have

$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x)$$

for $x \in \mathcal{O}_\alpha$ and $t \in T$. For $\alpha \in \Phi$, there is a unique homomorphism

$$\phi_\alpha : \text{SL}_{2,k} \to G_k$$

such that

$$\phi_\alpha \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = u_\alpha(x), \quad \phi_\alpha \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = u_{-\alpha}(x)$$

Downloaded from https://www.cambridge.org/core. IP address: 64.70.40.11, on 08 Aug 2021 at 15:38:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/fms.2021.32
as in [Spr98, §9.2.2]. Also note that \( \phi_{\alpha}(\text{diag}(t, t^{-1})) = a^\vee(t) \).

Let \( \alpha \in \Delta^P \). Set \( Y = E \times A^1 \) and

\[
\psi_{\alpha} : Y \to G_k; \quad ((x, y), t) \mapsto x\phi_{\alpha}(A(t))y^{-1}
\]

where \( A(t) = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}_{2,k} \).

Note that \( \phi_{\alpha}(A(0)) = s_{\alpha} \) in \( W \). The following identity will be crucial for later purposes:

\[
A(t) = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}.
\] (3.1.3)

Let \( \varphi : T \to T; \ g \mapsto g\varphi(g)^{-1} \) be the Lang torsor. Then \( \varphi \) induces the isomorphism

\[
\varphi_* : X_*(T) \sim \sim X_*(T) ; \ \delta \mapsto \varphi \circ \delta = \delta - q\sigma(\delta).
\]

We put \( \delta_{\alpha} = \varphi^{-1}(a^{\vee}) \). Recall that \( \sigma \) denotes the \( q \)th power Frobenius action on \( \Delta \). We put

\[
m_{\alpha} = \min\{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\}
\] (3.1.4)

and \( t_{\alpha} = t^{-1}\alpha(\varphi(\delta_{\alpha}(t)))^{-1} = t\alpha(\delta_{\alpha}(t))^{-1} \in \mathbb{I} \), where \( t \) is an indeterminate.

**Proposition 3.1.4.** The following properties hold:

1. The image of \( \psi_{\alpha} \) is contained in \( X_{\alpha} \).
2. For any \((x, y) \in E \) and \( t \in \mathbb{A}^1 \), one has \( \psi_{\alpha}((x, y), t) \in U_{\mu} \iff t \neq 0 \).
3. For all \((x, y) \in E \), we have \( \psi_{\alpha}((x, y), 0) \in E \cdot s_{\alpha} \).

**Proof.** It suffices to show (2) and (3). If \( t = 0 \), we have \( \phi_{\alpha}(A(0)) = s_{\alpha} \) in \( W \). Hence, \( \psi_{\alpha}((x, y), 0) \in E \cdot s_{\alpha} \). Assume that \( t \neq 0 \). We put

\[
u_{t, \alpha} = \prod_{i=1}^{m_{\alpha}-1} \phi_{\sigma^{-i}(\alpha)} \left( \begin{pmatrix} 1 & \frac{-q^t}{q^t} \\ 0 & 1 \end{pmatrix} \right),
\]

where the products are taken in the increasing order of indices. By (3.1.3) and the definitions of \( \delta_{\alpha}, t_{\alpha} \) and \( u_{t, \alpha} \), we have

\[
\phi_{\alpha}(A(t)) = \phi_{\alpha} \left( \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \right) \delta_{\alpha}(t)\phi_{\alpha}(\delta_{\alpha}(t))^{-1} \phi_{\alpha} \left( \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \right)
\]

\[
= \phi_{\alpha} \left( \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \right) \delta_{\alpha}(t)\phi_{\alpha} \left( \begin{pmatrix} 1 & t_{\alpha} \\ 0 & 1 \end{pmatrix} \right) \phi_{\alpha}(\delta_{\alpha}(t))^{-1}
\]

\[
= \phi_{\alpha} \left( \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \right) \delta_{\alpha}(t)u_{t, \alpha} \left( \begin{pmatrix} 1 & -t_{\alpha} \\ 0 & 1 \end{pmatrix} \right) u_{t, \alpha}^{-1}.
\] (3.1.5)

We have

\[
\left( \phi_{\alpha} \left( \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \right) \delta_{\alpha}(t)u_{t, \alpha}, \varphi(\delta_{\alpha}(t))\phi_{\alpha} \left( \begin{pmatrix} 1 & -t_{\alpha} \\ 0 & 1 \end{pmatrix} \right) u_{t, \alpha} \right) \in E
\] (3.1.6)

because

\[
\phi_{\alpha} \left( \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \right) \in R_u(P), \quad \phi_{\sigma^{-m_{\alpha}-1}(\alpha)} \left( \begin{pmatrix} 1 & \frac{-t_{\alpha}}{q^{m_{\alpha}-1}} \\ 0 & 1 \end{pmatrix} \right) \in R_u(Q)
\]

by \( \alpha \notin I \) and \( \sigma^{-m_{\alpha}-1}(\alpha) \notin \sigma(I) \). Hence, we have \( \psi_{\alpha}((x, y), t) \in U_{\mu} \) if \( t \neq 0 \).

\[\square\]

Set \( Y_0 := E \times \mathbb{G}_m \subset Y \). We obtain a map \( \psi_{\alpha} : Y_0 \to U_{\mu} \).
Corollary 3.1.5. Let \( f : U_\mu \rightarrow \mathbb{A}^n \) be a regular map. Then \( f \) extends to a regular map \( G_k \rightarrow \mathbb{A}^n \) if and only if for all \( \alpha \in \Delta^P \), the map \( f \circ \psi_{\alpha} : Y_0 \rightarrow \mathbb{A}^n \) extends to a map \( Y \rightarrow \mathbb{A}^n \).

Proof. Applying Lemma 3.1.2 and Lemma 3.1.3 to the coordinate functions of \( f \), we can extend \( f \) to \( \bigcup_{\alpha \in \Delta^P} X_\alpha \). Because the complement of \( \bigcup_{\alpha \in \Delta^P} X_\alpha \) in \( G \) has codimension \( \geq 2 \), we can extend \( f \) to \( G \) by normality.

\[ \square \]

3.2. The space of \( \mu \)-ordinary sections

Recall that \( \mathcal{U}_\mu = [E \setminus U_\mu] \subset G\text{-Zip}^\mu \) denotes the \( \mu \)-ordinary locus (see Subsection 2.2.4). The open substack \( \mathcal{U}_\mu \subset G\text{-Zip}^\mu \) is dense and hence induces an obvious injective map

\[ H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) \rightarrow H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) \]

for any \( (V, \rho) \in \text{Rep}(P) \). This will give an upper bound approximation of the space \( H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) \).

We claim that \( 1 \) is the identity component of \( L_\varphi \). Indeed, by Theorem 2.2.4, \( U_\mu \) coincides with the \( E \)-orbit of the element \( w_{0,l}w_0z^{-1} \). Because \( z = \sigma(w_{0,l})w_0 \), we obtain \( w_{0,l}w_0z^{-1} = w_{0,l}\sigma(w_{0,l}) \). This element is in the same \( E \)-orbit as \( 1 \), because \( (w_{0,l}, \sigma(w_{0,l})) \in E \). This proves the claim.

We denote by \( L_\varphi \subset E \) the scheme-theoretical stabiliser of the element 1. Note that

\[ L_\varphi = E \cap \{(x, x) \mid x \in G_k\} \quad \text{(3.2.1)} \]

is a 0-dimensional algebraic group. In general it is nonsmooth. Denote by \( L_0 \subset L \) the largest algebraic subgroup defined over \( \mathbb{F}_q \). In other words,

\[ L_0 = \bigcap_{n \geq 0} L(q^n). \]

In view of (3.2.1), it is clear that the restriction of the first projection \( E \rightarrow P \) induces a closed immersion \( L_\varphi \rightarrow P \). Hence, we will identify \( L_\varphi \) with its image and view it as a subgroup of \( P \).

Lemma 3.2.1 ([KW18, Lemma 3.2.1]).

(1) One has \( L_\varphi \subset L \).

(2) The group \( L_\varphi \) can be written as a semidirect product

\[ L_\varphi = L_\varphi^0 \rtimes L_0(\mathbb{F}_q) \]

where \( L_\varphi^0 \) is the identity component of \( L_\varphi \). Furthermore, \( L_\varphi^0 \) is a finite unipotent algebraic group.

(3) Assume that \( P \) is defined over \( \mathbb{F}_q \). Then \( L_0 = L \) and \( L_\varphi = L(\mathbb{F}_q) \), viewed as a constant algebraic group.

Proposition 3.2.2. The stack \( \mathcal{U}_\mu \) is isomorphic to \( B(L_\varphi) = [1/L_\varphi] \), the classifying stack of \( L_\varphi \).

Proof. The action map \( E \rightarrow U_\mu \), \( e \mapsto e \cdot 1 \) induces an isomorphism \( E/L_\varphi \cong U_\mu \). Hence, \( \mathcal{U}_\mu = [E \setminus U_\mu] \cong [E \setminus (E/L_\varphi)] \cong [1/L_\varphi] \). \[ \square \]

Corollary 3.2.3. The category of vector bundles on \( \mathcal{U}_\mu \) is equivalent to the category \( \text{Rep}(L_\varphi) \) of representations of \( L_\varphi \). Furthermore, for all \( (V, \rho) \in \text{Rep}(L_\varphi) \), the space of global sections of the attached vector bundle \( \mathcal{V}(\rho) \) on \( \mathcal{U}_\mu \) identifies with the space of \( L_\varphi \)-invariants of \( V \):

\[ H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}. \quad \text{(3.2.2)} \]

Furthermore, this identification is functorial in \( (V, \rho) \).

The identity (3.2.2) can be seen as an isomorphism between two functors \( \text{Rep}(L_\varphi) \rightarrow \text{Vec}_k \). The notation \( V^{L_\varphi} \) for the space of invariants is to be understood in a scheme-theoretical way as the set of...
$v \in V$ such that for any $k$-algebra $R$, one has $\rho(x)v = v$ in $V \otimes_k R$ for all $x \in L_{\varphi}(R)$. In particular, if $(V, \rho) \in \text{Rep}(P)$ and $V(\rho)$ is the attached vector bundle on $G$-$\text{Zip}^\mu$, the restriction of $V(\rho)$ to $U_\mu$ is attached to the restriction of $\rho$ to $L_{\varphi}$, and the formula (3.2.2) applies similarly.

By (2.4.1), any $f \in V^L_{\varphi} = H^0(U_\mu, V(\rho))$ corresponds bijectively to a unique function

$$\tilde{f} : U_\mu \to V$$

satisfying $\tilde{f}(1) = f$ and $\tilde{f}(axb^{-1}) = \rho(a)\tilde{f}(x)$ for all $(a, b) \in E$ and all $x \in U_\mu$. The strategy to determine the space $H^0(G$-$\text{Zip}^\mu, V(\rho))$ will be to characterise which of these functions extends to a function $G_k \to V$. We will use Corollary 3.1.5 for this purpose. As another preliminary, we introduce (a generalisation of) the Brylinski-Kostant filtration in the next section.

### 3.3. Brylinski-Kostant filtration

**Lemma 3.3.1.** Let $\alpha \in \Phi$. Let $V$ be a finite-dimensional algebraic representation of $TU_\alpha$. Let $v \in V_\nu$ for $v \in X^*(T)$. Then we have

$$u_\alpha(x)(v) - v = \sum_{j=1}^{\infty} x^j v_j,$$

where $v_j \in V_{\nu+j\alpha}$.

**Proof.** This is proved in the proof of [Don85, Proposition 3.3.2]. We recall the argument. We write $u_\alpha(x)v$ as $\sum_{j \geq 0} x^j v_j$ for some $v_j \in V$. We note that $v_0 = v$. By (3.1.2), we have $v_j \in V_{\nu+j\alpha}$. \qed

For $\alpha \in \Phi$, we define $E^{(j)}_\alpha : V \to V$ by

$$u_\alpha(x)v = \sum_{j \geq 0} x^j E^{(j)}_\alpha(v)$$

for $j \geq 0$ and put $E^{(j)}_\alpha = 0$ if $j < 0$. By Lemma 3.3.1, we have $E^{(j)}_\alpha(v) \in V_{\nu+j\alpha}$ for $v \in V_\nu$.

Let $\Xi = (a_1, \ldots, a_m) \in \Phi^m$. Let $H$ be a closed subgroup scheme of $G$ containing $T$ and $U_{a_i}$ for $1 \leq i \leq m$. Let $V$ be a finite-dimensional algebraic representation of $H$. Let $a = (a_1, \ldots, a_m) \in (k^*)^m$ and $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$. We put

$$\mathbb{Z}_r^m = \left\{ (n_1, \ldots, n_m) \in \mathbb{Z}^m \mid \sum_{i=1}^m n_i r_i = 0 \right\},$$

$$\Lambda_{\Xi, r} = \left\{ \sum_{i=1}^m n_i a_i \mid (n_1, \ldots, n_m) \in (\mathbb{Z}_r^m) \right\}.$$

For $[\nu] \in X^*(T)/\Lambda_{\Xi, r}$, we put

$$V_{[\nu]} = \bigoplus_{v \in [\nu]} V_v.$$

We use the notation $j$ for $(j_1, \ldots, j_m) \in \mathbb{Z}^m$. For $[j] \in \mathbb{Z}^m/(\mathbb{Z}^m)_r$ and $[\nu] \in X^*(T)/\Lambda_{\Xi, r}$, we put

$$[j] \cdot r = \sum_{i=1}^m j_i r_i \in \mathbb{R},$$

$$[\nu] + [j] \cdot \Xi = \left[ \nu + \sum_{i=1}^m j_i a_i \right] \in X^*(T)/\Lambda_{\Xi, r},$$

$$j = (j_1, \ldots, j_m) \in \mathbb{Z}^m.$$
which are well defined. For \([v] \in X^*(T)/\Lambda_{\Xi,r}\) and a function \(\delta : X^*(T) \to \mathbb{R}\), we define \(\text{Fil}^\Xi_{\delta} a_r V_v\) by

\[
\bigcap_{[j] \in \mathbb{Z}^m/(\mathbb{Z}^m)_{\chi}} \bigcap_{x \in [v] + [j] \in \Xi, \chi \in \Delta} \ker \left( \sum_{[j] \in [j]} \text{pr}_x \circ a_1^{(j_1)} E_{a_1}^{(j_1)} \circ \ldots \circ a_m^{(j_m)} E_{a_m}^{(j_m)} : V_v \to V_x \right),
\]

where \(\text{pr}_x : V_v + [j] \in \Xi \to V_x\) denotes the projection.

**Example 3.3.2.** Assume that \(\Xi = (\alpha) \in \Phi, r_1 = 1\) and \(\delta\) is a constant function \(c \in \mathbb{R}\). Then \(\Lambda_{\Xi,r} = 0\) and \(V_v = V_{\nu}\) for \(\nu \in X^*(T)\). In this case,

\[
\text{Fil}^\Xi_{\delta} a_r V_v = \bigcap_{j > c} \ker \left( E_{a}^{(j)} : V_v \to V_{v + j a} \right),
\]

which we simply write \(\text{Fil}^\Xi_\delta V_v\). This is a Brylinski-Kostant filtration (cf. [XZ19, (3.3.2)])

### 3.4. Main result

We now investigate the space of global sections over \(G\)-Zip\(\mu\) of the vector bundle \(\mathcal{V}(\rho)\) for \((V, \rho) \in \text{Rep}(P)\). By (3.2.2), this space is contained in \(V_{L^\phi}\). Conversely, the problem is to determine which \(f \in V_{L^\phi}\) correspond to sections of \(\mathcal{V}(\rho)\) that extend from \(U_\mu\) to \(G\)-Zip\(\mu\). Equivalently, we ask for which \(f \in V_{L^\phi}\) the regular function \(\tilde{f} : U_\mu \to V\) defined in (3.2.3) extends to a regular function \(G_k \to V\).

Recall the definition of the integer \(m_a\) in (3.1.4) for each \(\alpha \in \Delta^P\). For example, if \(P\) is defined over \(\mathbb{F}_q\), then \(m_a = 1\) for all \(\alpha \in \Delta^P\). We put \(a_\alpha = (-1, \ldots, -1) \in (k^\times)^{m_a}\). For \(\alpha \in \Delta^P\), we put \(\Xi_{a_\alpha} = (-\alpha, \sigma^{-1}(\alpha), \ldots, \sigma^{-(m_a - 1)}(\alpha))\) and \(r_{a_\alpha} = (r_{a_\alpha, 1}, \ldots, r_{a_\alpha, m_a})\), where \(r_{a_\alpha, 1} = 1 - \langle \alpha, \delta_\alpha \rangle\)

where \(2 \leq i \leq m_a\). We view \(\delta_\alpha\) as a function \(X^*(T) \to \mathbb{R}\) by \(\chi \mapsto \langle \chi, \delta_\alpha \rangle\).

**Theorem 3.4.1.** Let \((V, \rho) \in \text{Rep}(P)\). Via the inclusion \(H^0(G\text{-Zip}\mu, \mathcal{V}(\rho)) \subset V_{L^\phi}\) (see Corollary 3.2.3) one has an identification

\[
H^0(G\text{-Zip}\mu, \mathcal{V}(\rho)) = V_{L^\phi} \cap \bigcap_{\alpha \in \Delta^P} \bigoplus_{[v] \in X^*(T)/\Lambda_{\Xi_{a_\alpha}, r_{a_\alpha}}} \text{Fil}^\Xi_{\delta_{\alpha}} a_r V_v.
\]

**Proof.** Let \(f \in V_{L^\phi}\), and let \(\tilde{f} : U_\mu \to V\) be the function defined in (3.2.3). It suffices to show that \(\tilde{f}\) extends to \(G\) if and only if

\[
\tilde{f} \in \bigoplus_{[v] \in X^*(T)/\Lambda_{\Xi_{a_\alpha}, r_{a_\alpha}}} \text{Fil}^\Xi_{\delta_{\alpha}} a_r V_v
\]

for all \(\alpha \in \Delta^P\). By Corollary 3.1.5, \(\tilde{f}\) extends to \(G_k\) if and only if \(\tilde{f} \circ \psi_\alpha : Y_0 \to V\) extends to a function \(Y \to V\). We now give an explicit formula for \(\tilde{f} \circ \psi_\alpha : (x, y) \mapsto Y_0 \to V\). Using (3.1.5) and (3.1.6), the element \(\psi_\alpha((x, y), t) \in U\) can be written as \(x_1 x_2^{-1}\) with \((x_1, x_2) \in E\) and

\[
x_1 = x \phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t_\alpha^{-1} & 1 \end{array} \right) \delta_\alpha(t) u_{t, \alpha}, \quad x_2 = y \varphi(\delta_\alpha(t)) \phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) u_{t, \alpha}.
\]

It follows that

\[
(\tilde{f} \circ \psi_\alpha)((x, y), t) = \tilde{f}(x_1 x_2^{-1}) = \rho(x_1) f = \rho(x) \rho(\phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t_\alpha^{-1} & 1 \end{array} \right) \delta_\alpha(t) u_{t, \alpha}) f.
\]
Hence, the function \( \tilde{f} \circ \psi_{\alpha} \) extends to \( Y \) if and only if the function

\[
F_{\alpha}: t \mapsto \rho \begin{pmatrix} \phi_{\alpha} \left( \begin{array}{cc} 1 & 0 \\ -t^{-l} & 1 \end{array} \right) \end{pmatrix} \delta_{\alpha}(t) u_{t,\alpha} f
\]

lies in \( k[t] \otimes V \). Write \( f = \sum_{v \in X^*(T)} f_v \) by the weight decomposition of \( f \). We put

\[
f^1_{v,\Xi_{\alpha}} = E_{-\alpha}^{(j_1)} E_{\alpha^{-1}(a)}^{(j_2)} \cdots E_{\alpha^{-m_\alpha - 1}(a)}^{(j_{m_\alpha})} f_v \in V_{\nu + j_{\Xi_{\alpha}}}
\]

for \( j = (j_1, \ldots, j_{m_\alpha}) \in \mathbb{Z}^{m_\alpha} \) and \( \nu \in X^*(T) \). We obtain

\[
F_{\alpha}(t) = \rho \begin{pmatrix} \delta_{\alpha}(t) \phi_{\alpha} \left( \begin{array}{cc} 1 & 0 \\ -t^{-l} & 1 \end{array} \right) \end{pmatrix} u_{t,\alpha} f
\]

\[
= \sum_{v} \rho \begin{pmatrix} \delta_{\alpha}(t) \phi_{\alpha} \left( \begin{array}{cc} 1 & 0 \\ -t^{-l} & 1 \end{array} \right) \end{pmatrix} \prod_{i=2}^{m_\alpha} \phi_{\alpha^{-i}(a)} \left( \begin{array}{cc} 1 & 0 \\ -t^{-l}_{a} & 1 \end{array} \right) f_v
\]

\[
= \sum_{v} \rho \big( \delta_{\alpha}(t) \big) \sum_{j \in \mathbb{Z}^{m_\alpha}} \left( -t^{(\alpha, \delta_{\alpha})-1} j_1 \prod_{i=2}^{m_\alpha} (-t^{-l}_{a} j_i) \right) f^1_{v,\Xi_{\alpha}}
\]

\[
= \sum_{v} \sum_{j \in \mathbb{Z}^{m_\alpha}} t^{\nu + j_{\Xi_{\alpha}}, \delta_{\alpha}} \left( -t^{(\alpha, \delta_{\alpha})-1} j_1 \prod_{i=2}^{m_\alpha} (-t^{-l}_{a} j_i) \right) f^1_{v,\Xi_{\alpha}}.
\]

For fixed \( \chi \in X^*(T) \), let \( F_{\alpha,\chi}(t) \) be the \( V_{\chi} \)-component of \( F_{\alpha}(t) \). Then we have

\[
F_{\alpha,\chi}(t) = \sum_{j \in \mathbb{Z}^{m_\alpha}} t^{\nu + j_{\Xi_{\alpha}}, \delta_{\alpha}} \left( -t^{(\alpha, \delta_{\alpha})-1} j_1 \prod_{i=2}^{m_\alpha} (-t^{-l}_{a} j_i) \right) f^1_{\chi,\nu - j_{\Xi_{\alpha}},\Xi_{\alpha}}
\]

\[
= \sum_{j \in \mathbb{Z}^{m_\alpha}} t^{\nu + j_{\Xi_{\alpha}}, \delta_{\alpha}} (-1)^{\sum_{i=1}^{m_\alpha} j_i} f^1_{\chi - j_{\Xi_{\alpha}},\Xi_{\alpha}}.
\]

The exponents of \( t \) in two terms in the last expression are equal if and only if the indices belong to the same coset in \( \mathbb{Z}^{m_\alpha} / (\mathbb{Z}^{m_\alpha})_\alpha \). Therefore, \( F_{\alpha,\chi}(t) \) lies in \( k[t] \otimes V_{\chi} \) for all \( \chi \in X^*(T) \) if and only if we have

\[
\sum_{j \in [j]} (-1)^{\sum_{i=1}^{m_\alpha} j_i} f^1_{\chi - j_{\Xi_{\alpha}},\Xi_{\alpha}} = 0
\]

for all \( \chi \in X^*(T) \) and \( [j] \in \mathbb{Z}^{m_\alpha} / (\mathbb{Z}^{m_\alpha})_\alpha \) such that \( j \cdot \alpha > \langle \chi, \delta_{\alpha} \rangle \). This condition is equivalent to that \( f \) belongs to \( \bigoplus_{\nu \in X^*(T) / \Lambda_{\alpha,\alpha}} \text{Fil}_{\delta_{\alpha}}^{-\alpha, \nu} V_{\nu} \). Hence, the claim follows.

We now give some corollaries of Theorem 3.4.1 in that case where the formula (3.4.1) becomes simpler. For \( \nu \in X^*(T) \) and \( \chi \in X^*(T)_\alpha \), we put

\[
\text{Fil}_{\chi, \alpha}^\nu V_{\nu} = \bigcap_{\alpha \in \Delta^\nu} \text{Fil}_{\delta_{\alpha}}^{-\alpha, \nu} V_{\nu}
\]

where \( \text{Fil}_{\delta_{\alpha}}^{-\alpha, \nu} V_{\nu} \) was defined in Example 3.3.2. The morphism \( \phi: T \rightarrow T \) induces the isomorphism

\[
\phi^*: X^*(T)^\nu \rightarrow X^*(T)^\nu ; \lambda \mapsto \lambda - q\alpha_{\nu}^{-1}(\lambda).
\]
Corollary 3.4.2. Assume that \( P \) is defined over \( \mathbb{F}_q \). Let \((V, \rho) \in \text{Rep}(P)\). Via the inclusion \( H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) \subset V^L(\mathbb{F}_q) \) one has
\[
H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) = V^L(\mathbb{F}_q) \cap \bigoplus_{\nu \in X^*(T)} \text{Fil}^P_{g^{r-1}(\nu)} V_\nu.
\]

Proof. For \( \alpha \in \Delta^P \) and \( \nu \in X^*(T) \), we have
\[
\text{Fil}^{\nu, a_\alpha, \tau_{\alpha}} V_\nu = \text{Fil}^{\nu, \alpha}_V V_\nu = \text{Fil}^{\nu, \alpha}_{(g^{r-1}(\nu), \alpha^\vee)} V_\nu.
\]
Hence, the claim follows from Lemma 3.2.1(3) and Theorem 3.4.1. \( \square \)

Assume again that \( P \) is defined over \( \mathbb{F}_q \). To simplify further, assume that \((V, \rho) \in \text{Rep}(P)\) is trivial on the unipotent radical \( R_u(P) \). Then we have \( E^{-j}(P) = 0 \) for all \( \alpha \in \Delta^P \) and all \( j > 0 \). It follows that \( \text{Fil}_c^{-\alpha} V_\nu = V_\nu \) for \( c \geq 0 \) and \( \text{Fil}_c^{-\alpha} V_\nu = 0 \) for \( c < 0 \). We obtain that for all \( \chi \in X^*(T) \), one has
\[
\text{Fil}^\nu_\chi V_\nu = \begin{cases} V_\nu & \text{if for all } \alpha \in \Delta^P \text{ one has } \langle \chi, \alpha^\vee \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]
Define a subspace \( V_{\Delta^P}^{\geq 0} \subset V \) as follows:
\[
V_{\Delta^P}^{\geq 0} = \bigoplus_{\langle \nu, \delta_\alpha \rangle \geq 0, \forall \alpha \in \Delta^P} V_\nu. \tag{3.4.3}
\]
For example, if \( T \) is split over \( \mathbb{F}_q \), then \( \delta_\alpha = -\alpha^\vee / (q - 1) \), and therefore \( V_{\Delta^P}^{\geq 0} \) is the direct sum of the weight spaces \( V_\nu \) for those \( \nu \in X^*(T) \) satisfying \( \langle \nu, \alpha^\vee \rangle \leq 0 \) for all \( \alpha \in \Delta^P \).

Corollary 3.4.3. Assume that \( P \) is defined over \( \mathbb{F}_q \) and furthermore that \((V, \rho) \in \text{Rep}(P)\) is trivial on the unipotent radical \( R_u(P) \). Then one has an equality
\[
H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) = V^L(\mathbb{F}_q) \cap V_{\Delta^P}^{\geq 0}.
\]
This formula recovers the result [Kos19, Theorem 1] (with slightly different notation). In [Kos19, Theorem 1], only the special case when \( G \) is split over \( \mathbb{F}_p \) and \( V \) is of the form \( V_I(\lambda) \) was considered.

3.5. Perfection

As noted in Remark 2.5.2, the perfection of the stack of \( G \)-zips appears in connection with the moduli of local shtukas. In [XZ17, Lemma 5.3.6], the zip datum that appears satisfies that \( P \) is defined over \( \mathbb{F}_q \). We do not make this assumption here. For a scheme \( X \) over \( k \), define the perfection of \( X \) as the projective limit
\[
X^{\text{pf}} := \lim_{\varphi_X} X
\]
where \( \varphi_X \) denotes the absolute \( q \)th power Frobenius endomorphism of \( X \). There is a natural map \( X^{\text{pf}} \to X \). We have an isomorphism
\[
X^{\text{pf}} \cong \lim \left( \cdots \varphi X^{(q^{-2})} \varphi X^{(q^{-1})} \varphi X \right),
\]
where \( \varphi \) denotes the relative \( q \)th power Frobenius endomorphism. The perfection of \( G\text{-Zip}^\mu \) is then given by
\[
G\text{-Zip}^{\mu, \text{pf}} = [E^{\text{pf}}(\mathcal{G}^\text{pf}_K)].
\]
Similar to Proposition 3.2.2, the perfection of the $\mu$-ordinary locus $U^\mu_\mu$ is isomorphic to $[1/L^\mu_\varphi]$. Because $L_\varphi = L_\varphi^* \cong L_0(\mathbb{F}_q)$ by Lemma 3.2.1(2), we obtain

$$U^\mu_\mu = [1/L_0(\mathbb{F}_q)]. \quad (3.5.1)$$

If $(V, \rho)$ is a $P$-representation, then we obtain a $P^\mu$-representation by pullback, which we denote by $\rho^{P^\mu}$. This yields a vector bundle $\mathcal{V}(\rho^{P^\mu})$ on $G\text{-Zip}^{\mu,P^\mu}$, which also coincides with the pullback of $\mathcal{V}(\rho)$ under the natural map

$$G\text{-Zip}^{\mu,P^\mu} \to G\text{-Zip}^\mu.$$

By equation (3.5.1), we see that the space $H^0(G\text{-Zip}^{\mu,P^\mu}, \mathcal{V}(\rho^{P^\mu}))$ is naturally a subspace of $V^{L_0(\mathbb{F}_q)}$.

**Corollary 3.5.1.** Let $(V, \rho) \in \text{Rep}(P)$. We have

$$H^0(G\text{-Zip}^{\mu,P^\mu}, \mathcal{V}(\rho^{P^\mu})) = V^{L_0(\mathbb{F}_q)} \cap \bigoplus_{\alpha \in \Delta^P} \bigcap_{\{\nu \in \mathcal{X}^*(T)/\Lambda\}} \text{Fil}_{\delta^\alpha}^{\Xi, a_\alpha, r_\alpha} V_{[\nu]}.$$

**Proof.** Let $d$ be the smallest positive integer such that $\mu$ is defined over $\mathbb{F}_{q^d}$. We show that $H^0(G\text{-Zip}^{\mu,P^\mu}, \mathcal{V}(\rho^{P^\mu}))$ is given by the subspace of elements $f \in V$ such that there exists $n \geq 1$ with $f \in H^0(G\text{-Zip}^{\mu}, \mathcal{V}(\rho^{(q^n)^d}))$. Indeed, such a section is given by a map $f : G^\mu \to V$ satisfying an $E^{pf}$-equivariance condition with respect to $\rho^{pf}$. Because $V$ is a scheme of finite type, such a map is given by a map $f_n : G_k \to V$ at a finite level of the system $(\cdots \overset{q^d}{\to} G_k \overset{q^d}{\to} G_k)$. We have

$$\text{Fil}_{\delta^\alpha}^{\Xi, a_\alpha, r_\alpha} V^{(q^d)^n} = \text{Fil}_{\delta^\alpha}^{\Xi, a_\alpha, r_\alpha} V_{[\nu]}.$$

Hence, changing $\rho$ to $\rho^{(q^n)}$ only affects $V^{L_\varphi}$. The result follows. □

### 3.6. $L$-semisimplification

If $\rho : P \to \text{GL}(V)$ is an arbitrary representation, we can attach a $P$-representation $(V, \rho^{L-ss})$ that is trivial on $R_\mu(P)$. The representation $\rho^{L-ss}$ is defined as the composition

$$\rho^{L-ss} : P \xrightarrow{\theta_\mu^P} L \xrightarrow{\rho} \text{GL}(V),$$

where $\theta_\mu^P : P \to L$ is the natural projection map whose kernel is $R_\mu(P)$, as defined in Subsection 2.2.1. We call $\rho^{L-ss}$ the $L$-semisimplification of $\rho$. We sometimes write $V^{L-ss}$ to denote this representation (even though the underlying vector space is the same as $V$).

One obvious property of $V^{L-ss}$ is $(V^{L-ss})^{L_\varphi} = V^{L_\varphi}$ because $L_\varphi \subset L$ by Lemma 3.2.1(1). In particular, by Corollary 3.2.3, we have for all $(V, \rho) \in \text{Rep}(P)$ the equality

$$H^0(U^\mu, \mathcal{V}(\rho^{L-ss})) = H^0(U^\mu, \mathcal{V}(\rho)). \quad (3.6.1)$$

Note that this identification is somewhat indirect: It is not induced by a morphism between the sheaves $\mathcal{V}(\rho)$ and $\mathcal{V}(\rho^{L-ss})$. For $f \in H^0(U^\mu, \mathcal{V}(\rho))$, we will write $f^{L-ss}$ for its image under the identification (3.6.1) and call it the $L$-semisimplification of $f$. As an element of $V$, $f^{L-ss}$ is the same as $f$, but we want to emphasise the fact that the representation has changed.

We now give another interpretation of $L$-semisimplification when $P$ is defined over $\mathbb{F}_q$. Write again $U^\mu_\mu \subset G_k$ for the unique open $E$-orbit and recall that $1 \in U^\mu_\mu$ (see Subsection 3.2).
Lemma 3.6.1. Assume that $P$ is defined over $\mathbb{F}_q$. There exists a unique regular map $\Theta: U_\mu \to L$ such that for any $(a, b) \in E$, one has
\[
\Theta(ab^{-1}) = \theta_L^P(a)\theta_L^Q(b)^{-1}.
\] (3.6.2)
Furthermore, we have $L \subset U_\mu$ and the inclusion $L \subset U_\mu$ is a section of $\Theta$.

Proof. First, note that because $P$ is defined over $\mathbb{F}_q$, one has $L = M$; hence, the formula (3.6.2) makes sense. The unicity of $\Theta$ is obvious. For the existence, consider the map $\tilde{\Theta}: E \to L; (a, b) \mapsto \theta_L^P(a)\theta_L^Q(b)^{-1}. $ Because $P$ is defined over $\mathbb{F}_q$, one has $L_\varphi = L(\mathbb{F}_q)$ (Lemma 3.2.13). For all $(a, b) \in E$ and all $x \in L(\mathbb{F}_q)$, one has $\tilde{\Theta}(ax, bx) = \tilde{\Theta}(a, b)$. Hence, $\tilde{\Theta}$ factors to a map $\Theta: E/L(\mathbb{F}_q) \to U_\mu \to L$. This proves the first result. Now, if $x \in L$, we can write $x = a\varphi(a)^{-1}$ with $a \in L$ by Lang’s theorem. Hence, $x \in U_\mu$ and $\Theta(x) = a\varphi(a)^{-1} = x$, so the second statement is proved. □

Example 3.6.2. Consider the case $G = \text{Sp}(2n)_{\mathbb{F}_q}$ for $n \geq 1$. We write an element of $G_k$ as
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
with $A, B, C, D$ square matrices of size $n \times n$. Let $P \subset G_k$ be the parabolic subgroup defined by the condition $B = 0$ and $Q \subset G_k$ the parabolic subgroup defined by the condition $C = 0$. We put $L = P \cap Q$. This gives a zip datum $(G, P, L, Q, L, \varphi)$. The Zariski open subset $U_\mu \subset G_k$ is the set of matrices in $G_k$ for which $A$ is invertible. The map $\Theta: U_\mu \to L$ is given by
\[
\Theta: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}.
\]

Proposition 3.6.3. Assume that $P$ is defined over $\mathbb{F}_q$. Let $(V, \rho) \in \text{Rep}(P)$ and let $f \in V^{L(\mathbb{F}_q)}$. Let $\tilde{f}$ be the corresponding function $U_\mu \to V$ defined in (3.2.3). Then the function $f_{L-ss}: U_\mu \to V$ that corresponds to the $L$-semisimplification $f_{L-ss}$ is the composition
\[
U_\mu \xrightarrow{\Theta} L \hookrightarrow U_\mu \xrightarrow{\tilde{f}} V.
\]

Proof. Put $f' = \tilde{f} \circ \Theta$. For $(a, b) \in E$ and $g \in U_\mu$ such that $g = ab^{-1}$, we have
\[
f'(g) = f'(ab^{-1}) = \tilde{f}(\Theta(ab^{-1})) = \tilde{f}(\theta_L^P(a)\theta_L^Q(b)^{-1}) = \rho(\theta_L^P(a))f = \rho_{L-ss}(a)f = f_{L-ss}(g).
\]
Hence, $f' = f_{L-ss}$. □

Let $f \in H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho))$ be a global section. We may view its restriction $f|_{U_\mu}$ as a section of $\mathcal{V}(\rho_{L-ss})$ over $U_\mu$ by the identification (3.6.1). It is thus natural to ask whether $(f|_{U_\mu})_{L-ss}$ extends to a global section over $G-\text{Zip}^\mu$. We prove that this holds when $P$ is defined over $\mathbb{F}_q$ in the following proposition.

Proposition 3.6.4. Assume that $P$ is defined over $\mathbb{F}_q$. The identification (3.6.1) extends to a commutative diagram
\[
\begin{array}{ccc}
H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) & \xrightarrow{\sim} & H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho_{L-ss})) \\
\downarrow & & \downarrow \\
H^0(U_\mu, \mathcal{V}(\rho)) & \xrightarrow{=} & H^0(U_\mu, \mathcal{V}(\rho_{L-ss})).
\end{array}
\]
Proof. Let \( f \in H^0(G, \text{Zip}^\mu, \mathcal{V}(\rho)) \). Because \( P \) is defined over \( \mathbb{F}_q \), we can apply Corollary 3.4.2 to the representation \((V, \rho)\). Furthermore, because \( R_u(P) \) acts trivially on \((V^{L-ss}, \rho^{L-ss})\), we can apply Corollary 3.4.3 to \((V^{L-ss}, \rho^{L-ss})\). Therefore, it suffices to show that for each \( \nu \in X^*(T) \),

\[
V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^*(T)} \text{Fil}^P_{\psi^\nu-1(\nu)} V_\nu \subset V^{L(\mathbb{F}_q)} \cap V_{\Delta^P}.
\]

By (3.4.3), it suffices to show the following: For any fixed \( \nu \in X^*(T) \), if \( \text{Fil}^P_{\psi^\nu-1(\nu)} V_\nu \neq 0 \), then \( \langle \psi^\nu^{-1}(\nu), \alpha^{\psi^\nu} \rangle \geq 0 \) for all \( \alpha \in \Delta^P \). More generally, using (3.4.2), it suffices to show that for any \( \alpha \in \Delta^P \) and any integer \( c \in \mathbb{Z} \) such that \( \text{Fil}^{-\alpha}_c V_\nu \neq 0 \), one has \( c \geq 0 \). This is trivial by (3.3.1) because \( E_{-\alpha}^{(0)} \) is the identity map. \( \square \)

Remark 3.6.5. Proposition 3.6.4 does not hold in general without the assumption that \( P \) is defined over \( \mathbb{F}_q \), as an example in Subsection 6.2 shows.

4. The case of \( G = \text{SL}_2, \mathbb{F}_q \)

4.1. Notation for \( \text{SL}_2 \)

Let \( B_2 \) and \( B_2^+ \) be the lower-triangular and upper-triangular Borel subgroup of \( \text{SL}_2, \mathbb{F}_q \). Let \( T_2 \) be the diagonal torus of \( \text{SL}_2, \mathbb{F}_q \). We put

\[
u_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B_2(\mathbb{F}_q).
\]

For \( r \in \mathbb{Z} \), let \( \chi_r \) be the character of \( B_2 \) defined by

\[
\begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} \mapsto x^r.
\]

Let \( \text{Std} : \text{SL}_2, \mathbb{F}_q \to \text{GL}_2, \mathbb{F}_q \) be the standard representation. Restrictions of \( \chi_r \) and \( \text{Std} \) to subgroups are denoted by the same notations.

4.2. Zip datum

Let \( G = \text{SL}_2, \mathbb{F}_q \) and \( \mu : G_m, \mathbb{F}_q \to G_k \); \( x \mapsto \text{diag}(x, x^{-1}) \). Let \( \Omega_{\mu} = (G, P, L, Q, M, \varphi) \) be the associated zip datum. We have \( P = B_2 \), \( Q = B_2^+ \) and \( L = M = T_2 \). We take \( (B, T) = (B_2, T_2) \) as a Borel pair and take a frame as in Lemma 2.2.3. Denote by \( \alpha \) the unique element of \( \Delta \). In our convention of positivity, \( \alpha = \chi_2 \). Note that \( I = \emptyset \) and \( \Delta^P = \{\alpha\} \). Identify \( X^*(T) = \mathbb{Z} \) such that \( r \in \mathbb{Z} \) corresponds to the character \( \chi_r \). The zip group \( E \) is equal to

\[
\left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \begin{pmatrix} a^q & b \\ 0 & a^{-q} \end{pmatrix} \right\} \in B_2 \times B_2^+.
\]

The unique open \( E \)-orbit \( U_{\mu} \subset G_k \) is given by

\[
U_{\mu} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \text{SL}_2, \mathbb{F}_q \bigg| x \neq 0 \right\}.
\]
4.3. The space \( H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) \)

Let \( \rho : B \rightarrow \text{GL}(V) \) be a representation. We write the weight decomposition as \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) where \( T \) acts on \( V_i \) by the character \( \chi_i \) for all \( i \in \mathbb{Z} \). We have

\[
H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L(\overline{\mathbb{F}}_q)} = \bigoplus_{i \in (q-1)\mathbb{Z}} V_i
\]

by Corollary 3.2.3. Because in this case the parabolic \( P = B \) is defined over \( \mathbb{F}_q \), we can apply Corollary 3.4.2 to compute the space of global section

\[
H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)).
\]

Also, because \( T \) is split over \( \mathbb{F}_q \), the map \( \varphi^* \) is given by \( \nu \mapsto -(q-1)\nu \); hence, \( \varphi^{-1}(\nu) = \frac{-\nu}{q-1} \). We obtain

\[
H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\overline{\mathbb{F}}_q)} \cap \bigoplus_{i \in \mathbb{Z}} \text{Fil}^\mu_{\varphi^{-1}} V_i = \bigoplus_{i \in (q-1)\mathbb{N}} \text{Fil}^-_{\frac{i}{q-1}} V_i,
\]

where we used that \( \text{Fil}^-_{\frac{i}{q-1}} V_i = 0 \) for \( i > 0 \). In particular, \( H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) \) is stable by \( T \) and is entirely determined by its weight spaces \( \text{Fil}^-_{\frac{i}{q-1}} V_i \subset V_i \) for \( i \in -(q-1)\mathbb{N} \). Let \( (V, \rho) \in \text{Rep}(B) \) and set \( n = \dim(V) \). Set \( V_{\leq i} = \bigoplus_{j \leq i} V_j \) and \( V_{\geq i} = \bigoplus_{j \geq i} V_j \). Then using Lemma 3.3.1, we have a \( B \)-stable filtration

\[
\cdots \subset V_{\leq i-1} \subset V_{\leq i} \subset V_{\leq i+1} \subset \cdots.
\]

For all \( i \in -(q-1)\mathbb{N} \), we have

\[
H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho))_i = \left\{ f \in V_i \middle| \rho(u_2)f \in V_{\geq \frac{(q+1)i}{q-1}} \right\}
\]

by the definition of \( \text{Fil}^-_{\frac{i}{q-1}} V_i \).

**Lemma 4.3.1.** Let \( (V, \rho) \in \text{Rep}(B) \) and \( m \in \mathbb{Z} \) be the smallest weight of \( \rho \). Then one has an inclusion

\[
\bigoplus_{i \in -(q-1)\mathbb{N}, (q+1)m \geq i \geq (q-1)m} V_i \subset H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)).
\]

**Proof.** Let \( f \in V_i \) with \( i \in -(q-1)\mathbb{N} \) and \( (q+1)i \leq (q-1)m \). Then we have \( V_{\geq \frac{(q+1)i}{q-1}} = V \), so we have \( f \in H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho))_i \).

The following example shows that \( H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho)) \) is not a sum of weight spaces of \( V \) in general.

**Example 4.3.2.** For \( i \in \{1, -1\} \), let \( e_i \) be a nonzero vector of weight \( i \) of Std. Consider \( \rho := \text{Std} \otimes \text{Std} \) with basis \( e_i \otimes e_j \) for \( i, j \in \{1, -1\} \). The weights of \( \rho \) are \( \{2, 0, -2\} \), and \( \dim(V_0) = \dim(V_{-2}) = 1 \), \( \dim(V_2) = 2 \). Then we have

\[
H^0(G\text{-}\text{Zip}^\mu, \mathcal{V}(\rho))_0 = \text{Span}(e_1 \otimes e_{-1} - e_{-1} \otimes e_1).
\]

4.4. **Property (P)**

**Proposition 4.4.1.** Let \( \rho : B \rightarrow \text{GL}(V) \) be an algebraic representation. Let \( m_1, \ldots, m_n \) be the weights of \( V \) ordered so that \( m_1 > m_2 > \cdots > m_n \). The following properties are equivalent:

(i) The subspace \( V_b(B) \) is one-dimensional (and hence is equal to \( V_{m_0} \)).
(ii) The intersection of all nonzero $B$-subrepresentations in $V$ is nonzero.
(iii) For all $1 \leq i \leq n$, we have $\dim(V_{m_i}) = 1$ and for any $v \in V_{m_i} \setminus \{0\}$, the projection of $\rho(u_2)v$ onto $V_{m_n}$ is nonzero.

Proof. We show (i) $\Rightarrow$ (ii). If $W \subset V$ is a nonzero $B$-subrepresentation, then $W^{R_n(B)} \subset V^{R_n(B)}$. Because $W^{R_n(B)} \neq 0$, we have $W^{R_n(B)} = V^{R_n(B)}$ and hence $V^{R_n(B)} \subset W$.

We show (ii) $\Rightarrow$ (iii). We show that for any nonzero $v \in V_{m_i}$, the projection of $\rho(u_2)v$ onto $V_{m_n}$ is nonzero. For a contradiction, assume that it is zero. Because $B = R_n(B)T$, the $B$-subrepresentation generated by $v$ is generated by $v$ as an $R_n(B)$-representation. Hence, this representation does not have a nontrivial intersection with $V_{m_n}$ by Lemma 3.3.1. This contradicts (ii). Hence, the claim follows. We note that $\dim V_{m_n} = 1$ by (ii). Assume that $\dim V_{m_i} \geq 2$ for some $i$. Then there is a nonzero $v \in V_{m_i}$ such that the projection to $V_{m_n}$ of $\rho(u_2)v$ is zero. This is a contradiction.

We show (iii) $\Rightarrow$ (i). Assume $\dim V^{R_n(B)} \geq 2$. Then $V^{R_n(B)}$ contains $V_{m_i}$ for some $i \neq n$. For any nonzero $v \in V_{m_i} \subset V^{R_n(B)}$, the projection of $\rho(u_2)v$ onto $V_{m_n}$ is zero. This is a contradiction. $\Box$

We say that $(V, \rho) \in \text{Rep}(B)$ satisfies the property (P) if the equivalent conditions of Proposition 4.4.1 are satisfied.

Example 4.4.2. For $\lambda \in X_+^*(T)$, the restriction to $B$ of $\text{Ind}^G_B(\lambda)$ satisfies the property (P) by the last sentence of Subsection 2.3.

Proposition 4.4.3. Assume that $(V, \rho) \in \text{Rep}(B)$ satisfies the property (P). Then the inclusion (4.3.2) is an equality; that is,

$$H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) = \bigoplus_{i \in -(q-1)\mathbb{N}, (q+1)i \leq (q-1)m} V_i.$$

Proof. In this case, the element $\rho(u_2)f$ in equation (4.3.1) has a nonzero projection onto $V_m$ by Proposition 4.4.1(iii). Thus, if $f \in H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho))$, then we must have $m \geq \frac{(q+1)i}{q-1}$. This shows that (4.3.2) is an equality. $\Box$

5. Category of automorphic vector bundles on $G\text{-Zip}^\mu$

5.1. The category $\mathfrak{BB}(G\text{-Zip}^\mu)$

Recall the functor $\text{Rep}(P) \rightarrow \mathfrak{BB}(G\text{-Zip}^\mu)$ (Subsection 2.4.2). This functor is not fully faithful even after restricting to the full subcategory $\text{Rep}(L) \subset \text{Rep}(P)$ (see Subsection 2.4.3). Indeed, consider the following example.

Example 5.1.1. Assume that $P$ is defined over $\mathbb{F}_q$. Let $1 \in \text{Rep}(L)$ be the trivial $L$-representation and $(V, \rho) \in \text{Rep}(L)$. Then $\text{Hom}_{\text{Rep}(L)}(1, V) = V^L$, whereas we have

$$\text{Hom}_{\mathfrak{BB}(G\text{-Zip}^\mu)}(\mathcal{V}(1), \mathcal{V}(\rho)) = H^0(G\text{-Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap V^\Delta_p$$

by Corollary 3.4.3.

To overcome the problem, we introduce $L_\varphi$-modules with additional structures.

Definition 5.1.2. An $L_\varphi$-module with $\Delta^P$-monodromy is a pair $((\tau, V), N)$ where $\tau : L_\varphi \rightarrow GL_k(V)$ is a finite-dimensional representation of $L_\varphi$ with a decomposition $V = \bigoplus_{v \in X^*_+} V_v$ as $k$-vector spaces and $N = \{N_{\alpha}^{(j)}\}_{\alpha \in \Delta^P, \alpha' \in \Delta^P, j \in \mathbb{Z}}$ is a set of $k$-linear endmorphisms of $V$ such that $N_{\alpha}^{(j)}(V_v) \subset V_{v+j\alpha'}$, $N_{\alpha'}^{(j)} = 0$ for $j < 0$.

Morphisms are given as follows: Let $((\tau, V), N)$ and $((\tau', V'), N')$ be two $L_\varphi$-modules with $\Delta^P$-monodromy. Then a morphism $((\tau, V), N) \rightarrow ((\tau', V'), N')$ is a $k$-linear map $f : V \rightarrow V'$ that satisfies the following:
(1) $f$ is an $L_{\varphi}$-equivariant morphism.

(2) For $\alpha \in \Delta^P$, $[j] \in \mathbb{Z}^{m_{\alpha}} / (\mathbb{Z}^{m_{\alpha}})_{r_{\alpha}}$ and $\chi \in X^*(T)$ such that $[j] \cdot r_{\alpha} > \delta_{\alpha}(\chi)$, we have

$$\sum_{j \in [j]} \sum_{j' \in \mathbb{Z}^{m_{\alpha}}} (-1)^{\sum_{\alpha} m_{\alpha}} j' \text{pr}_\chi \left( N_{\alpha_1}^{\ell(j'_1)} \cdots N_{\alpha_m}^{\ell(j'_m)} f N_{\alpha_1}^{\ell[j_1 - j'_1]} \cdots N_{\alpha_1}^{\ell[j_1 - j'_1]} \right) = 0,$$

where $\text{pr}_\chi$ denotes the projection

$$\text{pr}_\chi : \text{Hom}(V, V') \cong \bigoplus_{\nu, \nu' \in X^*(T)} \text{Hom}(V_\nu, V'_{\nu'}) \rightarrow \bigoplus_{\nu \in X^*(T)} \text{Hom}(V_\nu, V_{\nu + \chi}).$$

We denote by $L_{\varphi} - \text{MN}_{\Delta^P}$ the category of $L_{\varphi}$-modules with $\Delta^P$-monodromy.

**Remark 5.1.3.** The condition (2) in Definition 5.1.2 means that $f$ is compatible with $N$ and $N'$ in some sense. Assume that $P$ is defined over $\mathbb{F}_q$. Then the condition (2) in Definition 5.1.2 is simplified as follows: For $\alpha \in \Delta^P$, $\chi \in X^*(T)$ and $j \in \mathbb{N}$ such that $jr_{\alpha, 1} > \delta_{\alpha}(\chi)$, we have

$$\text{pr}_\chi \left( \sum_{0 \leq j' \leq j} (-1)^{j'} N_{\alpha}^{\ell(j')} f N_{\alpha}^{\ell[j - j']} \right) = 0.$$

The morphism $N_{\alpha}^{\ell(j)}$ is an analogue of $N^j / j!$ for a monodromy operator $N$ in characteristic zero. In this sense,

$$f \mapsto \sum_{0 \leq j' \leq j} (-1)^{j'} N_{\alpha}^{\ell(j')} f N_{\alpha}^{\ell[j - j']}$$

is an analogue of $j$th iterate of

$$f \mapsto fN - N'f$$

divided by $j!$ for monodromy operators $N$ and $N'$ in characteristic zero.

We have the functor

$$F_{\text{MN}} : \text{Rep}(P) \rightarrow L_{\varphi} - \text{MN}_{\Delta^P} ; \ (V, \rho) \mapsto \left( (V, \rho |_{L_{\varphi}}), \{E_{\alpha}^{(j)} \}_{\alpha \in \Delta^P}, \sigma' \in \Sigma, j \in \mathbb{Z} \right)$$

where we equip $V$ with the natural $T$-weight decomposition $V = \bigoplus_{\nu} V_\nu$.

**Definition 5.1.4.** An $L_{\varphi}$-module with $\Delta^P$-monodromy is called admissible if it is in the essential image of $F_{\text{MN}}$. We denote by $L_{\varphi} - \text{MN}_{\Delta^P}^{\text{adm}}$ the category of admissible $L_{\varphi}$-modules with $\Delta^P$-monodromy.

**Theorem 5.1.5.** The functor $\nabla : \text{Rep}(P) \rightarrow \mathfrak{BB}(G - \text{Zip})$ factors through the functor $F_{\text{MN}} : \text{Rep}(P) \rightarrow L_{\varphi} - \text{MN}_{\Delta^P}^{\text{adm}}$ and induces an equivalence of categories

$$L_{\varphi} - \text{MN}_{\Delta^P}^{\text{adm}} \longrightarrow \mathfrak{BB} P(G - \text{Zip}).$$

**Proof.** For two $P$-representations $(V, \rho)$ and $(V', \rho')$, one has

$$\text{Hom}_{\mathfrak{BB}(G - \text{Zip})}(V(\rho), V(\rho')) = \text{Hom}_{\mathfrak{BB}(G - \text{Zip})}(V(\mathbf{1}), V(\rho') \otimes V(\rho'))$$

$$= \text{Hom}_{\mathfrak{BB}(G - \text{Zip})}(V(\mathbf{1}), V(\rho' \otimes \rho'))$$

$$= H^0(G - \text{Zip}, V(\rho' \otimes \rho'))$$

$$= (V' \otimes V')_{L_{\varphi}} \cap \bigoplus_{a \in \Delta^P \{v \in X^*(T) / \Lambda_{\Sigma, r_{\alpha}} \} \tn{Fil}_{\delta_{\varphi} \cdot a \cdot r_{\alpha}} (V' \otimes V')_{[v]}.$$
where we used Theorem 3.4.1 in the last line. We can see from the definition that this space coincides with the space of homomorphisms $F_{\text{MN}}(V, \rho) \to F_{\text{MN}}(V', \rho')$ using that the action of $u_{\alpha'}(x)$ on $V \otimes V'$ is given by $f \mapsto \rho'(u_{\alpha'}(x)) f \circ \rho(u_{\alpha'}(-x))$ for $\alpha' \in \Xi_{\alpha}$. \hfill \Box

Let $S_K$ denote the good reduction special fibre of a Hodge-type Shimura variety, with the same notations and assumptions as in Subsection 2.5. Recall that there is a functor $\mathcal{V} : \text{Rep}(P) \to \mathfrak{GB}(S_K)$ (see (2.5.1)), which induces functors

$$
\text{Rep}(P) \xrightarrow{\mathcal{V}} \mathfrak{GB}_P(G\text{-Zip}^\mu) \xrightarrow{\xi^*} \mathfrak{GB}_P(S_K),
$$

where $\mathfrak{GB}_P(S_K)$ also denotes the essential image of $\text{Rep}(P)$ in $\mathfrak{GB}(S_K)$. We obtain the following corollary in the context of Shimura varieties.

**Corollary 5.1.6.** The functor $\mathcal{V} : \text{Rep}(P) \to \mathfrak{GB}_P(S_K)$ factors as

$$
\text{Rep}(P) \xrightarrow{F_{\text{MN}}} L_{\varphi^*}\text{-MN}_{\Delta^P} \xrightarrow{\xi^*} \mathfrak{GB}_P(S_K).
$$

5.2. The category $\mathfrak{GB}_L(G\text{-Zip}^\mu)$

We assume that $P$ is defined over $\mathbb{F}_q$. Hence, in what follows, we have $L_{\varphi} = L(\mathbb{F}_q)$.

**Definition 5.2.1.** Let $\mathfrak{GB}_L(G\text{-Zip}^\mu)$ denote the full subcategory of $\mathfrak{GB}(G\text{-Zip}^\mu)$, which is equal to the essential image of the functor $\text{Rep}(L) \to \mathfrak{GB}(G\text{-Zip}^\mu)$. We call it the category of $L$-vector bundles on $G\text{-Zip}^\mu$.

For example, the automorphic vector bundles $(\mathcal{V}(\lambda))_{\lambda \in X^*} (T)$ (see Subsection 2.4.3) lie in the subcategory of $L$-vector bundles on $G\text{-Zip}^\mu$.

**Definition 5.2.2.** A $\Delta^P$-filtered $L_{\varphi}$-module is a pair $((\tau, V), \mathcal{F})$ where $\tau : L_{\varphi} \to \text{GL}_k(V)$ is a finite-dimensional representation of $L_{\varphi}$ and $\mathcal{F} = \{V^{\alpha}_{\geq r}\}_{\alpha \in \Delta^P}$ is a set of filtrations on $V$. Here, $V^{\alpha}_{\geq r}$ denotes a descending filtration $(V^{\alpha}_{\geq r})_{r \in \mathbb{R}}$.

Morphisms are given as follows. Let $((\tau, V), \mathcal{F})$ and $((\tau', V'), \mathcal{F}')$ be two $\Delta^P$-filtered $L_{\varphi}$-modules. Then a morphism $((\tau, V), \mathcal{F}) \to ((\tau', V'), \mathcal{F}')$ is a $k$-linear map $f : V \to V'$ that satisfies the following:

1. $f$ is an $L_{\varphi^*}$-equivariant morphism.
2. For each $\alpha \in \Delta^P$, the map $f$ is compatible with the filtrations $V_{\geq r}^{\alpha}$ and $V_{\geq r}^{\alpha'}$ in the sense that $f(V^{\alpha}_{\geq r}) \subset V^{\alpha'}_{\geq r}$ for any $r \in \mathbb{R}$.

We denote by $L_{\varphi^*}\text{-MF}_{\Delta^P}^{\text{adm}}$ the category of $\Delta^P$-filtered $L_{\varphi}$-modules.

Let $((\tau, V), N) \in L_{\varphi^*}\text{-MN}_{\Delta^P}$. For $\alpha \in \Delta^P$, define the $\alpha$-filtration $(V^{\alpha}_{\geq r})$ of $V$ as follows: Let $V = \bigoplus V_\nu$ be the weight decomposition of $V$. For all $r \in \mathbb{R}$, let $V^{\alpha}_{\geq r}$ be the direct sum of $V_\nu$ for all $\nu$ satisfying $\langle \nu, \delta_\alpha \rangle \geq r$. We call $V^{\alpha}_{\geq r}$ the $\alpha$-filtration of $V$. Thus, we have a functor $L_{\varphi^*}\text{-MN}_{\Delta^P} \to L_{\varphi^*}\text{-MF}_{\Delta^P}$. Taking composition, we obtain

$$
F_{\text{MF}} : \text{Rep}(L) \to \text{Rep}(P) \xrightarrow{F_{\text{MN}}} L_{\varphi^*}\text{-MN}_{\Delta^P} \to L_{\varphi^*}\text{-MF}_{\Delta^P}. \quad (5.2.1)
$$

**Definition 5.2.3.** A $\Delta^P$-filtered $L_{\varphi}$-module is called admissible if it is in the essential image of $F_{\text{MF}}$. We denote by $L_{\varphi^*}\text{-MF}_{\Delta^P}^{\text{adm}}$ the category of admissible $\Delta^P$-filtered $L_{\varphi}$-modules.

**Theorem 5.2.4.** The functor $\mathcal{V} : \text{Rep}(L) \to \mathfrak{GB}(G\text{-Zip}^\mu)$ factors through the functor $F_{\text{MF}} : \text{Rep}(L) \to L_{\varphi^*}\text{-MF}_{\Delta^P}^{\text{adm}}$ and induces an equivalence of categories

$$
L_{\varphi^*}\text{-MF}_{\Delta^P}^{\text{adm}} \to \mathfrak{GB}_L(G\text{-Zip}^\mu).
$$
Proof. By Theorem 5.1.5, it suffices to show

$$\text{Hom}_{L^\varphi - \text{MN}_{\Delta}^p} (F_{\text{MN}}(\rho), F_{\text{MN}}(\rho')) = \text{Hom}_{L^\varphi - \text{MF}_{\Delta}^p} (F_{\text{MF}}(\rho), F_{\text{MF}}(\rho'))$$

for $$(V, \rho), (V', \rho') \in \text{Rep}(L)$$. This follows from Remark 5.1.3 and the definitions of morphisms in $L^\varphi - \text{MN}_{\Delta}^p$ and $L^\varphi - \text{MF}_{\Delta}^p$.

6. Examples

6.1. The algebras $R_I$ and $R_\Delta$

Fix a connected reductive group $G$ over $\mathbb{F}_q$, a cocharacter $\mu: \mathbb{G}_{m,k} \to G_k$ and a frame $(B, T, z)$ for $\mathbb{Z}_\mu$ (Subsection 2.2.3). For $\lambda \in X^*_T(T)$, denote by $V_\Delta(\lambda)$ the $G$-representation $\text{Ind}_B^G(\lambda)$. We add a subscript $\Delta$ to avoid confusion with $V_I(\lambda) = \text{Ind}_{B_\lambda}^G(\lambda)$ for $\lambda \in X^*_T(T)$ (see §2.4.3). Let $V_\Delta(\lambda)$ be the vector bundle on $G$-Zip$^\mu$ attached to $V_\Delta(\lambda)$. We put

$$R_I = \bigoplus_{\lambda \in X^*_T(T)} H^0(G\text{-Zip}^\mu, V_I(\lambda)) \quad \text{and} \quad R_\Delta = \bigoplus_{\lambda \in X^*_T(T)} H^0(G\text{-Zip}^\mu, V_\Delta(\lambda)).$$

By (2.3.3), the $k$-vector spaces $R_I$ and $R_\Delta$ have a natural structure of $k$-algebra. They capture information about all $V_I(\lambda)$ and $V_\Delta(\lambda)$ at once.

Remark 6.1.1. In general, we do not know whether $R_I$ and $R_\Delta$ are finite-type algebras, but we conjecture that this is the case. The algebra $R_I$ was studied in [Kos19]. In the case of $G = \text{Sp}(4)$ with a cocharacter $\mu$ whose centraliser Levi subgroup is isomorphic to $G_2$, we showed that $R_I$ is a polynomial algebra in three indeterminates [Kos19, Theorem 5.4.1].

In this first example, we examine $R_\Delta$ in the case of $G = \text{SL}_2, \mathbb{F}_q$ with the zip datum explained in Subsection 4.2. In this case, the algebra $R_I$ is very simple: It is a polynomial algebra in one indeterminate, generated by the classical Hasse invariant. Let $n \in \mathbb{N}$. The representation $V_\Delta(\chi_n)$ identifies with $\text{Sym}^n(\text{Std})$. The weights of $V_\Delta(\chi_n)$ are $\{-n + 2i \mid 0 \leq i \leq n\}$. By Example 4.4.2 and Proposition 4.4.3, we have

$$H^0(G\text{-Zip}^\mu, V_\Delta(\chi_n)) = \bigoplus_{i \in -(q-1)\mathbb{N}, (q+1)i \leq -(q-1)n} V_\Delta(\chi_n)_i$$

(6.1.1)

for all $n \geq 0$. Let $x, y$ be indeterminates. Let $\text{SL}_2$ act on $k[x, y]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P = P(ax + cy, bx + dy).$$

Then $V_\Delta(\chi_n) = \text{Sym}^n(\text{Std})$ is the subrepresentation of $k[x, y]$ spanned by homogeneous polynomials in $x, y$ of degree $n$. The highest weight vector is $x^n$. By (6.1.1), we have

$$H^0(G\text{-Zip}^\mu, V_\Delta(\chi_n)) = \text{Span}_k (x^j y^{n-j} \mid j \geq 0, q - 1|n - 2j, (q + 1)j \leq n).$$

Similarly, $R_\Delta$ is the subalgebra of $k[x, y]$ generated by $x^j y^{n-j}$ for all $0 \leq j \leq n$ with $q - 1|n - 2j$ and $(q + 1)j \leq n$.

Proposition 6.1.2. The algebra $R_\Delta$ is generated by $y^{q-1}$ and $xy^q$. In particular, it is a polynomial algebra in two indeterminates.

Proof. It is clear that $y^{q-1}$ and $xy^q$ are elements of $R_\Delta$. Let $n \geq 0$ and $0 \leq j \leq n$ such that $x^j y^{n-j} \in R_\Delta$. We can write $x^j y^{n-j} = (xy^q)^j y^{n-(q+1)j}$. Note that $n \geq (q + 1)j$ and $q - 1$ divides $n - (q + 1)j = n - 2j - (q - 1)j$. It follows that $x^j y^{n-j}$ lies in the subalgebra of $k[x, y]$ generated by $y^{q-1}$ and $xy^q$. □
We give an interpretation of these sections. In the case of \( G = \text{SL}_{2,\mathbb{F}_q} \), recall that for an \( \mathbb{F}_q \)-scheme \( S \), the groupoid \( G\text{-Zip}^\mu(S) \) consists of tuples \( \mathcal{H} = (\mathcal{H}, \omega, F) \) where

1. \( \mathcal{H} \) is a locally free \( \mathcal{O}_S \)-module of rank 2 with a trivialisation \( \text{det}(\mathcal{H}) \cong \mathcal{O}_S \),
2. \( \omega \subset \mathcal{H} \) is a locally free \( \mathcal{O}_S \)-submodule of rank 1 such that \( \mathcal{H}/\omega \) is locally free and
3. \( F : \mathcal{H}^{(q)} \to \mathcal{H} \) and \( V : \mathcal{H} \to \mathcal{H}^{(q)} \) are \( \mathcal{O}_S \)-linear maps satisfying the conditions \( \text{Ker}(F) = \text{Im}(V) = \omega^{(q)} \) and \( \text{Ker}(V) = \text{Im}(F) \).

Consider the flag space \( \mathcal{F}_G \) over \( G\text{-Zip}^\mu \) parametrising pairs \( (\mathcal{H}, \mathcal{L}) \) with \( \mathcal{L} \subset \mathcal{H} \) a locally free \( \mathcal{O}_S \)-submodule of rank 1 such that \( \mathcal{H}/\mathcal{L} \) is locally free. The natural projection map \( \pi_G : \mathcal{F}_G \to G\text{-Zip}^\mu \) is a \( \mathbb{P}^1 \)-fibration. For \( n \in \mathbb{Z} \), the push-forward \( \pi_G_* (\mathcal{L}^{-n}) \) coincides with the vector bundle \( \mathcal{V}_\Delta(\chi_n) \). Consider the map

\[
\mathcal{L} \subset \mathcal{H} \xrightarrow{V} \mathcal{H}^{(q)} \to (\mathcal{H}/\mathcal{L})^{(q)} \cong \mathcal{L}^{-q},
\]

where we used that \( \mathcal{H}/\mathcal{L} \cong \mathcal{L}^{-1} \) by the trivialisation \( \text{det}(\mathcal{H}) \cong \mathcal{O}_S \). We obtain a section of \( \mathcal{L}^{-(q+1)} \). It corresponds to the element \( xy^q \) in Proposition 6.1.2. On the other hand, the classical Hasse invariant \( Ha \in H^0(S, \omega_2^{-1}) \) is given by the map \( V : \omega \to \omega^{(q)} \cong \omega^q \). By sending \( Ha \) under the morphism

\[
\omega \subset \mathcal{H} \to \mathcal{H}/\mathcal{L} \cong \mathcal{L}^{-1},
\]

we obtain a section of \( \mathcal{L}^{-(q-1)} \). This section corresponds to \( y^{q-1} \) in Proposition 6.1.2.

### 6.2. Example on \( L \)-semisimplification

We give an example that shows that Proposition 3.6.4 does not hold in general without the assumption that \( P \) is defined over \( \mathbb{F}_q \). Let \( G = \text{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \text{SL}_{2,\mathbb{F}_{q^2}} \) and

\[
\mu : \mathbb{G}_{m,k} \to G_k = \text{SL}_{2,k} \times \text{SL}_{2,k}; \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \( Z_\mu = (G, P, L, Q, M, \varphi) \) be the associated zip datum. We have \( P = B_2 \times \text{SL}_{2,k}, L = T_2 \times \text{SL}_{2,k}, Q = \text{SL}_{2,k} \times B_2^* \) and \( M = \text{SL}_{2,k} \times T_2 \). We take \( (B, T) = (B_2 \times B_2, T_2 \times T_2) \) as a Borel pair and take a frame as in Lemma 2.2.3. Then \( \Delta^P \) consists of one root \( \alpha = \chi_2 \otimes \chi_0 \). We have

\[
L_{\varphi} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} x^q & y \\ 0 & x^{-q} \end{pmatrix} \in L \mid x \in \mathbb{F}_q^\times, y^q = 0 \right\}.
\]

We have

\[
\delta_\alpha = -\alpha^\vee - q\sigma(\alpha^\vee), \quad r_\alpha = \left( \frac{q^2 + 1}{q^2 - 1}, -\frac{(q^2 + 1)}{q(q^2 - 1)} \right),
\]

\[
(Z_2^\times)_{r_\alpha} = \{(n_1, n_2) \in Z_2^2 \mid qn_1 = n_2\}.
\]

We define \( \rho : P \to \text{GL}(V) \) by

\[
\left( \text{Sym}^{q^2 - 1}(\text{Std} \otimes X_{q^2 - 1}) \right) \otimes \text{Sym}^{q^2 - 1}(\text{Std}^{(q)})\).
\]

We write \( (V', \rho') \) for \( (\text{V}^{L_{\text{ss}}}, \rho^{L_{\text{ss}}}) \). Then we have \( V_{L_{\varphi}} = V \) and \( V'^{L_{\varphi}} = V' \). We put \( \nu = \chi_0 \otimes \chi_{-q(q^2 - 3)} \). We have

\[
V_{[\nu]} = V_{\nu} \oplus V_{\nu + \alpha - q\sigma(\alpha)}.
\]
We parametrise elements \([j] \in \mathbb{Z}^2/(\mathbb{Z}^2)_{\alpha}\) by classes \([(0, j)]\) with \(j \in \mathbb{Z}\). Using this notation, we have

\[
\text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V_{[\nu]} = \bigcap_{j \in \mathbb{Z}} \bigcap_{\chi \in \mathbb{Z}^2/(\mathbb{Z}^2)_{\alpha}} \text{Ker} \left( \sum_{j, i \in \mathbb{Z}} \text{pr}_\chi \circ E^{(j_1)} \circ E^{(j_2+q)} : V_{[\nu]} \to V_{\chi} \right)
\]

because \((-1)_{j_1}(-1)_{j_2+q} = (-1)^j \in k\). We have \(V_\chi \neq 0\) if and only if \(\chi = \nu + i_1 \alpha + q_i \sigma(\alpha)\) for \(0 \leq i_1 \leq q_2 - 1\) and \(-1 \leq i_2 \leq q_2 - 2\). For \(\chi = \nu + i_1 \alpha + q_i \sigma(\alpha)\), the conditions \(\chi \in \mathbb{Z}^2/(\mathbb{Z}^2)_{\alpha}\) and \(j r_{\alpha, 2} > \delta_\alpha(\chi)\) hold if and only if \(j = q(i_1 + i_2)\) and \(i_2 - i_1 > q^2 - 2 - 2/(q^2 - 1)\). Hence,

\[
\chi \in [\nu + j \sigma(\alpha)], j r_{\alpha, 2} > \delta_\alpha(\chi), V_\chi \neq 0 \iff \chi = \nu + q(q^2 - 2) \sigma(\alpha), j = q(q^2 - 2).
\]

We put \(\chi_0 = \nu + q(q^2 - 2) \sigma(\alpha)\) and \(j_0 = q(q^2 - 2)\). Then we have

\[
\text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V_{[\nu]} = \text{Ker} \left( \text{pr}_{\chi_0} \circ \left( E^{(j_0)}(\alpha) + E^{(1)}(\alpha) \right) : V_{[\nu]} \to V_{\chi_0} \right) = \left\{ (v_1, v_2) \in V_{\nu} \oplus V_{\nu + q \sigma(\alpha)} \mid E^{(j_0)}(\alpha)(v_1) + E^{(1)}(\alpha)(v_2) = 0 \right\}.
\]

We note that

\[
E^{(j_0)}(\alpha) : V_{\nu} \to V_{\chi_0}, \quad E^{(1)}(\alpha) : V_{\nu + q \sigma(\alpha)} \to V_{\chi_0}
\]

are isomorphisms. In the same way, we have

\[
\text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V'_{[\nu]} = \text{Ker} \left( \text{pr}_{\chi_0} \circ E^{(j_0)}(\alpha) : V'_{[\nu]} \to V'_{\chi_0} \right) = V'_{\nu + q \sigma(\alpha)}
\]

using \(E^{(1)}(\alpha) = 0\) for \((V', \rho')\). Hence, \(\text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V_{[\nu]} \subset \text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V'_{[\nu]}\). Therefore, we have \(H^0(G, \text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V_{[\nu]}, \rho) \neq H^0(G, \text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V'_{[\nu]}, \rho)\).

### 6.3. The case of the unitary group \(U(2, 1)\) with \(p\) inert

In this subsection, we examine an example that arises in the study of Picard surfaces. These are Shimura varieties of PEL type (in particular, of Hodge type) attached to unitary groups \(G\) over \(\mathbb{Q}\) with respect to some totally imaginary quadratic extension \(E/\mathbb{Q}\). We impose that \(G_{\mathbb{R}} = \text{GU}(2, 1)\). We choose a rational prime \(p\) that is inert in \(E\) and consider the attached zip datum \((G, P, Q, L, M, \phi)\). Because \(p\) is inert, the parabolic \(P\) is not defined over \(\overline{\mathbb{F}}_p\). We study the space \(H^0(G, \text{Fil}^{\tilde{E}_{\alpha}, a_{\alpha}, r_{\alpha}} V_{[\nu]})\). To simplify, we will work with a unitary group \(U\), instead of a group of unitary similitudes \(\text{GU}\). The case of \(\text{GU}\) is very similar.

Let \((V, \psi)\) be a 3-dimensional vector space over \(\mathbb{F}_q\) endowed with a nondegenerate Hermitian form \(\psi : V \times V \to \mathbb{F}_q^2\) (in the context of Shimura varieties, take \(q = p\)). Write \(\text{Gal}(\mathbb{F}_q^2/\mathbb{F}_q) = \{\text{Id}, \sigma\}\). We take a basis \(B = (v_1, v_2, v_3)\) of \(V\) where \(\psi\) is given by the matrix

\[
J = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

We define a reductive group \(G\) by

\[
G(R) = \{ f \in \text{GL}_{\mathbb{F}_q^2} (V \otimes_{\mathbb{F}_q} R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y), \forall x, y \in V \otimes_{\mathbb{F}_q} R \}
\]

for any \(\mathbb{F}_q\)-algebra \(R\). One has an identification \(G_{\mathbb{F}_q^2} \cong \text{GL}(V)\), given as follows: For any \(\mathbb{F}_q^2\)-algebra \(R\), we have an \(\mathbb{F}_q^2\)-algebra isomorphism \(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q^2} R \to V \otimes_{\mathbb{F}_q} R \), \(a \otimes x \mapsto (ax, \sigma(a)x)\). By tensoring with \(V\), we obtain an isomorphism \(V \otimes_{\mathbb{F}_q} R \to (V \otimes_{\mathbb{F}_q} R) \oplus (V \otimes_{\mathbb{F}_q} R)\). Then any element of \(G(R)\)
stabilises this decomposition and is entirely determined by its restriction to the first summand. This yields an isomorphism as claimed. Using the basis $\mathcal{B}$, we identify $G_{\mathbb{F}_q^2}$ with $\text{GL}_3,\mathbb{F}_q^2$. The action of $\sigma$ on the set $\text{GL}_3(k)$ is given as follows: $\sigma \cdot A = J\sigma(J)A^{-1}J$. Let $T$ denote the maximal diagonal torus and $B$ the lower-triangular Borel subgroup of $G_k$. Note that by our choice of the basis $\mathcal{B}$, the groups $B$ and $T$ are defined over $\mathbb{F}_q$. Identify $X'(T) = \mathbb{Z}^3$ such that $(k_1, k_2, k_3) \in \mathbb{Z}^3$ corresponds to the character $\text{diag}(x_1, x_2, x_3) \mapsto \prod_{i=1}^3 x_i^{k_i}$. The simple roots are $\Delta = \{e_1 - e_2, e_2 - e_3\}$, where $(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{Z}^3$.

Define a cocharacter $\mu : \mathbb{G}_{m,k} \to G_k$ such that $\mu$ is given by $x \mapsto \text{diag}(x, x, 1)$ via the identification $G_k \simeq \text{GL}_3,k$. Let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. Note that $P$ is not defined over $\mathbb{F}_q$. One has $I = \{e_1 - e_2\}$ and $\Delta^P = \{\alpha\}$ with $\alpha = e_2 - e_3$.

**Lemma 6.3.1.** Let $H$ be the function on $G_k$ defined by

$$H((x_i,j)_{1 \leq i, j \leq 3}) = x_1^q \Delta_1 - x_2^q \Delta_2 \quad \text{with} \quad \begin{cases} 
\Delta_1 = x_1,1x_2,2 - x_1,2x_2,1, \\
\Delta_2 = x_1,1x_2,3 - x_2,1x_1,3.
\end{cases}$$

The $\mu$-ordinary stratum $U_\mu \subset G_k$ is equal to the complement of the vanishing locus of $H$.

**Proof.** In this case, there is a unique $E$-orbit of codimension 1 by the first part of Theorem 2.2.4. Furthermore, this $E$-orbit is dense in $G_k \setminus U_\mu$ by the closure relation. Hence, it suffices to show that $H$ does not vanish on $U_\mu$. The group $E$ consists of pairs $(x, y) \in P \times Q$ with

$$x = \begin{pmatrix} a & b & 0 \\
0 & c & d \\
e & f & g \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} g^q & h & i \\
0 & d^q & b^q \\
0 & c^q & a^q \end{pmatrix}^{-1}.$$

Because $1 \in U_\mu$, the open $U_\mu$ consists of elements of the form $xy^{-1}$. We find

$$H(xy^{-1}) = (ag^q)^q g^q d^q (ad - bc) - (cg^q)^q g^q b^q (ad - bc) = g^{q^2+q} (ad - bc)^{q+1}.$$  

This expression is nonzero, so the result is proved. \hfill $\square$

We have

$$L_\varphi = \left\{ \begin{pmatrix} a & b \\
d & a^{-q} \end{pmatrix} \in L \left| a, d \in \mathbb{F}_q \setminus \{0\}, \quad d^{q+1} = 1, \quad b^q = 0 \right. \right\}.$$

The endomorphism $\varphi_* : X_*(T)_{\mathbb{R}} \to X_*(T)_{\mathbb{R}}$ is given by the matrix

$$\varphi_* = \begin{pmatrix} 1 & q \\
q & 1 \end{pmatrix}.$$

Hence, it follows that $\delta_\alpha = \varphi_*^{-1}(\alpha^\vee) = \frac{1}{q^2-1}(-q, q - 1, 1)$. We have $m_\alpha = 2$, $a_\alpha = (-1, -1)$, $\Xi_\alpha = (-\alpha, \sigma(\alpha))$ and

$$r_\alpha = \begin{pmatrix} q^2 - q + 1 \\
q^2 - 1 \\
-q^2 + q - 1 \\
q^2 - 1 \end{pmatrix}, \quad (\mathbb{Z}^2)_{r_\alpha} = \{(n_1, n_2) \in \mathbb{Z}^2 \mid qn_1 = n_2\}.$$

The group $\Lambda_{\Xi_\alpha, r_\alpha}$ is

$$\Lambda_{\Xi_\alpha, r_\alpha} = \mathbb{Z}(q, -(q + 1), 1).$$

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 08 Aug 2021 at 15:38:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/fms.2021.32
Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) be an \( L \)-dominant character (i.e., \( \lambda_1 \geq \lambda_2 \)) and consider the \( L \)-representation \( V_l(\lambda) \). We simply write \( V \) for \( V_l(\lambda) \) sometimes. Under the isomorphism

\[
\text{GL}_2 \times \mathbb{G}_m \to L; \ (A, z) \mapsto \begin{pmatrix} A & \cdot \\ \cdot & z \end{pmatrix},
\]

the representation \( V \) corresponds to the representation

\[
\text{det}_{\text{GL}_2}^\lambda \otimes \text{Sym}^{\lambda_1-\lambda_2}(\text{Std}_{\text{GL}_2}) \otimes \xi_{\lambda_3},
\]

where \( \xi_r \) is the character of \( \text{GL}_2 \times \mathbb{G}_m \) given by \( (A, z) \mapsto z^r \). Hence, \( V \) is a representation of dimension \( \lambda_1 - \lambda_2 + 1 \) and it has weights

\[
v_i := (\lambda_1 - i, \lambda_2 + i, \lambda_3), \quad 0 \leq i \leq \lambda_1 - \lambda_2.
\]

Note that the difference \( v_i - v_{i'} \) of two weights is never in \( \Lambda_{\xi_\alpha, r_\alpha} = \mathbb{Z}^2/(\mathbb{Z}^2)_{r_\alpha} \) unless \( i = i' \). Therefore, \( V_{|\nu|} = V_{\nu} \) for all \( \nu \in \mathbb{Z}^3 \). One deduces

\[
V^{L_\nu} = \bigoplus_{q|\nu, q^2-1<\lambda_1-i-q\lambda_3} V_{v_i}.
\]

It remains to determine \( \text{Fil}^{\xi_\alpha, a_\alpha, r_\alpha} V_\nu \), which is either 0 or \( V_\nu \). We parametrise elements \( [j] \in \mathbb{Z}^2/(\mathbb{Z}^2)_{r_\alpha} \) by classes \( [(0, j)] \) with \( j \in \mathbb{Z} \). Then, an element \( j \in [j] \) can be written as \( (0, j) + j_1(1, q) \) with \( j_1 \in \mathbb{Z} \). Using this notation, we obtain

\[
\text{Fil}^{\xi_\alpha, a_\alpha, r_\alpha} V_\nu = \bigcap_{j>q(\nu, \delta_\alpha)} \ker \left( \sum_{j_1 \in \mathbb{Z}} \text{pr}_\chi \circ E_{\nu}^{(j_1)} \circ E_{r_\alpha}^{(j+q_\nu j_1)} : V_\nu \to V_\chi \right)
\]

because \((-1)^j (-1)^{q_\nu j_1} = (-1)^j \in k \). We have \( E_{\nu}^{(j_1)} = 0 \) unless \( j_1 = 0 \) because \( \alpha \in \Delta P \) and \( V \) is trivial on \( R_u(P) \). Hence, in the sum appearing in the above formula, only the case \( j_1 = 0 \) contributes. Furthermore, \( E_{r_\alpha}^{(j)}(V_\nu) \subset V_{\nu+j(\sigma(\alpha))} \). Hence, we have

\[
\text{Fil}^{\xi_\alpha, a_\alpha, r_\alpha} V_\nu = \bigcap_{j>q(\nu, \delta_\alpha)} \ker \left( E_{r_\alpha}^{(j)} \circ E_{r_\alpha}^{(j+q_\nu j_1)} : V_\nu \to V_{\nu+j(\sigma(\alpha))} \right).
\]

Take \( \nu = v_i \) for some \( 0 \leq i \leq \lambda_1 - \lambda_2 \). We deduce \( \text{Fil}^{\xi_\alpha, a_\alpha, r_\alpha} V_{v_i} = V_{v_i} \) if and only if for all \( j \geq 0 \) such that \( j > q(\nu, \delta_\alpha) \), one has \( E_{r_\alpha}^{(j)}(V_{v_i}) = 0 \). Computing explicitly the representation \( V \), one sees that this space is zero if and only if the binomial coefficient \( \binom{i}{j} \) is divisible by \( p \). In particular, it is never zero for \( j = i \). We deduce that

\[
\text{Fil}^{\xi_\alpha, a_\alpha, r_\alpha} V_{v_i} = V_{v_i} \iff i \leq q(\nu, \delta_\alpha).
\]

Furthermore, we find

\[
\langle v_i, \delta_\alpha \rangle = \frac{i(2q - 1)}{q^2 - 1} + \frac{1}{q^2 - 1} (-q \lambda_1 + (q - 1) \lambda_2 + \lambda_3).
\]

For \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \in X_{\ast, r}(T) \), we put

\[
F(\lambda) = \frac{q}{q^2 - q + 1} (q \lambda_1 - (q - 1) \lambda_2 - \lambda_3).
\]

We deduce the following.
Proposition 6.3.2. We have

\[ H^0(G\text{-}\text{Zip}^\mu, V_1(\lambda)) = \bigoplus_{q | i, q+1|\lambda_2+i, q^2-1|\lambda_1-i-q\lambda_3, i \geq F(\lambda)} V_I(\lambda)_v. \] (6.3.1)

(1) For example, take \( \lambda = (1 + q, 1, q) \). Then one sees that \( V_I(\lambda)_{L^c} = V_I(\lambda)_{V_q} \), where \( v_q = (1, 1 + q, q) \). One finds \( F(\lambda) = q \); hence, \( H^0(G\text{-}\text{Zip}^\mu, V_I(\lambda)) = V_I(\lambda)_{V_q} \).

(2) Similarly, take \( \lambda = (1, 0, q) \). Then we find \( V_I(\lambda)_{L^c} = V_I(\lambda)_{V_0} \), where \( v_0 = \lambda = (1, 0, q) \). We have \( F(\lambda) = 0 \); hence, again \( H^0(G\text{-}\text{Zip}^\mu, V_I(\lambda)) = V_I(\lambda)_{V_0} \).

(3) Take \( \lambda = (q + 1, q + 1, q^2 + q) \). Then \( V_I(\lambda) \) is a 1-dimensional representation of \( L \) (i.e., a character), and \( V_I(\lambda)_{L^c} = V_I(\lambda) \). Because \( F(\lambda) = -q(q^2 - 1)/(q^2 - q + 1) < 0 \), we have \( H^0(G\text{-}\text{Zip}^\mu, V_I(\lambda)) = V_I(\lambda) \). It is spanned by the \( \mu \)-ordinary (nonclassical) Hasse invariant \( H \) given by Lemma 6.3.1, also constructed in [GN17] and [KW18].

Recall the cone \( C_{\text{zip}} \subset X^*_{+, L}(T) \) studied in [Kos19], [GK18], defined as the set of \( \lambda \in X^*(T) \) such that \( H^0(G\text{-}\text{Zip}^\mu, V_I(\lambda)) \neq 0 \). In this example, we deduce that it is the set of \( \lambda \in X^*_{+, L}(T) \) such that there exists \( 0 \leq i \leq \lambda_1 - \lambda_2 \) satisfying the four conditions listed below the direct sum sign of (6.3.1). For a cone \( C \subset X^*(T) \), write \( \langle C \rangle \) for the saturated cone of \( C \); that is, the set of \( \lambda \in X^*(T) \) such that \( N\lambda \) lies in \( C \) for some positive integer \( N \).

Corollary 6.3.3. We have

\[ \langle C_{\text{zip}} \rangle = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 \mid \lambda_1 \geq \lambda_2, (q - 1)\lambda_1 + \lambda_2 - q\lambda_3 \leq 0 \}. \]

Proof. Assume that \( \lambda \in C_{\text{zip}} \). Then, in particular, \( \lambda_1 - \lambda_2 \geq F(\lambda) \), which amounts to \( (q - 1)\lambda_1 + \lambda_2 - q\lambda_3 \leq 0 \). Conversely, assume that \( \lambda \in X^*_{+, L}(T) \) satisfies \( \lambda_1 - \lambda_2 \geq F(\lambda) \). Then, after changing \( \lambda \) to \( q(q^2 - 1)\lambda \), we find that \( i = \lambda_1 - \lambda_2 \) satisfies the four conditions below the direct sum sign of (6.3.1); hence, \( \lambda \in \langle C_{\text{zip}} \rangle \). This terminates the proof. \( \square \)

Remark 6.3.4. The two sections of weight \( (1 + q, 1, q) \) and \( (1, 0, q) \) given in (1) and (2) are partial Hasse invariants (viewing them as a section of the stack of zip flags \( G\text{-}\text{ZipFlag}^\mu \), their vanishing locus is a single flag stratum; see [Kos19, §1.3] for details). Their weights generate the cone \( \langle C_{\text{Sm}} \rangle \) defined in [Kos19, Definition 1.7.1]. The cone \( \langle C_{\text{zip}} \rangle \) is not spanned by these weights because \( G \) does not satisfy the equivalent conditions of [Kos19, Lemma 2.3.1]. We also refer to [GIK21] for a general study of the cone \( C_{\text{zip}} \) as well as related results.

Acknowledgements. The authors thank the referee for helpful comments and suggestions. This work was supported by JSPS KAKENHI Grant Numbers 18F18311 and 18H01109.

Conflict of Interest: None.

References


