A DISTRIBUTION FUNCTION OF CANTOR-VITALI TYPE

D. B. Sumner

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1. Introduction. In his 1922 article [1] on functions of bounded variation, Vitali gave a method for constructing monotone non-absolutely continuous functions, generalizing ideas from the ternary set introduced in another connection by Cantor. In [2], Hille and Tamarkin gave a full account of the "middle-third" function, showing it to be a singular distribution function, and finding its characteristic function. In [3], Evans obtained a generalization of the middle-third function by discarding middle intervals of length other than one-third, and obtained algorithms by which the moments of his function could be calculated. In various papers, among them [4], Wintner studied infinite convolutions of symmetric Bernoulli distributions, finding a great variety of distributions whose characteristic functions were of the form $\prod_{1}^{\infty} \cos(xa_k)$.

In the present paper the Cantor ternary set will be generalized as a $(2N+1)$-ary set, and a Cantor-Vitali distribution function will be defined on it. An algorithm for calculating its moments will be given, while its characteristic function will turn out to be a natural generalization of the preceding infinite product of cosines.

About terms and symbols. If a finite or infinite fraction is written in Greek letters, it will be in the scale $2N+1$, and if in Latin letters, in the scale $N+1$. The letter α will represent any one of the odd integers $1, 3, 5, \ldots, 2N-1$; β will represent any even integer $0, 2, \ldots, 2N$; and γ will represent any integer $0, 1, 2, 3, \ldots, 2N$. For convenience we write

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$r = (N+1)^{-1}$, $\rho = (2N+1)^{-1}$, $\varepsilon = 2N$, $\alpha + 1 = \alpha^{+} = 2a$, $a - 1 = a^{-}$, $\beta + 1 = \beta^{+}$, $\beta - 1 = \beta^{-}$, $\beta = 2b$. Thus a_i and b_i are integers satisfying

(1.1) $$0 \le b_i \le N, \quad 1 \le a_i \le N.$$

A suffix attached to any one of these letters will indicate its position, e.g., $0.\alpha_1\beta_2 = \alpha_1\rho + \beta_2\rho^2$.

2. The sets R and Δ. A discard is defined as the process of dividing each closed interval of a set of closed intervals into 2N+1 equal parts, and of removing from each closed interval the interiors of its 2^{nd}, 4^{th}, 6^{th}, ..., $(2N)^{th}$ parts.

The first discard is applied to the interval $I = [0, 1]$. The removed open intervals are

$$\delta_{1\alpha_1} = (0.\alpha_1, 0.\alpha_1^{+}), \quad (\alpha_1 = 1, 3, 5, \ldots, 2N-1),$$

and the residual closed subintervals are

$$\eta_{1\beta_1} = [0.\beta_1, 0.\beta_1^{+}], \quad (\beta_1 = 0, 2, 4, \ldots, 2N).$$

The second discard is applied to the set of intervals $\eta_{1\beta_1}$, the removed and the residual intervals being designated as $\delta_{2\alpha_2}$, $(\alpha_2 = 1, 3, 5, \ldots, 2N-1)$ and $\eta_{2\beta_2}$ respectively.

The process of making successive discards is continued, the set from which the (m+1)-th discard is made being the (previous) residual set $\eta_{m\beta_m}$. A little consideration will show that any interval $\delta_{m\alpha_m}$ of the m^{th} discard is

$$(0.\beta_1 \ldots .\beta_{m-1}\alpha_m, \; 0.\beta_1 \ldots .\beta_{m-1}\alpha_m^{+})$$

the number of these intervals being $N(N+1)^{m-1}$, since each digit β can take $N+1$ and α_m can take N possible values.

Let

$$\delta_m = \bigcup_{\alpha_m} \delta_{m\alpha_m}, \quad \Delta = \bigcup_m \delta_m, \quad R = I - \Delta .$$

The following assertions can easily be verified, since each is a generalization (from 3 to $2N+1$) of a statement in [2]:

(2.1) The set R is perfect and nowhere dense in I;

(2.2) Each interval $\delta_{m\alpha_m}$ has length ρ^m; the measure of δ_m is $\frac{N}{(N+1)}\left[\frac{N+1}{2N+1}\right]^m$; the measure of Δ is 1; and R is a set of measure zero;

(2.3) A number x is in Δ if and only if there occurs at least one odd digit in its representation in the scale $2N+1$, and at least one of the digits following this odd digit is neither 0 nor $2N$.

3. The distribution function $F(x)$. Definition. Let $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$. When $0 \leq x \leq 1$, we may, by introducing if necessary recurring groups of digits, represent x in the scale $2N+1$ as an infinite fraction

$$x = 0.\gamma_1\gamma_2 . \ldots . \gamma_n \ldots .$$

When all γ are even, (i.e., $x = 0.\beta_1 . \ldots . \beta_n \ldots$) let $F(x) = 0.b_1 b_2 . \ldots . b_n \ldots$. When at least one γ is odd, say, $x = 0.\beta_1 . \ldots . \beta_{p-1}\alpha_p\gamma_{p+1} \ldots$, with the first odd digit occurring in the p-th position, let $F(x) = 0.b_1 . \ldots . b_{p-1}a_p$.

Properties of $F(x)$ similar to those given for the case $N=1$ by Hille and Tamarkin in [2] and by Evans in [3] are as follows:

(3.1) F(x) <u>is constant over the closure of any removed interval.</u> Consider the removed interval $(0.\beta_1.\ \dots\ .\beta_{m-1}\alpha_m,\ 0.\beta_1.\ \dots\ .\beta_{m-1}\alpha^+_m)$ or $(0.\beta_1.\ \dots\ .\beta_{m-1}\alpha^-_m\dot{\varepsilon},\ 0.\beta_1.\ \dots\ .\beta_{m-1}\alpha_m\dot{\varepsilon})$. By definition

$$F(0.\beta_1.\ \dots\ .\beta_{m-1}\alpha_m) = 0.b_1.\ \dots\ .b_{m-1}a^-_m\dot{N}$$

$$= 0.b_1.\ \dots\ .b_{m-1}a_m,$$

$$F(0.\beta_1.\ \dots\ .\beta_{m-1}\alpha^+_m) = 0.b_1.\ \dots\ .b_{m-1}a_m.$$

When x is interior to this removed interval, say

$$x = 0.\beta_1.\ \dots\ .\beta_{m-1}\alpha_m\gamma_{m+1}\ \dots,$$

the digits $\gamma_{m+1}, \dots$ cannot all be zero, nor all equal to 2N: then by definition $F(x) = 0.b_1.\ \dots\ .b_{m-1}a_m$.

(3.2) F(x) <u>is non-decreasing</u>. It is sufficient to prove that $F(x') > F(x)$ when $x' > x$ and $x, x' \in R$. Let $x = 0.\beta_1.\ \dots\ .\beta_q\beta_{q+1}\ \dots$, $x' = 0.\beta_1.\ \dots\ .\beta_q\beta'_{q+1}\ \dots$ where $\beta'_{q+1} > \beta_{q+1}$. Then

$$F(x') = 0.b_1.\ \dots\ .b_q b'_{q+1}\ \dots > 0.b_1.\ \dots\ .b_q b_{q+1}\dots = F(x).$$

(3.3) F(x) <u>is continuous but not absolutely continuous</u>. It is sufficient to show that as $x' \to x$ through values greater than x, then $F(x') \to F(x)$, both x and x' being in R. This is easily deduced from (2.3) and the definition of F.

To show that F is not AC, consider the residual intervals η_{mq}, where $q = 1, 2, \dots, (N+1)^m$. We note that $R \subset \cup_q \eta_{mq}$. Let η_{mq} be the interval $\kappa_q \leq x \leq \lambda_q$. Then

$$\sum_q |F(\lambda_q) - F(\kappa_q)| = \sum_q \{F(\lambda_q) - F(\kappa_q)\} = 1,$$

since the interval $(\kappa_{q-1}, \lambda_q)$ is part of the m-th discard.

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(3.4) $F'(x) \stackrel{o}{=} 0$.

(3.5) $\int_0^x F'(t)dt = 0$ <u>but</u> $F(x) - F(0) > 0$ <u>when</u> $x > 0$.

(3.6) $F(x) + F(1-x) = 1$ <u>for all</u> x.

This is obvious when $x < 0$ or when $x > 1$. When $0 \le x \le 1$, and $x = 0.\beta_1 . \ldots . \beta_m \ldots$, then

$$1 - x = \sum_{q=1}^{\infty} (2N-\beta_q)\rho^q, \text{ and}$$

$$F(x) + F(1-x) = \sum_{q=1}^{\infty} (b_q + N - b_q)r^q = 1.$$

When $0 \le x \le 1$ and $x = 0.\beta_1, \ldots, \beta_{p-1}\alpha_p\gamma_{p+1} \ldots$, then

$$1 - x = \sum_{q=1}^{p-1} (2N - \beta_q)\rho^q + (2N - \alpha_p)\rho^p + \sum_{q=p+1}^{\infty} (2N - \gamma_q)\rho^q ,$$

and

$$F(x) + F(1-x) = \sum_{q=1}^{p-1} b_q r^q + \tfrac{1}{2}(\alpha_p + 1)r^p + \sum_{q=1}^{p-1} (N - b_q)r^q$$
$$+ \tfrac{1}{2}(2N + 1 - \alpha_p)r^p$$
$$= 1 .$$

(3.7) $rF(x) = F(\rho x)$, $(0 \le x \le 1)$;

(3.8) $r + F(x-2\rho) = F(x)$, $(2\rho \le x \le 1)$.

(3.7) and (3.8) follow easily from the definition of F.

4. <u>Moments of</u> $F(x)$. It is easily verified that the moments $M_n = \int_0^1 x^n F(x)dx$, and the associated moments

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$\mu_n = \int_{-\infty}^{\infty} x^n dF(x)$ are connected by

$$(4.1) \quad \mu_o = 1; \quad \mu_{n+1} = 1 - (n+1)M_n, \quad (n \geq 0).$$

Following the ideas of Evans in [3], we shall prove:

$$(4.2) \quad M_n + \sum_0^n (-)^p \binom{n}{p} M_p = 1/(n+1), \quad (n \geq 0).$$

$$(4.3) \quad [1-\rho^{n+1}]M_n = \frac{Nr}{n+1} - \frac{r(2\rho)^{n+1}}{(n+1)(n+2)}\left[B_{n+2}(N+\tfrac{1}{2})-B_{n+2}(\tfrac{1}{2})\right]$$

$$+ r\rho^{n+1} \sum_{p=1}^{n} \binom{n}{p} 2^p M_{n-p} \left\{\frac{B_{p+1}(N+1)-B_{p+1}(1)}{p+1}\right\}.$$

Formula (4.2) does not give a value for M_1, and is therefore not sufficient for calculating all M_n. However the value

$$M_1 = (8N+7)/24(N+1)$$

can be deduced from (4.3); and this value together with (4.2) affords a means of calculating all M_n.

Proof of (4.2): Follows from (3.6).

Proof of (4.3): Here we shall use (3.7) and (3.8), and the Bernoulli polynomials $B_m(x)$ and Bernoulli numbers B_m defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_0^{\infty} \frac{B_m(x)t^m}{m!}$$

$$B_m(0) = B_m ,$$

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with the difference relation

$$B_m(x+1) - B_m(x) = mx^{m-1} .$$

Now

$$M_n = \sum_{p=0}^{N} \int_{2p\rho}^{(2p+1)\rho} x^n F(x)dx + \sum_{p=1}^{N} \int_{(2p-1)\rho}^{2p\rho} x^n F(x)dx .$$

But since

$$\int_{2p\rho}^{(2p+1)\rho} x^n F(x)dx = \int_{2p\rho}^{(2p+1)\rho} x^n [pr + F(x-2p\rho)]dx$$

$$= \frac{pr\rho^{n+1}}{n+1} [(2p+1)^{n+1} - (2p)^{n+1}]$$

$$+ r\rho^{n+1} \int_0^1 (t+2p)^n F(t)dt ,$$

and

$$\int_{(2p-1)\rho}^{2p\rho} x^n F(x)dx = \frac{pr\rho^{n+1}}{(n+1)} [(2p)^{n+1} - (2p-1)^{n+1}] ,$$

it follows that

$$M_n = \frac{r\rho^{n+1}}{n+1} \sum_{p=1}^{N} p[(2p+1)^{n+1} - (2p-1)^{n+1}]$$

$$+ \sum_{p=0}^{N} r\rho^{n+1} \int_0^1 (t+2p)^n F(t)dt$$

$$= \frac{Nr}{n+1} - \frac{r(2\rho)^{n+1}}{(n+1)(n+2)} \{B_{n+2}(N+\tfrac{1}{2}) - B_{n+2}(\tfrac{1}{2})\} + \rho^{n+1} M_n$$

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$$+ r\rho^{n+1} \sum_{p=1}^{n} \binom{n}{p} 2^p M_{n-p} \left\{ \frac{B_{p+1}(N+1) - B_{p+1}(1)}{p+1} \right\},$$

from which (4. 3) follows.

5. Characteristic function of $F(x)$. This section is devoted to the proof of the

THEOREM: Let $f_p(t) = (N+1)^{-1} \sum_{k=0}^{N} \exp(2ikt\rho^p)$: then the infinite product $\prod_{p=1}^{\infty} f_p(t)$ converges for all real t to a function $f(t)$, which is the characteristic function corresponding to the distribution function $F(x)$.

We need to prove that $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_0^1 e^{itx} dF(x)$, an integral known to exist since $F(x)$ is bounded and increasing. Let the interval $[0, 1]$ be partitioned into $q = (2N+1)^n$ equal subintervals. These subintervals will consist either of partitions of the removed intervals $\delta_{1\alpha_1}, \delta_{2\alpha_2}, \ldots, \delta_{n\alpha_n}$, or of residual intervals $\eta_{n\beta_n}$. Then the integral is equal to

$$\lim_{n\to\infty} \sum_{j=1}^{q} \exp(it\xi_j)[F(x_j) - F(x_{j-1})] .$$

The increments of F are zero when x_{j-1} and x_j belong to the same removed interval, non-vanishing terms in the sum occurring only when the interval (x_{j-1}, x_j) is one of the residual intervals $\eta_{n\beta_n}$. In this case

$$F(x_j) - F(x_{j-1}) = (N+1)^{-n}$$

for $j = 1, 2, \ldots, s$, where $s = (N+1)^n$. Let us take ξ_j to be the left end point of the corresponding residual interval $\eta_{n\beta_n}$: it is evident that the set of numbers ξ_j, $(j = 1, 2, \ldots, s)$, consists of all finite fractions of the form

$$0.\beta_1 \ldots \beta_n .$$

Designating summation over all possible arrangements of $\beta_1 \ldots \beta_n$, (repetitions being allowed) by the symbol $\Sigma_{(\beta)}$, it follows that we have to calculate the limit as $n \to \infty$ of

$$\sigma_n = (N+1)^{-n} \Sigma_{(\beta)} \exp(it\xi_j) .$$

Now

$$\sigma_n = (N+1)^{-n} \Sigma_{(\beta)} \exp\{it \sum_{p=1}^{n} \beta_p \rho^p\}$$

$$= (N+1)^{-n} \prod_{p=1}^{n} [1 + \exp(2it\rho^p) + \exp(4it\rho^p) + \ldots$$

$$+ \exp(2Nit\rho^p)] ,$$

as can be verified by induction, when (1.1) and the meaning of $\Sigma_{(\beta)}$ are taken into account. Thus

$$\sigma_n = \prod_{p=1}^{n} f_p(t) .$$

But

$$f_p(t) = (N+1)^{-1} [(N+1) + \sum_{k=1}^{N} \{\exp(2kit\rho^p) - 1\}]$$

$$= 1 + it\,(N+1)^{-1} \sum_{k=1}^{N} \int_0^{2k\rho^p} e^{itx}\,dx\,,$$

$$\ll 1 + \frac{|t|}{N+1} \sum_{k=1}^{N} \frac{2k}{(2N+1)^p}\,, \qquad (t \text{ real});$$

$$= 1 + \frac{N|t|}{(2N+1)^p}.$$

Thus $\prod_{p=1}^{\infty} f_p(t)$ is absolutely convergent for all real t; and is the value of our integral.

We add some remarks about $f(t)$, most of which are self-evident, or easily derived from known properties of singular distributions and their characteristic functions.

(5.1) If $f_p(t)$ is the characteristic function of $F_p(x)$, then

$$F_p(x) = (N+1)^{-1} \sum_{k=0}^{N} H(x-2k\rho^p)\,,$$

$H(x)$ being the Heaviside unit function.

(5.2) $F_p(x)$ are discrete distributions, while $f_p(t)$ are periodic of period $\pi(2N+1)^p$.

(5.3) $\sum_{p=1}^{n} f_p(t)$ has period $\pi(2N+1)^n$.

(5.4) $f(t) = \lim_{n\to\infty} \prod_{1}^{n} f_p(t)$ is not periodic, but for any positive integer P,

$$f[\pi(2N+1)^P] = f(\pi)\,.$$

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$$(5.5)\quad F(x) = \lim_{n\to\infty} F_1 * F_2 * \cdots * F_n .$$

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McMaster University

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