THE ITERATED EQUATION OF GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY

IV. CIRCLE THEOREMS

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1. Introduction

Milne-Thomson's well-known circle theorem [1] gives the stream function for steady two-dimensional irrotational flow of a perfect fluid past a circular cylinder when the flow in the absence of the cylinder is known. Butler's sphere theorem [2] gives the corresponding result for axially symmetric irrotational flow of a perfect fluid past a sphere. Collins [3] has obtained a sphere theorem for axially symmetric Stokes flow of a viscous liquid which gives a stream function satisfying the appropriate viscous boundary conditions on the surface of a sphere when the stream function for irrotational flow in the absence of the sphere is known.

In each of these theorems, the stream function satisfies an equation which is a special case of the iterated equation of generalized axially symmetric potential theory (GASPT). If r, θ are polar coordinates in the plane of the flow for two-dimensional flow or in any meridian plane in axially symmetric flow (θ being measured from the axis of symmetry), the operator L_k can be defined as

$$L_k \equiv \frac{\partial^2}{\partial r^2} + \frac{1+k}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{(1+k)\mu}{r^2}\frac{\partial}{\partial \mu}, \tag{1}$$

where $\mu = \cos\theta$. The iterated equation of GASPT is then

$$L_k^n(f) = 0. \tag{2}$$

In each of the three theorems mentioned above the stream function satisfies an equation obtained from (2) by giving k and n special values: $n = 1$, $k = 0$, -1 for Milne-Thomson's and Butler's theorems respectively, $n = 2$, $k = -1$ for Collins's theorem. In this paper a generalized circle theorem which holds for equation (2) is given and this includes these three theorems among its special cases.

Collins [4] has also given a sphere theorem in which the stream function for axially symmetric Stokes flow past a sphere is given in terms of the stream function for the corresponding Stokes flow in the absence of the sphere. This theorem can be extended so that it applies to solutions of equation (2) for general k and with $n = 2, 3, 4$ and it is conjectured that the theorem can be extended further to a general value of n.

These circle theorems give solutions of equation (2) satisfying given boundary conditions on the circle $r = a$ and valid in the region outside this circle; there are corresponding theorems which hold in the region inside this circle.

Circle theorems can be regarded as applications of the method of images which is more easily applied when the rigid boundary involved is a straight line rather than a circle and theorems applicable to a region bounded by a rigid straight line boundary are also obtained.

A discussion of the use of these circle theorems in specific physical problems is outside the scope of this paper but some well-known examples are mentioned briefly in the concluding section.

2. Generalized circle theorem

THEOREM 2.1. *If the function $f_0(r, \theta)$ is a solution of the equation $L_k(f) = 0$ which has all its singularities outside the circle $r = a$, then a solution $f_n(r, \theta)$ of the equation $L_k^n(f) = 0$ which has the same singularities outside $r = a$ as $f_0(r, \theta)$ and which satisfies the conditions*

$$f_n(a, \theta) = \frac{\partial f_n}{\partial r}(a, \theta) = \cdots = \frac{\partial^{n-1} f_n}{\partial r^{n-1}}(a, \theta) = 0 \tag{3}$$

is given by

$$f_n(r, \theta) = f_0(r, \theta) - f_0^*(r, \theta) \tag{4}$$

where

$$f_0^*(r, \theta) = \sum_{t=0}^{n-1} \frac{(-a)^t}{t!} \frac{\partial^t f_0}{\partial r^t}\left(\frac{a^2}{r}, \theta\right)\left(\frac{r}{a}\right)^{-k-t}\left(1-\frac{r^2}{a^2}\right)^t \Lambda^k_{n-1-t}\left(1-\frac{r^2}{a^2}\right) \tag{5}$$

and $\Lambda^k_m(\xi)$ is defined as follows:

for $k \neq 0$, $$\Lambda^k_m(\xi) \equiv \sum_{l=0}^{m} \binom{k/2}{l} (-\xi)^l; \; \Lambda^0_m(\xi) \equiv 1.$$

The expression for $f_n(r, \theta)$ can be obtained, in simple cases at least, by assuming a solution of the form

$$f_n(r, \theta) = f_0(r, \theta) - \sum_{s=0}^{n-1} \sum_{t=0}^{\infty} A_{st} \left(\frac{r}{a}\right)^{-k+2s-t} \frac{\partial^t f_0}{\partial r^t}\left(\frac{a^2}{r}, \theta\right),$$

each term of which is known to be a solution of equation (2) (theorem 5.7 of [5]). The boundary conditions (3) lead to equations which are linear relations between

$$f_0(a, \theta), \frac{\partial f_0}{\partial r}(a, \theta) \cdots.$$

Equating the coefficients of each of these terms in each of the equations leads to equations which can be solved for the coefficients A_{st} at any rate when n is a small integer. This is essentially the method used by Collins [3] in the case $n = 2$, $k = -1$.

The general solution given by (4) and (5) is suggested by the form of the solution for $n = 2, 3, \cdots$ and it remains to prove that this is the required solution. The proof falls into three parts.

(i) *$f_n(r, \theta)$ is a solution of the equation $L_k^n(f) = 0$.*

In fact, each term in the expression for $f_n(r, \theta)$ is a solution of the equation (2). The first term, $f_0(r, \theta)$, being a solution of $L_k(f) = 0$, is certainly a solution of the iterated equation. The general term in $f_0^*(r, \theta)$ involves $\xi^t \Lambda_{n-1-t}^k(\xi)$ where ξ replaces $1-r^2/a^2$. For $k \neq 0$, this is a polynomial in ξ of degree $n-1$ while for $k = 0$ it is equal to ξ^t. Since $t \leqq n-1$, $\xi^t \Lambda_{n-1-t}^k(\xi)$ is in both cases a polynomial in ξ of degree at most $n-1$. Replacing ξ by $1-r^2/a^2$ gives a polynomial in r^2/a^2 of degree at most $n-1$ which will be denoted by $P_{n-1}(r^2/a^2)$. It follows that the general term in the expression (5) for $f_0^*(r, \theta)$ is of the form

$$\frac{\partial^t f_0}{\partial r^t}\left(\frac{a^2}{r}, \theta\right)\left(\frac{r}{a}\right)^{-k-t} P_{n-1}\left(\frac{r^2}{a^2}\right),$$

which, by theorem 5.7 of [5], is a solution of $L_k^n(f) = 0$.

(ii) *$f_n(r, \theta)$ has the same singularities outside $r = a$ as $f_0(r, \theta)$.*

Since the singularities of $f_0(r, \theta)$ all lie outside the circle $r = a$, it is clear that those of $f_0^*(r, \theta)$ all lie inside this circle so $f_n(r, \theta)$ and $f_0(r, \theta)$ have the same singularities outside the circle.

(iii) *$f_n(r, \theta)$ satisfies the boundary conditions* (3) *on $r = a$.*

It must be proved that

$$\frac{\partial^s f_n}{\partial r^s}(a, \theta) = 0$$

for $0 \leqq s \leqq n-1$. It is obvious that $f_0^*(a, \theta) = f_0(a, \theta)$ so $f_n(a, \theta) = 0$ as required but to evaluate the derivatives of $f_n(r, \theta)$ at $r = a$ it is helpful first to separate each term in the expression for $f_0^*(r, \theta)$ into two parts.

It is easily seen that for $|\xi| < 1$,

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$$\Lambda_m^k(\xi) = (1-\xi)^{k/2} + R_{m+1}(\xi)$$

where $R_{m+1}(\xi)$ is used here (and later) to denote a function of ξ which vanishes, together with its first m derivatives, when $\xi = 0$.

In a small neighbourhood of the sphere, $|1-r^2/a^2| < 1$ so

$$\Lambda_{n-1-t}^k\left(1-\frac{r^2}{a^2}\right) = \left(\frac{r}{a}\right)^k + R_{n-t}\left(1-\frac{r}{a}\right).$$

Since the general term of $f_0^*(r, \theta)$ already contains a factor $(1-r/a)^t$, $f_n(r, \theta)$ can be written in the form

$$f_n(r, \theta) = f_0(r, \theta) - \sum_{t=0}^{n-1} \frac{1}{t!}\left(r-\frac{a^2}{r}\right)^t \frac{\partial^t f_0}{\partial r^t}\left(\frac{a^2}{r}, \theta\right) + R_n\left(1-\frac{r}{a}\right), \tag{6}$$

from which it is again apparent that $f_n(a, \theta) = 0$.

If (6) is now differentiated with respect to r, it is easily shown that the result is to replace f_0 everywhere it occurs in (6) by $\partial f_0/\partial r$, to reduce by one the number of terms in the sum and to change the remainder term from $R_n(1-r/a)$ to $R_{n-1}(1-r/a)$. This process can be repeated so that for $0 \leqq s \leqq n-1$,

$$\frac{\partial^s f_n}{\partial r^s}(r, \theta) = \frac{\partial^s f_0}{\partial r^s}(r, \theta) - \sum_{t=0}^{n-s-1} \frac{1}{t!}\left(r-\frac{a^2}{r}\right)^t \frac{\partial^{s+t} f_0}{\partial r^{s+t}}\left(\frac{a^2}{r}, \theta\right) + R_{n-s}\left(1-\frac{r}{a}\right).$$

Since $R_{n-s}(1-r/a) = 0$ when $r = a$ for $0 \leqq s \leqq n-1$, it is apparent that $f_n(r, \theta)$ and its first $n-1$ derivatives with respect to r all vanish when $r = a$.

A similar proof can be given for the corresponding internal sphere theorem:

THEOREM 2.2. *If the function $f_0(r, \theta)$ is a solution of the equation $L_k(f) = 0$ which has all its singularities inside the circle $r = a$, then a solution of the equation $L_k^n(f) = 0$ which has the same singularities inside $r = a$ as $f_0(r, \theta)$ and which satisfies the condition* (3) *is given by* (4) *and* (5).

3. Reflection in a straight line boundary

A general theorem applicable to a straight line boundary can be obtained which corresponds to the general circle theorem just proved. The simplest examples of such a theorem are well-known (e.g. in classical hydrodynamics or electrostatics) and Collins [3] has given the theorem for axially symmetric Stokes flow of a viscous fluid.

In discussing the solution of equation (2) in a half-plane it is natural to use rectangular cartesian coordinates (x, y) chosen so that $x = r\cos\theta$, $y = r\sin\theta$ and the axis of symmetry is the axis $y = 0$. The half-plane under consideration will be taken to be the region $x \geqq 0$. In these coor-ordinates, the operator L_k becomes

$$L_k \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y}.$$

THEOREM 3.1. *If the function $f_0(x, y)$ is a solution of the equation $L_k(f) = 0$ which has all its singularities in the region $x > 0$, then a solution $f_n(x, y)$ of the equation $L_k^n(f) = 0$ which has the same singularities in $x > 0$ as $f_0(x, y)$ and which satisfies the conditions*

$$f_n(0, y) = \frac{\partial f_n}{\partial x}(0, y) = \cdots = \frac{\partial^{n-1} f_n}{\partial x^{n-1}}(0, y) = 0 \tag{7}$$

is given by

$$f_n(x, y) = f_0(x, y) - f_0^*(x, y) \tag{8}$$

where

$$f_0^*(x, y) = \sum_{t=0}^{n-1} \frac{2^t}{t!} x^t \frac{\partial^t f_0}{\partial x^t}(-x, y). \tag{9}$$

As in the case of the circle theorem, the solution $f_n(x, y)$ can be found, for small values of n, by assuming a solution of appropriate form and using the boundary condition to deduce the unknown coefficients. Once the general solution given by (8) and (9) has been recognised it remains only to verify that it is the required solution.

In this case it is evident from Theorem 2.5 of [5] that $f_n(x, y)$ given by (8) and (9) does satisfy the equations $L_k^n(f) = 0$ and it is clear that $f_n(x, y)$ and $f_0(x, y)$ have the same singularities in the region $x > 0$. As for the boundary conditions (7) on $x = 0$, it is seen at once that $f_n(0, y) = 0$. When $f_n(x, y)$ is differentiated with respect to x, the resulting expression can be written as

$$\frac{\partial f_n}{\partial x}(x, y) = \frac{\partial f_0}{\partial x}(x, y) - \sum_{t=0}^{n-2} \frac{2^t}{t!} x^t \frac{\partial^{t+1} f_0}{\partial x^{t+1}}(-x, y) + R_{n-1}(x),$$

where the remainder term $R_{n-1}(x)$ vanishes together with its first $n-2$ derivatives when $x = 0$. Thus the result of differentiation of $f_n(x, y)$ with respect to x is to replace f_0 everywhere it appears by $\partial f_0/\partial x$, to reduce by one the number of terms in the sum and to introduce a remainder term of the form $R_{n-1}(x)$. Since differentiation of $R_{n-1}(x)$ leads to a function of the form $R_{n-2}(x)$, repeated differentiation of $f_n(x, y)$ gives

$$\frac{\partial^s f_n}{\partial x^s}(x, y) = \frac{\partial^s f_0}{\partial x^s}(x, y) - \sum_{t=0}^{n-s-1} \frac{2^t}{t!} x^t \frac{\partial^{t+s} f_0}{\partial x^{t+s}}(-x, y) + R_{n-s}(x).$$

Since $R_{n-s}(0) = 0$ for $0 \leqq s \leqq n-1$, it follows that $f_n(x, y)$ and its first $n-1$ derivatives vanish when $x = 0$.

4. Further generalization of the circle theorem

Collins [4] has obtained a sphere theorem for Stokes flow which takes as its starting point a known stream function for Stokes flow which is a solution of the equation $L^2_{-1}(\psi) = 0$ but need not be a solution also of the equation $L_{-1}(\psi) = 0$ as is required for his original theorem [3]. Collins's result is easily extended to give a theorem which applies to solutions of the equation $L^2_k(f) = 0$ for any value of k and further extension to the general iterated equation $L^n_k(f) = 0$ can be carried out, solutions having been obtained for the cases $n = 2, 3, 4$.

The general problem is as follows: given a solution $f_0(r, \theta)$ of the equation $L^n_k(f) = 0$ which has a known distribution of singularities all of which lie outside the circle $r = a$, to find a solution $f_n(r, \theta)$ of the same equation which has the same singularities as $f_0(r, \theta)$ in the region outside $r = a$ and which satisfies the condition that f_n and its first $n-1$ derivatives with respect to r vanish on $r = a$. (There will be a corresponding problem for the case in which the singularities of $f_0(r, \theta)$ are all inside $r = a$.)

When $n = 1$, the problem is solved by theorem 2.1. When $n = 2$, the result (given by Collins [4] for the special case $k = -1$) is

$$f_2(r, \theta) = F_2(r, \theta) - \frac{r^2}{4}\left(1-\frac{r^2}{a^2}\right)^2 L_k\left[\left(\frac{r}{a}\right)^{-k} f_0\left(\frac{a^2}{r}, \theta\right)\right] \tag{10}$$

where $F_2(r, \theta)$ is formally identical with the solution $f_2(r, \theta)$ given by (4) and (5) of theorem 2.1 in the case $n = 2$ but with the $f_0(r, \theta)$ which appears in that expression now having the meaning associated with it for this theorem, namely a solution of $L^2_k(f) = 0$ which need not be a solution of $L_k(f) = 0$.

When $n = 3$, the result is

$$\begin{aligned} f_3(r, \theta) = F_3(r, \theta) - \frac{r^2}{4}\left(1-\frac{r^2}{a^2}\right)^3 \Bigg\{ & L_k\left[\left(\frac{r}{a}\right)^{-k} f_0\left(\frac{a^2}{r}, \theta\right)\right] \\ & - \frac{r^2}{16} L^2_k\left[\left(\frac{r}{a}\right)^{-k} f_0\left(\frac{a^2}{r}, \theta\right)\right] \\ & - \tfrac{1}{16}(r^2-2a^2) L^2_k\left[\left(\frac{r}{a}\right)^{-k+2} f_0\left(\frac{a^2}{r}, \theta\right)\right]\Bigg\} \end{aligned} \tag{11}$$

where $F_3(r, \theta)$ is defined in the same way as $F_2(r, \theta)$.

In both (10) and (11) (and in the corresponding solution for $n = 4$

which has not been quoted) it is clear from theorem 5.7 of [5] that these solutions reduce to those given by theorem 2.1 when $f_0(r, \theta)$ is a solution of $L_k(f) = 0$.

A proof of the case $n = 2$ for general k can be modelled on Collins's proof [4] for the case $k = -1$. The procedures used to obtain the solutions for $n = 3$ and $n = 4$ could be used to obtain the solutions for higher values of n and presumably it would be possible eventually to anticipate the form for general n. Once this is done, the procedures used for verifying the solutions for the cases $n = 3, 4$ could be applied to verify the solution in the general case.

The extension of the theorem from $n = 2$ to higher values is not immediate and new features appear in the proof which are not needed when $n = 2$. However, as the general result has not yet been obtained, no account of the proof will be given.

5. Applications of circle theorems

A full discussion of the applications of these circle theorems to particular problems is outside the scope of this paper but some examples of the first theorem (2.1) will be given which lead to well-known results.

As mentioned in the introduction, the stream function for irrotational flow of a perfect fluid or Stokes flow of a viscous fluid satisfies equation (2) with $n = 1, 2$ respectively and the flow in each case is two-dimensional or axially symmetric according as $k = 0, -1$. For flow past a circular cylinder or sphere, the stream function satisfies the boundary conditions (3) in each case.

The stream function for an undisturbed uniform flow parallel to the axis of symmetry is given by

$$\psi_0(r, \theta) = \frac{Ur^{1-k}\sin^{1-k}\theta}{1-k}$$

and theorem 2.1 can now be used to give the stream function when the flow is disturbed by the rigid boundary $r = a$. When $n = 1$, the theorem gives

$$\psi(r, \theta) = \frac{Ur^{1-k}\sin^{1-k}\theta}{1-k}\left\{1-\left(\frac{a}{r}\right)^{2-k}\right\}$$

and putting $k = 0, -1$ gives the results obtained from Milne-Thomson's and Butler's theorems respectively. When $n = 2$, theorem 2.1 gives the stream function for Stokes flow when a uniform stream is disturbed by the circle $r = a$ as

$$\psi(r, \theta) = \frac{Ur^{1-k}\sin^{1-k}\theta}{1-k}\left\{1-\frac{k}{2}\left(\frac{a}{r}\right)^{2-k}-\left(1-\frac{k}{2}\right)\left(\frac{a}{r}\right)^{-k}\right\}. \tag{12}$$

When $k = -1$ this is the familiar Stokes result for a sphere in a uniform stream and when $k = 0$ the result is $\psi \equiv 0$. It is of course well known that the Stokes flow problem for a circular cylinder in a uniform stream has only this trivial solution.

Butler [2] and Collins [3] have given other applications of their theorems to flow problems such as the flow due to a source or dipole in the presence of a sphere and the general theorem 2.1 can be used to derive formal expressions which reduce to their solutions when $k = -1$. When k is put equal to 0 in these same expressions, the functions obtained are formal solutions for the stream functions for the corresponding two-dimensional flow problems.

For both axially symmetric and two-dimensional flow, the stream functions obtained in this way certainly satisfy the appropriate differential equations and the right boundary conditions on $r = a$. They also have the same singularities in the region of flow as the original stream function. The conditions at infinity however remain to be examined in particular cases and both Butler and Collins have made reference to this aspect of the problem. In the case of the Stokes solution obtained from (12) for the flow past a sphere set in a uniform stream, the stream function does have the right behaviour at infinity. However, in other cases it is more difficult to decide on the validity of the solutions obtained and further study of these questions is required.

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