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WEAK INJECTIVITY AND CONGRUENCE EXTENSION IN CONGRUENCE-DISTRIBUTIVE EQUATIONAL CLASSES

BRIAN A. DAVEY

There are many concepts which arise naturally in a discussion of injectivity in an equational class; for example, weak injective algebras, absolute subretracts, essential extensions, the congruence extension property, and the amalgamation property (see [**3**; **9**; **17**; **18**]). It has already been demonstrated in several papers, notably [**9**; **17**; **26**; **27**; **28**], that the study of these concepts is greatly enriched by the assumption that the algebras under consideration have distributive congruence lattices. In this work attention is focused on weak injective algebras (Section 2) and the congruence extension property (Section 3).

In Section 1 our terminology is introduced, Jónsson's lemma and its immediate corollaries are stated, and a diagramatic interpretation of Jónsson's lemma is given; this *Jónsson Diagram* is the basis of all of our proofs.

Our aim in Sections 2 and 3 is to reduce considerations of weak injectivity and congruence extension to the subdirectly irreducible algebras. For example, we prove (Theorem 2.2) that if **K** is a congruence distributive equational class whose subdirectly irreducible members form an axiomatic class, then a subdirectly irreducible member of **K** is a weak injective in **K** provided it is a weak injective "within" the class of subdirectly irreducible algebras. This result is then applied to prove (Theorem 2.5) that if **K** is a congruence-distributive equational class generated by a finite simple algebra A, then the weak injectives in **K** are precisely the Boolean extensions of A by complete Boolean algebras. The main result of Section 3 (Theorem 3.3) states that if **K** is a congruence-distributive equational class whose subdirectly irreducible algebras form an axiomatic class, then **K** satisfies the congruence extension property if and only if the subdirectly irreducible members of **K** satisfy the congruence extension property. The paper closes with a discussion, in Section 4, of some applications of the results.

1. Preliminaries. For any class **K** of universal algebras (of the same type), let Equ(**K**) be the equational class generated by **K**, let Si(**K**) be the class consisting of all subdirectly irreducible members of **K**, let $I(\mathbf{K})$, $H(\mathbf{K})$, and $S(\mathbf{K})$ be the classes consisting of all isomorphic copies, homomorphic images, and subalgebras of members of **K**, and let $P(\mathbf{K})$, $P_S(\mathbf{K})$ and $P_U(\mathbf{K})$ be the classes consisting of all direct products, subdirect products, and ultraproducts

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of members of **K**; if $\mathbf{K} = \{A\}$ we shall drop the brackets from around A. If $\mathscr{F}$ is a filter on X, then $\theta_{\mathscr{F}}$ is the congruence on $\prod(A_x|x \in X)$ given by

$$a \equiv b(\theta_{\mathscr{F}}) \Leftrightarrow \{x \in X | a(x) = b(x)\} \in \mathscr{F}.$$

The reduced product $\prod(A_x|x \in X)/\theta_{\mathscr{F}}$ will be denoted by $\prod_{\mathscr{F}}(A_x|x \in X)$. For a discussion of reduced products and ultraproducts (otherwise known as *prime products*) we refer to T. Frayne, A. C. Morel, and D. Scott [**11**] and to G. Grätzer [**16**], to which we also refer for all the standard terminology and results of universal algebra. If $\alpha : B \rightarrowtail C$ is a monomorphism and θ is a congruence on C, then the restriction, $\theta \upharpoonright B$, of θ to B is defined by

$$a \equiv b(\theta \upharpoonright B) \Leftrightarrow a\alpha \equiv b\alpha(\theta).$$

All of our results rely heavily on Jónsson's lemma and its many powerful corollaries (see B. Jónsson [**22**]).

1.1 Jónsson's Lemma. *Let B be a subalgebra of the direct product of the family $(A_x|x \in X)$ and assume that B has a distributive congruence lattice. If θ is a completely meet-irreducible congruence on B, then there is an ultrafilter $\mathscr{F}$ on X such that $\theta_{\mathscr{F}} \upharpoonright B \leqq \theta$.*

If every algebra in a class **K** has a distributive congruence lattice, then we say that **K** is *congruence distributive.* The immediate corollaries of Jónsson's lemma are collected together in the following remark.

1.2 *Remark.* Let **A** be a class of algebras and assume that $\mathbf{K} = \mathrm{Equ}(\mathbf{A})$ is congruence distributive.

(i) Since a congruence θ on B is completely meet irreducible if and only if B/θ is subdirectly irreducible, it follows that $\mathrm{Si}(\mathbf{K}) \subseteq HSP_U(\mathbf{A})$ and hence $\mathbf{K} = IP_SHSP_U(\mathbf{A})$.

(ii) If **A** is an axiomatic class (i.e., there is a set Σ of first-order sentences such that **A** is the class consisting of all models of Σ), then $P_U(\mathbf{A}) = \mathbf{A}$, and hence $\mathrm{Si}(\mathbf{K}) \subseteq HS(\mathbf{A})$ and $\mathbf{K} = IP_SHS(\mathbf{A})$.

(iii) If **A** is a finite set of finite algebras, then $P_U(\mathbf{A}) = I(\mathbf{A})$, and again $\mathrm{Si}(\mathbf{K}) \subseteq HS(\mathbf{A})$ and $\mathbf{K} = IP_SHS(\mathbf{A})$.

(iv) If **A** is a finite set of finite algebras, then up to isomorphism **K** has only a finite set of subdirectly irreducible algebras and they are all finite.

The situation described below is of fundamental importance to our considerations. Assume that **K** is congruence distributive; let $B, C \in \mathbf{K}$, and let $\alpha : B \rightarrowtail C$ be a monomorphism. By Birkhoff's subdirect-representation theorem we may express C as a subdirect product of subdirectly irreducible algebras; let $\beta : C \rightarrowtail \prod(A_x|x \in X)$ be such a representation. Now let θ be a completely meet-irreducible congruence on B. By Jónsson's lemma there is an ultrafilter $\mathscr{F}$ on X such that $\theta_{\mathscr{F}} \upharpoonright B \leqq \theta$. Thus we obtain the commutative diagram below,

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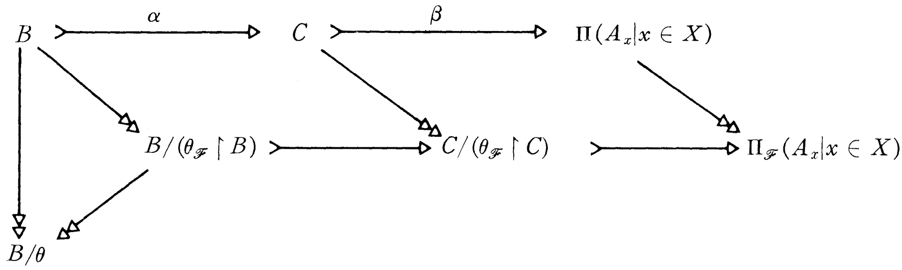
hereafter called a *Jónsson Diagram*; all unlabelled homomorphisms are "natural".



A Jónsson Diagram

1.3 *Remark.* We are often interested in extending congruences on B to congruences on C or $\prod(A_x|x \in X)$. The following observations are most relevant.

(i) Since the congruences on B form an algebraic lattice, for every congruence θ on B there is a family $(\theta_j|j \in J)$ of completely meet-irreducible congruences such that $\theta = \bigwedge(\theta_j|j \in J)$; which amounts to saying that B/θ is a subdirect product of the family $(B/\theta_j|j \in J)$ of subdirectly irreducible algebras.

(ii) For every family $(\Phi_j|j \in J)$ of congruences on C,

$$\bigwedge(\Phi_j|j \in J) \restriction B = \bigwedge(\Phi_j \restriction B|j \in J).$$

(iii) For every family $(\mathscr{F}_j|j \in J)$ of filters on X,

$$\bigwedge(\theta_{\mathscr{F}_j}|j \in J) = \theta_{\bigwedge(\mathscr{F}_j|j \in J)}.$$

N.B. Unless otherwise stated, all classes of algebras considered below are assumed to be *congruence distributive.*

2. Weak Injectives and absolute subretracts. The concept of weak injectivity was introduced by G. Grätzer and H. Lakser in [**18**] in order to extend the results of R. Balbes and G. Grätzer [**2**] on injective Stone algebras to other equational classes of distributive pseudocomplemented lattices. In both [**2**] and [**18**] the (weak) injectives are described indirectly in terms of Boolean extensions. The first explicit use of Boolean extensions to describe injective algebras occurs in A. Day's thesis (see [**9**]). The results of this section were inspired by Day's work. The approach taken here is greatly influenced by the author's belief that Boolean extensions should be viewed as algebras of continuous functions, and hence our techniques are quite different to those used in [**9**].

Let **K** be a class of algebras; then an algebra $I \in$ **K** is an *injective* [a *weak injective*] in **K** if for each algebra $C \in$ **K**, each subalgebra B of C, and every homomorphism [epimorphism] $\lambda: B \to I$, there exists an extension $\bar{\lambda}: C \to I$; that is, $\bar{\lambda} \restriction B = \lambda$. And I is an *absolute subretract* in **K** if it is a retract of each of its extensions in **K**. Clearly, if I is an injective then it is a weak injective, and if I is a weak injective then it is an absolute subretract. For discussions of various aspects of injectivity in equational classes we refer the reader to [**3**; **8**; **9**; **17**; **18**].

2.1 PROPOSITION. *A subdirectly irreducible member of an equational class* $\mathbf{K}$ *is a weak injective in* $\mathbf{K}$ *if and only if it is a weak injective in* $P_U\mathrm{Si}(\mathbf{K})$.

Proof. Assume that A is subdirectly irreducible and is a weak injective in $P_U\mathrm{Si}(\mathbf{K})$. Let $C \in \mathbf{K}$ with B a subalgebra of C and let $\lambda : B \twoheadrightarrow A$ be an epimorphism. Then $\theta = \ker \lambda$ is a completely meet-irreducible congruence on B and A is isomorphic to B/θ; thus a Jónsson Diagram results.

Clearly it is sufficient to find a homomorphism $\delta : \prod_{\mathscr{F}}(A_x|x \in X) \rightarrow B/\theta$ such that the lower triangle commutes; but, since B/θ is isomorphic to A, such a homomorphism is provided by the weak injectivity of A in $P_U\mathrm{Si}(\mathbf{K})$.

Our first theorem follows immediately.

2.2 THEOREM. *Let* $\mathbf{K}$ *be an equational class and assume that* $\mathrm{Si}(\mathbf{K})$ *is axiomatic. Then a subdirectly irreducible member of* $\mathbf{K}$ *is a weak injective in* $\mathbf{K}$ *if and only if it is weak injective in* $\mathrm{Si}(\mathbf{K})$.

2.3 COROLLARY. *Assume that* $\mathbf{K} = \mathrm{Equ}(\mathbf{A})$ *where* $\mathbf{A}$ *is a finite set of finite algebras. Then a subdirectly irreducible member of* $\mathbf{K}$ *is a weak injective in* $\mathbf{K}$ *if and only if it is a weak injective in* $\mathrm{Si}(\mathbf{K})$.

A simple application of 1.2 (iii) yields our final corollary.

2.4 COROLLARY. *Assume that* $\mathbf{K} = \mathrm{Equ}(A)$ *for some finite subdirectly irreducible algebra* A. *Then* A *is a weak injective in* $\mathbf{K}$.

In general (without the assumption of congruence distributivity) a maximal subdirectly irreducible member of $\mathbf{K}$ is an absolute subretract; by 1.2 (iii) congruence distributivity guarantees that A is maximal in $\mathbf{K} = \mathrm{Equ}(A)$ and hence A is an absolute subretract in $\mathbf{K}$. In general the congruence extension property is required in order to guarantee that an absolute subretract is a weak injective. Corollary 2.4 shows that if $\mathbf{K}$ is congruence distributive, then, as far as A is concerned, the congruence extension property is unnecessary; indeed, as the equational class generated by the lattice M_3 illustrates, the assumptions of 2.4 may hold while the congruence extension property fails. The interrelation between congruence distributivity and the congruence extension property will be investigated in the next section.

If A is a finite algebra and B is a Boolean Algebra, then the algebra $C(X, A)$ of continuous functions from the Boolean space X of ultrafilters of B into the discrete space A is called the *Boolean extension* of A by B and is denoted by $A[B]$. This differs from, but is equivalent to, the definition given in [**16**]. For the relevant results on Boolean algebras and Boolean spaces we refer the reader to P. R. Halmos [**20**]. In his study [**9**], of injectivity in congruence-distributive equational classes, A. Day employed a categorical approach; we could do the same here, but little would be gained. For the interested reader we make the observation that for any finite algebra A an adjoint to the (contravariant) functor $C(-, A) : \mathbf{BSp} \to \mathbf{K}$ from the category **BSp** of Boolean spaces to $\mathbf{K} = \mathrm{Equ}(A)$, is provided by the hom-set functor $\mathbf{K}(-, A) : \mathbf{K} \to \mathbf{BSp}$.

We now state the main theorems of this section.

2.5 THEOREM. *Assume that* $\mathbf{K} = \mathrm{Equ}(A)$ *for some finite simple algebra* A. *Then* I *is a weak injective in* $\mathbf{K}$ *if and only if it is isomorphic to* $A[B]$ *for some complete Boolean algebra* B.

The particular case of this result where A is the triangle is proved in E. Fried and G. Grätzer [**14**]. In R. W. Quackenbush [**28**], where several of the results of [**14**] are placed in a universal-algebraic setting, the theorem above is stated with the added assumption that A is a weak injective in $\mathbf{K}$. Corollary 2.4 above shows that this assumption is unnecessary, and hence the theorem follows from Quackenbush's result. Quackenbush calls on the results of A. Day [**9**]; below we give a proof of Theorem 2.5 which is independent of Day's work.

As was already mentioned, if $\mathbf{K}$ has the congruence extension property, then I is a weak injective in $\mathbf{K}$ if and only if it is an absolute subretract in $\mathbf{K}$. But since A is simple, if $\mathbf{K}$ satisfies the congruence extension property then by 1.2 (iii) we have $\mathrm{Si}(\mathbf{K}) \subseteq IS(A)$ and hence $\mathbf{K} = ISP(A)$. By restricting our attention to $ISP(A)$ rather than $\mathrm{Equ}(A) = HSP(A)$ we can avoid the congruence extension property. The following result was announced in [**7**]. The author would like to thank Harry Lakser for calling attention to the results of R. W. Quackenbush [**28**] and for pointing out that with one addition to the proof of 2.6 (namely Corollary 2.10) Theorem 2.5 could be obtained.

2.6 THEOREM. *Assume that* A *is a finite simple algebra and let* $\mathbf{K} = ISP(A)$. *Then the following are equivalent:*

(i) *I is a weak injective in* $\mathbf{K}$;
(ii) *I is an absolute subretract in* $\mathbf{K}$;
(iii) *I is isomorphic to $A[B]$ for some complete Boolean algebra B.*

We shall prove Theorems 2.5 and 2.6 simultaneously.

Let $(A_x | x \in X)$ be a family of algebras and let $\phi : A \rightarrowtail \prod(A_x | x \in X)$ be an embedding of A as a subdirect product; if ϕ also embeds A as a retract of $\prod(A_x | x \in X)$, then A is called a *subdirect retract* of the family $(A_x | x \in X)$. The following lemma and its corollary are the essence of Proposition 4.4 of B. A. Davey [**8**]; congruence distributivity is not required.

2.7 LEMMA. *Let A be a finite algebra in an equational class* $\mathbf{K}$. *For every complete Boolean algebra B, $A[B]$ is a subdirect retract of copies of A.*

A subdirect retract of a family of [weak] injectives in an equational class $\mathbf{K}$ is itself a [weak] injective in $\mathbf{K}$ (see [**18**]).

2.8 COROLLARY. *Let A be a finite [weak] injective in an equational class* $\mathbf{K}$. *For every complete Boolean algebra B, $A[B]$ is a [weak] injective in* $\mathbf{K}$.

Thus, if A is a finite simple algebra and $\mathbf{K} = \text{Equ}(A)$ is congruence distributive, then (by 2.4) for every complete Boolean algebra B, $A[B]$ is a weak injective in $\mathbf{K}$ and hence is a weak injective in $ISP(A)$.

2.9 LEMMA. *Assume that* $\mathbf{K} = \text{Equ}(\mathbf{A})$ *where* $\mathbf{A}$ *is a finite set of finite algebras. Let I be a weak injective in* $\mathbf{K}$*; if A is a subdirectly irreducible homomorphic image of I, then $A \in H(\mathbf{A})$, and hence $I \in IP_S H(\mathbf{A})$.*

Proof. Since I is a weak injective in $\mathbf{K}$ and $\mathbf{K} = HSP(\mathbf{A})$, we have $I \in HP(\mathbf{A})$. Thus $A \in H(I)$ implies $A \in HHP(\mathbf{A}) = HP(\mathbf{A})$. By Jónsson's lemma, $A \in HP_U(\mathbf{A}) = H(\mathbf{A})$, as required.

2.10 COROLLARY. *Let* $\mathbf{K} = \text{Equ}(A)$ *for some finite simple algebra A. If I is a weak injective in* $\mathbf{K}$, *then $I \in IP_S(A) \subseteq ISP(A)$.*

Since every weak injective is an absolute subretract, to complete the proofs of 2.5 and 2.6 we must prove that every absolute subretract I in $ISP(A)$ is of the form $A[B]$ for some complete Boolean algebra B; i.e. since a Boolean algebra is complete if and only if its space of ultrafilters is extremally disconnected, we must prove that I is isomorphic to $C(X, A)$ for some extremally disconnected Boolean space X. The result is trivial if $|A| = 1$, so for the remainder of this section we assume that A is a finite simple algebra with $|A| \geqq 2$.

2.11 LEMMA. *If B is a diagonal subalgebra of A^X (i.e. B contains the constant maps) and θ is a congruence on B, then there is a filter $\mathscr{F}$ on X such that $\theta = \theta_{\mathscr{F}} \upharpoonright B$.*

Proof. By Remark 1.3 it is sufficient to consider the case in which θ is completely meet irreducible. Consider a Jónsson Diagram with $\mathrm{C} = \prod(A_x|x \in X)$ and $A_x = A$ for all $x \in X$. Since B is a diagonal subalgebra and A is finite it follows that

$$B/(\theta_{\mathscr{F}} \upharpoonright B) \simeq \textstyle\prod_{\mathscr{F}}(A_x|x \in X) \simeq A,$$

and hence $B/(\theta_{\mathscr{F}} \upharpoonright B)$ is simple. But B/θ is a nontrivial homomorphic image of $B/(\theta_{\mathscr{F}} \upharpoonright B)$, and consequently $\theta = \theta_{\mathscr{F}} \upharpoonright B$.

For every subset Y of X define a congruence θ_Y on A^X by

$$a \equiv b(\theta_Y) \Leftrightarrow a(x) = b(x) \quad \text{for all } x \in Y;$$

i.e. $\theta_Y = \theta_{[Y)}$, where $[Y) = \{U \subseteq X | Y \subseteq U\}$ is the principal filter determined by Y.

Let X be a Boolean space and let $\mathscr{F}$ be a filter on the set X. Then $\overline{\mathscr{F}} = \{U \in \mathscr{F} | U \text{ is clopen in } X\}$ is a filter of the Boolean algebra B of clopen subsets of X, and hence, since X is a Boolean space, there is a (unique) closed subset Y of X such that $\overline{\mathscr{F}} = [Y) \cap B$. For all $a, b \in C(X, A)$ the set

$\{x \in X | a(x) = b(x)\}$ is clopen in X and thus

$$\theta_{\mathcal{F}} \upharpoonright C(X, A) = \theta_Y \upharpoonright C(X, A).$$

Thus 2.11 yields the following result.

2.12 COROLLARY. *For every Boolean space X and each congruence θ on $C(X, A)$ there is a closed subset Y of X such that $\theta = \theta_Y \upharpoonright C(X, A)$.*

2.13 PROPOSITION. *Assume that X is a Boolean space, I is a subalgebra of $C(X, A)$, and I is an absolute subretract in $ISP(A)$. Then there is a closed subset Y of X such that the restriction map*

$$\phi: I \to C(Y, A) \quad \textit{given by} \quad a\phi = a \upharpoonright Y,$$

is an isomorphism.

Proof. Since I is an absolute subretract there is a retraction $\tau: C(X, A) \twoheadrightarrow I$, and by 2.12 there is a closed subset Y of X with $\ker \tau = \theta_Y \upharpoonright C(X, A)$. Assume that $a, b \in I$ with $a \neq b$; then $a\tau \neq b\tau$, and so $a \upharpoonright Y \neq b \upharpoonright Y$ since $\ker \tau = \theta_Y \upharpoonright C(X, A)$. Hence ϕ is one-one. Let $c \in C(Y, A)$. Since finite discrete spaces are injective in the category of Boolean spaces there exists $b \in C(X, A)$ with $b \upharpoonright Y = c$. Let $a = b\tau$; then $a \equiv b$ $(\ker \tau)$ and so $a \upharpoonright Y = b \upharpoonright Y = c$. Thus ϕ is onto.

2.14 COROLLARY. *If I is an absolute subretract in $ISP(A)$, then there is a Boolean space X such that I is isomorphic to $C(X, A)$.*

Proof. Since $I \in ISP(A)$ there is a set S such that I is isomorphic to a subalgebra of A^S. But A^S is isomorphic to $C(\beta S, A)$, and the Stone-Čech compactification βS of the discrete space S is a Boolean space. Since a closed subspace of a Boolean space is itself Boolean, the result follows immediately from 2.13.

2.15 COROLLARY. *If X is a Boolean space and $C(X, A)$ is an absolute subretract in $ISP(A)$, then X is extremally disconnected.*

Proof. Let X be a Boolean space and assume that $C(X, A)$ is an absolute subretract in $ISP(A)$. Let $\epsilon : E \twoheadrightarrow X$ be a continuous map from an extremally disconnected space E onto X (e.g. let E be the Stone-Čech compactification of the underlying set of X). Then ϵ induces an embedding

$$\phi : C(X, A) \rightarrowtail C(E, A) \quad \text{given by} \quad a\phi(e) = a(\epsilon(e)) \quad \text{for all } e \in E.$$

By 2.13 there is a closed subset Y of E such that the map

$$\mu : C(X, A) \to C(Y, A) \quad \text{given by} \quad a\mu = (a\phi) \upharpoonright Y,$$

is an isomorphism.

Since a continuous map from a compact space to a hausdorff space is a homeomorphism if and only if it is a bijection, and since a retract of an extremally disconnected space is extremally disconnected, it is sufficient to prove that $\epsilon \upharpoonright Y : Y \rightarrow X$ is a bijection. That $\epsilon \upharpoonright Y$ is one-one follows from the fact that μ is onto, and that $\epsilon \upharpoonright Y$ is onto follows from the fact that μ is one-one; the arguments are standard and are omitted.

With these two corollaries we conclude the proof of Theorems 2.5 and 2.6.

By combining 2.8 with Theorem 2.5 we obtain the following result of R. W. Quackenbush [**28**].

2.16 THEOREM. *Assume that* $\mathbf{K} = \text{Equ}(A)$ *from some finite simple algebra* A. *If* A *is injective in* $\mathbf{K}$, *then the following are equivalent:*
(i) *I is injective in* $\mathbf{K}$;
(ii) *I is a weak injecive in* $\mathbf{K}$;
(iii) *I is isomorphic to* $A[B]$ *for some complete Boolean algebra* B.

A class $\mathbf{K}$ has *enough injectives* if every member of $\mathbf{K}$ has an injective extension in $\mathbf{K}$. If $\mathbf{K}$ has enough injectives, then in $\mathbf{K}$ the concepts of injective, weak injective, and absolute subretract are equivalent (see [**3**]). The proof of the following result is easy and is omitted.

2.17 PROPOSITION. *Assume that* $\mathbf{K} = \text{Equ}(A)$ *for some finite simple algebra* A. *Then* $\mathbf{K}$ *has enough injectives if and only if* A *is injective in* $\mathbf{K}$ *and every subdirectly irreducible member of* $\mathbf{K}$ *is isomorphic to a subalgebra of* A (i.e. $\mathbf{K} = ISP(A)$).

By an argument similar to our proof of Proposition 2.1, A. Day [**9**] has shown that if A is a finite subdirectly irreducible algebra all of whose subalgebras are either weak injective in $\mathbf{K} = ISP(A)$ or subdirectly irreducible, and $\mathbf{K}$ is congruence distributive, then the following are equivalent: (i) $\mathbf{K}$ has enough injectives; (ii) A is injective in $\mathbf{K}$; (iii) A is self injective (i.e. every homomorphism from a subalgebra of A into A extends to an endmorphism of A). Since no nontrivial proper subalgebra of a simple algebra is a weak injective we obtain the following combination of 2.17 and Day's result.

2.18 PROPOSITION. *Assume that* $K = \text{Equ}(A)$ *for some finite simple algebra* A *all of whose nontrivial subalgebras are subdirectly irreducible. Then the following are equivalent:*
(i) $\mathbf{K}$ *has enough injectives;*
(ii) A *is injective in* $\mathbf{K}$ *and* $\mathbf{K} = ISP(A)$;
(iii) A *is self injective and* $\mathbf{K} = ISP(A)$.

3. The congruence extension property. In part, the results of this section were inspired by the results of E. Fried, G. Grätzer, and H. Lakser [**15**] and A. Day [**10**].

Let $\mathbf{K}$ be a class of algebras and let $B, C \in \mathbf{K}$, with B a subalgebra of C; a congruence θ on B *extends* to C if there is a congruence Φ on C such that $\Phi \upharpoonright B = \theta$. If for every subalgebra B of C the congruences on B extend to C, then C satisfies the *congruence extension property*; if every member of $\mathbf{K}$ satisfies the congruence extension property then we say that $\mathbf{K}$ satisfies the *congruence extension property*. The congruences on B are *extensile in* $\mathbf{K}$ if for each extension C of B, with $C \in \mathbf{K}$, the congruences on B extend to C.

Let $\mathbf{K}$ be a congruence-distributive equational class; we shall show that, provided $\mathrm{Si}(\mathbf{K})$ is reasonably well behaved, the extendability of congruences in $\mathbf{K}$ is determined by the subdirectly irreducible members of $\mathbf{K}$.

A subdirectly irreducible algebra A in $\mathbf{K}$ is called an *extensor* if for every subdirectly irreducible member C of $\mathbf{K}$ and each subalgebra B of C each congruence θ on B, with B/θ isomorphic to A, extends to C. For example, if A cannot be obtained as a proper homomorphic image of a subalgebra of a subdirectly irreducible member of $\mathbf{K}$, it is an extensor.

3.1 THEOREM. *Assume that* $\mathrm{Si}(\mathbf{K})$ *is axiomatic. Let B and C be members of K, with B a subalgebra of C, and let θ be a congruence on B. If B/θ is a subdirect product of extensors, then θ extends to C.*

Proof. By Remark 1.3 it is sufficient to show that every completely meet-irreducible congruence θ on B, for which B/θ is an extensor, extends to C. Again, a Jónsson Diagram describes the situation. Since $\mathrm{Si}(\mathbf{K})$ is axiomatic, $\prod_{\mathscr{F}}(A_x|x \in X)$ is subdirectly irreducible, and since B/θ is an extensor, there is a congruence Φ on $\prod(A_x|x \in X)$, with $\theta_{\mathscr{F}} \leqq \Phi$, such that $\Phi/\theta_{\mathscr{F}}$ extends $\theta/(\theta_{\mathscr{F}} \restriction B)$. It follows that $\Phi \restriction C$ extends θ.

3.2 COROLLARY. *Assume that* $\mathrm{Si}(\mathbf{K})$ *is axiomatic. Let $B \in \mathbf{K}$; if every subdirectly irreducible homomorphic image of B is an extensor, then the congruences on B are extensile in* $\mathbf{K}$.

The final corollary of 3.1 deserves a more prestigious title. This theorem generalizes a result of A. Day [**10**].

3.3 THEOREM. *Assume that* $\mathrm{Si}(\mathbf{K})$ *is axiomatic. Then the following are equivalent:*

(i) $\mathbf{K}$ *satisfies the congruence extension property;*

(ii) *each subdirectly irreducible member of* $\mathbf{K}$ *satisfies the congruence extension property;*

(iii) *each subdirectly irreducible member of* $\mathbf{K}$ *is an extensor.*

Of course, we could also state 3.1, 3.2, and 3.3 with the assumption that $\mathbf{K} = \mathrm{Equ}(\mathbf{A})$ for some finite set $\mathbf{A}$ of finite algebras; this is left to the reader. In fact, in this case we can slightly improve 3.3; since the proof involves only an application of Remark 1.3 and a Jónsson Diagram, similar to the proof of 3.1, it is omitted.

3.4 THEOREM. *Assume that* $\mathbf{K} = \mathrm{Equ}(\mathbf{A})$ *for some finite set* $\mathbf{A}$ *of finite algebras. If C is a subdirect product of subdirectly irreducible algebras which satisfy the congruence extension property, then C satisfies the congruence extension property.*

4. Some applications. The results of Section 2 may be used to obtain R. W. Quackenbush's description of the injectives and weak injectives in an equational class generated by a quasiprimal algebra A (see [**26**; **27**]); for example, let A be a finite simple cylindric algebra of dimension 1 (see [**21**]), a finite field (regarded as a ring), a finite chain (regarded as a double Heyting algebra; see [**29**] ,where Heyting algebras are referred to as pseudo-Boolean algebras), or a finite simple monadic algebra (see [**19**]). Every lattice, and more generally every lattice-ordered algebra, has a distributive congruence lattice. Thus our results may be applied to any equational class generated by a finite simple lattice; for example, we obtain R. Balbes's description of injective [bounded] distributive lattices (see [**1**]), and E. Fried, G. Grätzer, and H. Lakser's description of the weak injectives in the equation class $\mathbf{M}_n$ generated by M_n (see [**15**]). There are three non-trivial equational classes of double Stone algebras (see [**23**]); indeed, the subdirectly irreducible double Stone algebras are the 2-element chain C_2, the 3-element chain C_3, both of which are simple, and the 4-element chain C_4, which is not simple. Clearly C_2 and C_3 are are self injective, whence 2.16 and 2.18 provide descriptions of the injective algebras in $\mathrm{Equ}(C_2)$ and $\mathrm{Equ}(C_3)$. The injective algebras in the equational class $\mathrm{Equ}(C_4)$ of all double Stone algebras are described in T. Katriňák [**23**]. When applied to De Morgan algebras and Kleene algebras our results yield R. Cignoli's description of the injective algebras (see [**6**]).

The results of Section 3 have many applications also. For example from 3.2 we obtain the result of E. Fried, G. Grätzer, and H. Lakser [**15**] that if B is a lattice in the equational class $\mathbf{M}_n$ and B has no prime ideals, then the congruences on B are extensile in $\mathbf{M}_n$; in fact, the result holds in $\mathbf{M}_\omega$. (Let M_ω be a countable lattice of length 2 and let $\mathbf{M}_\omega$ be the equational class it generates; then $A \in \mathrm{Si}(\mathbf{M}_\omega)$ if and only if A is a 2-element chain or has length 2 and at least five elements, whence $\mathrm{Si}(\mathbf{M}_\omega)$ is axiomatic.)

Let $\mathbf{K}$ be the class of distributive pseudocomplemented lattices; then $A \in \mathrm{Si}(\mathbf{K})$ if and only if it is isomorphic to a Boolean algebra with a new unit adjoined (see [**25**]), and hence $\mathrm{Si}(\mathbf{K})$ is axiomatic. That every algebra in $\mathrm{Si}(\mathbf{K})$ satisfies the congruence extension property follows trivially from the fact that Boolean algebras satisfy the congruence extension property; thus, by 3.3, $\mathbf{K}$ has the congruence exsension property (see [**17**]).

Two further classes where the use of our results would be fruitful are orthomodular lattices (see [**4**; **5**]) and weakly associative lattices (see [**12**; **13**; **14**]).

References

1. R. Balbes, *Projective and injective distributive lattices*, Pacific J. Math. *21* (1967), 405–435.

2. R. Balbes and G. Grätzer, *Injective and projective Stone algebras*, Duke Math. J. *38* (1971), 339–347.

3. B. Banaschewski, *Injectivity and essential extensions in universal algebra*, Proc. Conf. in Universal Algebra, Queen's Univ., Kingston, Ont., 1969 (Queen's Univ., Kingston, Ont., 1970), pp. 131–147.

https://doi.org/10.4153/CJM-1977-048-1 Published online by Cambridge University Press

4. G. Bruns and G. Kalmbach, *Varieties of orthomodular lattices*, Can. J. Math. *23* (1971), 802–810.
5. ——— *Varieties of orthomodular lattices*, II, Can. J. Math. *24* (1972), 328–337.
6. R. Cignoli, *Injective De Morgan and Kleene algebras*, Proc. Amer. Math. Soc. *47* (1975), 269–278.
7. B. A. Davey, *Weak injectives and Boolean extensions*, Notices Amer. Math. Soc., *22* (1975), A-618. Abstract #75T-A224.
8. ——— *Dualities for equational classes of Brouwerian algebras and Heyting algebras*, Trans. Amer. Math. Soc. *221* (1976), 119–146.
9. A. Day, *Injectivity in equational classes of algebras*, Can. J. Math. *24* (1972), 209–220.
10. ——— *The congruence extension property and subdirectly irreducible algebras—an example*, Algebra Univ. *3* (1973), 229–237.
11. T. Frayne, A. C. Morel, and D. Scott, *Reduced direct products*, Fund. Math. *51* (1962), 195–228.
12. E. Fried, *Tournaments and nonassociative lattices*, Ann. Univ. Sci. Budapest.Eötvös Sect. Math. *13* (1970), 151–164.
13. ——— *Weakly associative lattices with congruence extension property*, Algebra Univ. *4* (1974), 151–162.
14. E. Fried and G. Grätzer, *A nonassociative extension of the class of distributive lattices*, Pacific J. Math. *49* (1973), 59–78.
15. E. Fried, G. Grätzer, and H. Lakser, *Amalgamation and weak injectives in the equational class of modular lattices* $\mathbf{M}_n$, Notices Amer. Math. Soc. *18* (1971), 624. Abstract #71T-A101.
16. G. Grätzer, *Universal algebra* (Van Nostrand, Princeton, N. J., 1968).
17. G. Grätzer and H. Lakser, *The structure of pseudocomplemented distributive lattices*. II. *Congruence extension and amalgamation*, Trans. Amer. Math. Soc. *156* (1971), 343–358.
18. ——— *The structure of pseudocomplemented distributive lattices*. III. *Injectives and absolute subretracts*, Trans. Amer. Math. Soc. *169* (1972), 475–487.
19. P. R. Halmos, *Algebraic logic* (Chelsea, New York, 1962).
20. ——— *Lectures on Boolean algebras*, Math. Studies No. 1. (Van Nostrand, Princeton, N. J., 1963).
21. L. Henkin and A. Tarski, *Cylindric algebras*, in *Lattice Theory*, Proc. of Symp. in Pure Math., II (Amer. Math. Soc., Providence, R. I., 1961), pp. 83–113.
22. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. *21* (1967), 110–121.
23. T. Katriňák, *Injective double Stone algebras*, Algebra Univ. *4* (1974), 259–267.
24. ——— *Congruence extension property for distributive double p-algebras*, Algebra Univ. *4* (1974), 273–276.
25. H. Lakser, *The structure of pseudocomplemented distributive lattices*. I. *Subdirect decomposition*, Trans. Amer. Math. Soc. *156* (1971), 334–342.
26. R. W. Quackenbush, *Demi-semi-primal algebras and Mal'cev type conditions*, Math. Z. *122* (1971), 166–176.
27. ——— *Structure theory for equational classes generated by quasi-primal algebras*, Trans. Amer. Math. Soc. *187* (1974), 127–145.
28. ——— *Semi-simple equational classes with distributive congruence lattices*, Ann. Univ. Sci. Budapest.Eötvös Sect. Math. *17* (1974), 15–19.
29. H. Rasiowa and R. Sikorski, *The mathematics of metamathematics* (Monag. Mat. Vol. 41, Pańtowe Wydawn, Nankowe, Warsaw, 1963).

La Trobe University,
Bundoora, Victoria, Australia