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ON BIORTHOGONAL SYSTEMS AND MAZUR'S INTERSECTION PROPERTY

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We give a characterisation of Banach spaces X containing a subspace with a shrinking Markushevich basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. This gives a sufficient condition for X to have a renorming with Mazur's intersection property.

A *biorthogonal system* in a Banach space X is a subset $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ such that $f_\gamma(x_{\gamma'}) = \delta_{\gamma\gamma'}$ for $\gamma, \gamma' \in \Gamma$. The biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in X is called *fundamental* if $X = \overline{\operatorname{span}}\{x_\gamma; \gamma \in \Gamma\}$. A *Markushevich basis* is a fundamental biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in X such that $\{f_\gamma\}_{\gamma\in\Gamma}$ separates points of X. A Markushevich basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ is called *shrinking* if $X^* = \overline{\operatorname{span}}\{f_\gamma; \gamma \in \Gamma\}$. In this note we use Γ as a cardinal number.

A Banach space X is said to be an *Asplund space*, if every separable subspace of X has a separable dual. A Banach space X has *Mazur's intersection property* if every bounded closed convex set can be represented as an intersection of closed balls. A *density* of a topological space is the least cardinality of a dense set. We refer to [**2**] for undefined terms used in this paper.

It is known, [**9**, Theorem 7.18, Theorem 7.12], that if a dual unit ball of a Banach space X is a Corson compact, then $\operatorname{dens} X = w^*\text{-dens } X^*$ and the following are equivalent.

(i) X has a shrinking Markushevich basis,
(ii) X is an Asplund space,
(iii) X admits a Fréchet smooth norm.

Let us remark that if a norm on X is Fréchet smooth, then X has Mazur's intersection property, [**1**, Proposition 4.5].

When we do not assume that the dual unit ball is a Corson compact, then the above is no longer true. For example, the Banach space $C(K)$, where K is a Kunen's compact (see [**8, 5**]), is an Asplund space without shrinking Markushevich basis and without Mazur's intersection property ([**6**]).

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The aim of this note is to prove a theorem in the spirit of equivalences above but without assuming anything about a dual unit ball.

Theorem 1. *Let E be a Banach space. Then the following are equivalent.*

(i) *There is a subspace $Y \subset E$ with a shrinking Markushevich basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$.*

(ii) *There is an Asplund space $X \subset E$ with* $\operatorname{dens} X = w^*\text{-}\operatorname{dens} X^* = \Gamma$.

(iii) *There is a subspace $Z \subset E$ that admits a Fréchet smooth norm and such that* $\operatorname{dens} Z = w^*\text{-}\operatorname{dens} Z^* = \Gamma$.

Moreover, if one from the above occurs with $\Gamma = \operatorname{dens} E^*$, then

(iv) *E admits a norm with the Mazur intersection property.*

Remark. The condition $\operatorname{dens} E = \operatorname{dens} E^*$ is necessarily for renorming with Mazur intersection property due to [**3**].

Proof: Implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). If Y has a shrinking Markushevich basis, then Y admits a Fréchet differentiable norm [**2**, Theorem 11.23]. Thus it is an Asplund space [**2**, Theorem 8.24]. It remains to show that $w^*\text{-}\operatorname{dens} Y^* = \operatorname{dens} Y = \Gamma$. Let $\{g_\alpha; \alpha \in A\} \subset Y^*$ be a weak* dense set. As the basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ is shrinking, we may assume without loss of generality that $\{g_\alpha; \alpha \in A\} \subset \operatorname{span}\{f_\gamma; \gamma \in \Gamma\}$. For a contradiction, assume that $|A| < \Gamma$. Thus there is $\Gamma' < \Gamma$ such that

$$\{g_\alpha; \alpha \in A\} \subset \operatorname{span}\{f_\gamma; \gamma \in \Gamma'\}.$$

Hence, for $\gamma \in \Gamma \setminus \Gamma'$ and all $\alpha \in A$

$$\big|(f_\gamma - g_\alpha)(x_\gamma)\big| = 1,$$

a contradiction with the density of $\{g_\alpha; \alpha \in A\}$. ∎

Implication (i) $\Rightarrow$ (iv). Due to [**6**, Theorem 2.4], to show that E admits a norm with the Mazur intersection property, it is enough to construct a fundamental biorthogonal system $\{q_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset E^* \times E$. As we assume that $Y \subset E$ has a shrinking Markushevich basis, that is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset Y^* \times Y$, we only need to show the following.

Lemma 2. *Let E be a Banach space with $\operatorname{dens} E^* = \Gamma$ and $Y \subset E$ be a closed subspace. Assume that there is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset Y^* \times Y$. Then there is a fundamental biorthogonal system $\{q_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset E^* \times E$.*

Proof: By a relabeling and rescaling, we may have a fundamental system $\{f^n_\gamma, x^n_\gamma\}_{\gamma\in\Gamma, n\in\mathbf{N}} \subset Y^* \times Y$ such that for every $\gamma \in \Gamma$, $\lim_n \|f^n_\gamma\| = 0$. By the Hahn-Banach theorem, consider $f^n_\gamma \in E^*$. Let $\{g_\gamma\}_{\gamma\in\Gamma}$ be a dense set of $B_{E^*} \cap Y^\perp$.

We claim, that $A = \{g_\gamma + f^n_\gamma\}_{\gamma\in\Gamma, n\in\mathbf{N}}$ is linearly dense in E^*. Indeed, let $G \in E^{**}$ be such that $G(f) = 0$ for every $f \in A$. Then $G(g_\gamma) = \lim_n G(g_\gamma + f^n_\gamma) = 0$ and thus $G \in (Y^\perp)^\perp = Y^{**}$. Hence $G = 0$ as $\{f^n_\gamma\}_{\gamma\in\Gamma, n\in\mathbf{N}}$ are linearly dense in Y^*.

Hence $\{g_\gamma + f_\gamma^n, x_\gamma^n\}_{\gamma\in\Gamma, n\in\mathbb{N}} \subset E^* \times E$ is a fundamental biorthogonal system. ∎

REMARK. As $c_0(\Gamma) \subset C([0,\Gamma])$, Lemma 2 provides a direct proof of the fact that there is a fundamental biorthogonal system $\{f_\gamma, x_\gamma\}_{\gamma\in\Gamma} \subset C([0,\Gamma])^* \times C([0,\Gamma])$. Thus $C([0,\Gamma])$ admits a norm with Mazur's intersection property, see also [**6**, Lemma 3.5].

Thus it remains to prove the implication (ii) $\Rightarrow$ (i).

The proof goes in the spirit of [**7**, Theorem 1.a.5] and [**4**]. We shall use the concept of the Jayne-Rogers selector, see [**1**, Chapter 1]. The Jayne-Rogers selection map $\mathcal{D}^X$ on an Asplund space X is a multi-valued map that satisfies the following.

(i) $\mathcal{D}^X(x) = \{D_n^X(x); n \in \mathbb{N}\} \cup D_\infty^X(x) \subset X^*$,

(ii) D_n^X, for $n \in \mathbb{N}$, are continuous functions from X to X^*,

(iii) $D_\infty^X(x) = \lim_{n\to\infty} D_n^X(x)$ for every $x \in X$,

(iv) $D_\infty^X(x)(x) = \|x\|^2 = \|D_\infty^X(x)\|^2$,

(v) $X^* = \overline{\operatorname{span}}\mathcal{D}^X(X)$.

Such selector exists by [**1**, Theorem 1.5.2].

In order to construct $Y \subset X$ we shall define, by a transfinite induction, vectors $x_{\alpha+1} \in X$, subspaces $Y_\alpha \subset X$ and subsets $F_\alpha \subset X^*$, for all $\alpha < \Gamma$. Put $Y_0 = 0$ and $F_0 = 0$ and pick arbitrary nonzero $x_1 \in (F_0)_\perp = \{x \in X; f(x) = 0 \text{ for all } f \in F_0\}$. Then put $Y_1 = \operatorname{span}\{x_1\}$, and $F_1 = \{\mathcal{D}^X(x); x \in Y_1\}$. Let Y_α and F_α for $\alpha < \Gamma$ have been chosen. Notice that $\operatorname{dens} Y_\alpha < \Gamma$ and thus $\operatorname{dens} F_\alpha \leqslant \aleph_0 . \operatorname{dens} Y_\alpha < \Gamma$. Thus F_α is not w^*-dense and we can pick a nonzero vector $x_{\alpha+1} \in (F_\alpha)_\perp$. Set $Y_{\alpha+1} = \operatorname{span}\{Y_\alpha \cup \{x_{\alpha+1}\}\}$ and $F_{\alpha+1} = \{\mathcal{D}^X(x); x \in Y_{\alpha+1}\}$.

If $\alpha \leqslant \Gamma$ is a limit ordinal, define $Y_\alpha = \overline{\operatorname{span}} \cup_{\beta<\alpha} Y_\beta$ and $F_\alpha = \{\mathcal{D}^X(x), x \in Y_\alpha\}$.

Put $Y = \overline{\operatorname{span}} \cup_{\alpha<\Gamma} Y_\alpha$. We shall show that Y has a shrinking Markushevich basis $\{x_{\alpha+1}, f_{\alpha+1}\}_{\alpha<\Gamma}$, where $\{x_{\alpha+1}\}_{\alpha<\Gamma}$ have been already chosen and their biorthogonals $f_{\alpha+1}$ will be defined by projections.

Clearly $Y = \overline{\operatorname{span}}\{x_{\alpha+1}; a < \Gamma\}$. Let us define projections $P_\alpha : Y \to Y_\alpha$ for all $\alpha \leqslant \Gamma$. First define projections $\widetilde{P}_\alpha : \operatorname{span}\{x_{\alpha+1}; \alpha < \Gamma\} \to Y_\alpha$ by letting $P_\alpha(x_\beta) = x_\beta$ if $\beta \leqslant \alpha$ and 0 otherwise. $\widetilde{P}_\alpha$ are well defined and once we show that they all have norm 1, they will extend naturally onto desired projections on Y.

Take $x \in \operatorname{span}\{x_{\alpha+1}; \alpha < \Gamma\}$ and fix $\alpha \leqslant \Gamma$. Then by the properties of the Jayne-Rogers selector and due to the choice of $\{x_{\alpha+1}; \alpha < \Gamma\}$ we have

$$\begin{aligned}\|\widetilde{P}_\alpha(x)\|^2 &= D_\infty^X(\widetilde{P}_\alpha(x))(\widetilde{P}_\alpha(x)) = D_\infty^X(\widetilde{P}_\alpha(x))(x) \\ &\leqslant \|x\|.\left\|D_\infty^X(\widetilde{P}_\alpha(x))\right\| = \|x\|.\|\widetilde{P}_\alpha(x)\|.\end{aligned}$$

Thus $\|\widetilde{P}_\alpha\| = 1$.

Define $f_{\alpha+1} \in Y^*$ for $\alpha < \Gamma$ such that $\|f_{\alpha+1}\| = 1$ and $f_{\alpha+1} \in (P_{\alpha+1} - P_\alpha)^* Y^*$. Clearly $\{x_{\alpha+1}, f_{\alpha+1}\}_{\alpha<\Gamma}$ is a biorthogonal system.

We shall show that the projection $\{P_\alpha\}_{\alpha<\Gamma}$ are shrinking. From that it follows that $\overline{\operatorname{span}}\{f_{\alpha+1};\alpha<\Gamma\}=Y^*$.

Let $\alpha\leqslant\Gamma$ be a fixed limit ordinal and set $Z=P_\alpha Y$. Let $f\in Z^*$ be arbitrary. We need to show that there exist a sequence of ordinals $\beta_n\to\alpha$ and $g_n\in P^*_{\beta_n}Z^*$ such that $g_n\to f$ in Z^*. Fix $\varepsilon>0$. Denote $\mathcal{D}^Z$ the restriction of $\mathcal{D}^X$ on Z, that is $D^Z_k(z)=D^X_k(z)|_Z$ for all $z\in Z$. Clearly $\mathcal{D}^Z$ is the Jayne-Rogers selection map for Z. As $Z\subset X$ is an Asplund space, $Z^*=\overline{\operatorname{span}}\mathcal{D}^Z(Z)$. Thus

$$\left\|f-\left(\sum_{i=1}^{n}D^Z_{k_i}(z_i)+\sum_{i=n+1}^{m}D^Z_\infty(z_i)\right)\right\|<\varepsilon,$$

where $k_i\in\mathbb{N}$, for $i=1,\dots,n$ and $z_i\in Z$, for $i=1,\dots,m$. Because D^Z_∞ is a pointwise limit of D^Z_n, there are $k_i\in\mathbb{N}$, $i=n+1,\dots,m$ such that

$$\left\|f-\sum_{i=1}^{m}D^Z_{k_i}(z_i)\right\|<\varepsilon.$$

Because D^Z_n are continuous, there is $\beta<\alpha$ such that

$$\left\|f-\sum_{i=1}^{m}D^Z_{k_i}(z_i')\right\|<\varepsilon,$$

for $z_i'\in P_\beta Z$.

Thus it remains to show that $\mathcal{D}^Z\big(P_\beta(Z)\big)\subset P^*_\beta Z^*$ for $\beta<\alpha$. Let $z\in P_\beta Z$. By the choice of $\{x_{\alpha+1};\alpha<\Gamma\}$ we know that $\mathcal{D}^Z(z)(x_\gamma)=0$ for $\gamma>\beta$. Thus

$$P^*_\beta\big(\mathcal{D}^Z(z)\big)(x)=\mathcal{D}^Z(z)(P_\beta x)=\mathcal{D}^Z(z)(x),$$

for all $x\in Z$, and it was exactly what we needed to prove.

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