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Dynamical properties of minimal Ferenczi subshifts

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Abstract. We provide an explicit $\mathcal{S}$-adic representation of rank-one subshifts with bounded spacers and call the subshifts obtained in this way 'minimal Ferenczi subshifts'. We aim to show that this approach is very convenient to study the dynamical behavior of rank-one systems. For instance, we compute their topological rank, the strong and the weak orbit equivalence class. We observe that they have an induced system that is a Toeplitz subshift having discrete spectrum. We also characterize continuous and non-continuous eigenvalues of minimal Ferenczi subshifts.

Key words: Ferenczi subshifts, rank-one subshifts, S-adic subshifts, minimal Cantor systems
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1. *Introduction*

Cutting and stacking transformations have been used extensively for more than 50 years in ergodic theory to produce a wide variety of dynamical systems which exhibit different behaviors [**Ada98**, **Bou93**, **Cha67**, **CPR22**, **Cre22**, **Jun76**, **Kin86**, **Kin88**, **Nad98**, **Orn72**, **Ryz20**]. These articles mainly concern the spectral properties, the centralizer and the disjointness of these transformations.

To understand how simple these systems are, in [**ORW82**] the notion of (measurable) *rank* is introduced to formalize some constructions initiated by Chacon in [**Cha67**]. Roughly speaking, the measurable rank is the minimal number of 'stacks' needed in the cutting and stacking process. They are defined by two sequences, usually called *cutting* and *spacer* parameters. The systems requiring a unique stack are called *rank-one systems* and should be thought as the simplest systems with respect to this notion. It includes periodic systems and rotations on compact groups [**Jun76**], but also many other systems that have received a lot of attention since the late 1960s, as 'almost all' interval exchanges [**Fer97**, Vee84]. They have been mainly studied from a spectral and probabilistic point of view, and served to create examples and counterexamples in ergodic theory. For instance, the Chacon transformation [Cha67] is one of the first known examples of a measurable transformation which is weakly mixing but not mixing.

Ferenczi [Fer96, Fer97] proposed a different perspective representing these systems as subshifts, whereas they have a purely measure-theoretic and geometric origin. This combinatorial and topological model, that can be traced back to [Kal84], imposed a different framework and led to many different questions. For instance, these subshifts are known to have zero topological entropy. Moreover, they have non-superlinear symbolic complexity [Fer96, Proposition 2], but they may have peaks with any prescribed sub-exponential growth [Fer96, Proposition 3]. We refer to [AFP17, GH14, GH16a, GH16b, GH21, GZ19, GZ20] for recent results about the combinatorial and topological models of rank-one systems.

For minimal systems defined on Cantor spaces, there exists a different and well-established notion of rank, called the *topological rank* [BDM10, BKMS13, DM08, DP22, Dur10]. The class of systems of topological rank one coincides with the class of odometers, so we decided to refer to the symbolic construction of rank-one systems as *Ferenczi subshifts* to avoid any misleading definition. Moreover, with Ferenczi being the one that popularized this class of subshifts [Fer96, Fer97], we came naturally to coin his name to them.

This article is devoted to the study of minimal Ferenczi subshifts, that is, those defined by a uniformly bounded sequence of spacers. We attempt to create a comprehensive classification for minimal Ferenczi subshifts according to some dynamical properties that we find relevant. More specifically, we want to compute their topological rank and to describe their (strong and weak) orbit equivalence class, to describe their (continuous and measurable) spectrum, to explore its mixing properties and to compute their automorphism group.

We begin by making the crucial observation that a subshift is a minimal Ferenczi subshift if and only if it is an $\mathcal{S}$-adic subshift generated by a particular directive sequence of finite alphabet rank. The family of $\mathcal{S}$-adic subshifts, introduced by Ferenczi in [Fer96], is a rich family that has been studied intensively and many different behaviors have been proposed [BD14, BSTY19, DDMP21, Dur00, Ler14].

It is particularly desirable to have primitive, proper and recognizable directive sequences as this allows, without effort, to define a nested sequence of Kakutani–Rokhlin partitions in towers [DL12]. This is a central tool for the study of the dynamical properties. For instance, systems admitting such partitions with a uniform bound for the number of towers are of zero topological entropy [Dur10], have an explicit description of their ergodic invariant probability measures [BKMS13] and there exist necessary and sufficient conditions for a complex number to be a continuous or measurable eigenvalue [BDM10, DFM19].

The directive sequence of morphisms we obtain for minimal Ferenczi subshifts has some nice properties, however they are not *proper*. A recent result of Espinoza [Esp22] shows that this directive sequence can be chosen to be proper, but his general method deteriorates the nice structure of the morphisms we obtained and considerably increases

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the size of the alphabets. Nevertheless, we can perform a standard trick which guarantees properness, retaining a nice structure of the morphisms and the alphabets.

A direct consequence of the nice structure of the morphisms generating a minimal Ferenczi subshift is that we can compute the topological rank in terms of the cutting and spacer parameters, we recover the well-known fact that they are uniquely ergodic and we show that they have a Toeplitz subshift as an induced system. Moreover, we show that this induced system is mean equicontinuous and, thus, has discrete spectrum [**DG16**, **GRJY21**, **LTY15**].

We characterize the *exact finite rank* of the directive sequences for minimal Ferenczi subshifts, that is, when all towers decomposing the system have a measure bounded away from zero at each level [**BKMS13**]. This has an incidence in the study of measurable eigenvalues, as we give a general necessary condition for a complex number to be a measurable eigenvalue for $\mathcal{S}$-adic subshifts. We believe this result has its own interest for further studies. It extends to subshifts what it is often called the *Veech criterion* for interval exchange transformations [**Vee84**].

In order to understand the (strong and weak) orbit equivalence class of minimal Ferenczi subshifts and their infinitesimals (in the spirit of [**GPS95**]), we provide a one-to-one correspondence between the orbit equivalence classes and a family of dimension groups, that we call of *Ferenczi type*.

We then turn to the study of eigenvalues of minimal Ferenczi subshifts. The group of measurable eigenvalues of a given system gives useful information, as it defines the Kronecker factor that comes naturally with the result of Halmos and von Neumann [**HN42**], and also allows to study the weakly mixing property. In the topological dynamics counterpart, the group of continuous eigenvalues allows us to understand the maximal equicontinuous factor (in the minimal case) and the topological weakly mixing property.

In general, it is not true that measurable eigenvalues are continuous. Measurable eigenvalues coincide with continuous ones for the class of primitive substitution systems [**Hos86**]. However, there exist linearly recurrent minimal Cantor systems with measurable and non-continuous eigenvalues [**BDM05**].

In this article, we adopt the general framework of [**BDM10**, **DFM19**] to study eigenvalues of minimal Ferenczi subshifts. This allows to give an alternative proof about the description of continuous eigenvalues [**GH16a**, **GZ19**] and to show that all measurable eigenvalues are continuous in the *exact finite-rank* case, which extends a result in [**GH16a**]. We also provide some realization results in the non-exact finite-rank case with non-continuous eigenvalues.

We also explore the mixing properties of minimal Ferenczi subshifts. With this purpose, inspired by results in [**KSS05**], we give a general necessary condition for topological mixing of minimal subshifts defined on a binary alphabet. This gives an alternative proof to the fact that minimal Ferenczi subshifts are not topologically mixing [**GZ19**].

Finally, we show that subshifts in this family have a unique asymptotic class, which by a standard argument implies that the automorphism group is trivial. This gives an alternative proof of a result in [**GH16b**].

We expect that this $\mathcal{S}$-adic approach is convenient to investigate some other relevant questions in topological and measurable dynamics of subshifts.

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1.1. *Organization.* In the next section we give the basic background in topological dynamics and $\mathcal{S}$-adic subshifts needed in this article. We characterize minimal Ferenczi subshifts as those $\mathcal{S}$-adic subshifts generated by particular directive sequences in §3. Section 4 is devoted to the study of these subshifts from the topological dynamics viewpoint. We compute the topological rank and the dimension group of minimal Ferenczi subshifts and their strong and weak orbit equivalence classes. Then, we study the continuous eigenvalues, the maximal equicontinuous factor and the topological mixing of minimal Ferenczi subshifts. In the last part of the section, we show that minimal Ferenczi subshifts have a unique asymptotic class and a trivial automorphism group.

We study the measurable eigenvalues of minimal Ferenczi subshifts in §5. We illustrate these results with concrete examples.

In this article, we let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of non-negative integers and the set of integers numbers, respectively. For a finite set $\mathcal{A}$, we also denote by $\mathbb{R}_+^{\mathcal{A}}$ (respectively, $\mathbb{Z}_+^{\mathcal{A}}$) the set of non-negative vectors (respectively, non-negative integer vectors) indexed by $\mathcal{A}$. Similarly, we denote by $\mathbb{R}_{>0}^{\mathcal{A}}$ (respectively, $\mathbb{Z}_{>0}^{\mathcal{A}}$) to the set of positive vectors (respectively, positive integer vectors). For a vector v in $\mathbb{R}^{\mathcal{A}}$ the Euclidean norm of v is denoted by $\|v\|$ and we write $|||v||| = \inf_{w\in\mathbb{Z}^{\mathcal{A}}}\|v - w\|$.

2. *Preliminaries*

2.1. *Basics in topological dynamics and eigenvalues.* A *topological dynamical system* (or just a system) is a compact metric space X together with a homeomorphism $T : X \to X$. We use the notation (X, T). If X is a Cantor space (i.e., X has a countable basis of clopen sets and it has no isolated points) we say it is a *Cantor system*. The system (X, T) is *minimal* if for every point $x \in X$ the orbit $\{T^n x : n \in \mathbb{Z}\}$ is dense in X.

Let (X, T) and (X', T') be two topological dynamical systems. We say that (X', T') is a *topological factor* of (X, T) if there exists a continuous and surjective map $\phi : X \to X'$ such that

$$\phi \circ T = T' \circ \phi. \tag{1}$$

In this case, we say that ϕ a *factor map*. If, in addition, the map ϕ in (1) is a homeomorphism, we say that it is a *topological conjugacy* and that (X, T) and (X', T') are *topologically conjugate*.

Let (X, T) be a minimal Cantor system and $U \subseteq X$ be a non-empty clopen set. We can define the *return time function* $r_U : X \to \mathbb{N}$ by

$$r_U(x) = \inf\{n > 0 : T^n x \in U\}, \quad x \in X.$$

It is easy to see that the map r_U is locally constant and, hence, continuous. The *induced map* $T_U : U \to U$ is defined by

$$T_U(x) = T^{r_U(x)}x, \quad x \in U.$$

We have that $T_U : U \to U$ is a homeomorphism and that (U, T_U) is a minimal Cantor system. We call it the *induced system* of (X, T) on U.

We say that a complex number λ is a *continuous eigenvalue* of the system (X, T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* associated with λ. The system (X, T) is *topologically weakly-mixing* if it has no non-constant continuous eigenfunctions.

Let μ be a T-invariant probability measure defined on the Borel σ-algebra of X, that is, $\mu(T^{-1}(A)) = \mu(A)$ for every measurable set $A \subseteq X$. We say that a complex number λ is a *measurable eigenvalue* of the system (X, T) with respect to μ if there exists $f \in L^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *measurable eigenfunction* associated with λ. The system is *weakly mixing* for μ if it has no non-constant measurable eigenfunctions.

If the system (X, T) is minimal (respectively, if μ is ergodic for (X, T)), then every continuous eigenvalue (respectively, measurable eigenvalue with respect to μ) has modulus one and every continuous eigenfunction (respectively, measurable eigenfunction) has a constant modulus on X (respectively, a constant modulus μ-almost everywhere on X).

Whenever the measure μ is ergodic for (X, T) or when (X, T) is minimal, we write $\lambda = \exp(2\pi i\alpha)$ with $\alpha \in [0, 1)$ to denote eigenvalues of the system. If $\lambda = \exp(2\pi i\alpha)$ is an eigenvalue of the system with α an irrational number (respectively, rational number), we say that λ is an irrational eigenvalue (respectively, rational eigenvalue).

2.2. *Basics in symbolic dynamics*

2.2.1. *Subshifts.* Let $\mathcal{A}$ be a finite set that we call *alphabet*. Elements in $\mathcal{A}$ are called *letters* or *symbols*. The number of letters of $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The set of finite sequences or *words* of length $\ell \in \mathbb{N}$ with letters in $\mathcal{A}$ is denoted by $\mathcal{A}^\ell$ and the set of two-sided sequences $(x_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ is denoted by $\mathcal{A}^\mathbb{Z}$. A word $w = w_0 w_1 \ldots w_{\ell-1} \in \mathcal{A}^\ell$ can be seen as an element of the free monoid $\mathcal{A}^*$ endowed with the operation of concatenation (whose neutral element is ε, the empty word). The integer ℓ is the *length* of the word w and is denoted by $|w| = \ell$; the length of the empty word is zero. A word v is a *power* of a word u if $v = u^n$ for some $n \in \mathbb{N}$.

For finite words p and s in $\mathcal{A}^*$, we say that they are a *prefix* and a *suffix*, respectively, of the word ps. For $x \in \mathcal{A}^\mathbb{Z}$ and integers $N > n$ we define the word $x_{[n,N)} = x_n x_{n+1} \ldots x_{N-1}$. For a non-empty word $w \in \mathcal{A}^*$ and a point $x \in \mathcal{A}^\mathbb{Z}$, we say that w *occurs* in x if there exists $n \in \mathbb{Z}$ such that $x_n x_{n+1} \ldots x_{n+|w|-1} = w$. In this case, we say that the index n is an *occurrence* of w in x. We use the same notion for finite non-empty words x. We say that a non-empty word $w = w_0 w_1 \ldots w_{\ell-1} \in \mathcal{A}^*$ *starts* (respectively, *ends*) with a non-empty word $u \in \mathcal{A}^*$ if $u = w_0 \ldots w_{i-1}$ for some $i \leq \ell$ (respectively, $u = w_j \ldots w_{\ell-1}$ for some $j \geq 0$).

The *shift map* $S : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is defined by $S((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. A *subshift* is a topological dynamical system (X, S) where X is a closed and S-invariant subset of $\mathcal{A}^\mathbb{Z}$. Here, we consider the product topology on $\mathcal{A}^\mathbb{Z}$. Classically, one identifies (X, S) with X, so one says that X itself is a subshift. When we say that a sequence x in a subshift is *aperiodic*, we implicitly mean that x is aperiodic for the action of the shift.

Let (X, S) be a subshift. The *language* of (X, S) is the set $\mathcal{L}(X)$ containing all words $w \in \mathcal{A}^*$ such that $w = x_{[m,m+|w|)}$ for some $x = (x_n)_{n\in\mathbb{Z}} \in X$ and $m \in \mathbb{Z}$. In this case, we also say that w is a *factor* (also called *subword*) of x. We denote by $\mathcal{L}_\ell(X)$ the set of words of length ℓ in $\mathcal{L}(X)$. Given $x \in X$, the language $\mathcal{L}(x)$ is the set of all words that occur in x. As before, we define $\mathcal{L}_\ell(x)$. For two words $u, v \in \mathcal{L}(X)$, the *cylinder set* $[u.v]$ is the set $\{x \in X : x_{[-|u|,|v|)} = uv\}$. When u is the empty word we only write $[v]$, erasing the dot. We remark that cylinder sets are clopen sets and they form a base for the topology of the subshift.

2.2.2. *Morphisms.* Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets and $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a morphism. We say that τ is *erasing* whenever there exists a letter $a \in \mathcal{A}$ such that $\tau(a)$ is the empty word. Otherwise, we say it is *non-erasing*. When the morphism τ is non-erasing, it extends naturally to a map from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{B}^{\mathbb{Z}}$ by concatenation (we apply τ to positive and negative coordinates separately and we concatenate the results at coordinate zero). We continue to call this map τ. We observe that any map $\tau : \mathcal{A} \to \mathcal{B}^*$ can be naturally extended to a morphism (that we also denote by τ) from $\mathcal{A}^*$ to $\mathcal{B}^*$ by concatenation.

The *composition matrix* of a morphism $\tau : \mathcal{A}^* \to \mathcal{B}^*$ is given for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$ by $M_\tau(b, a) = |\tau(a)|_b$, where $|\tau(a)|_b$ counts the number of occurrences of the letter b in the word $\tau(a)$. The morphism τ is said to be *positive* if M_τ has positive entries and *proper* if there exist $p, s \in \mathcal{B}$ such that for all $a \in \mathcal{A}$ the word $\tau(a)$ starts with p and ends with s.

The minimum and maximal lengths of τ are, respectively, the numbers

$$\langle \tau \rangle = \min_{a \in \mathcal{A}} |\tau(a)| \quad \text{and} \quad |\tau| = \max_{a \in \mathcal{A}} |\tau(a)|.$$

We say that a morphism τ is of *constant length* if $\langle \tau \rangle = |\tau|$. Observe that if $\tau : \mathcal{A}^* \to \mathcal{B}^*$ and $\tau' : \mathcal{B}^* \to \mathcal{C}^*$ are two constant length morphisms, then $\tau' \circ \tau$ is also of constant length and

$$|\tau' \circ \tau| = |\tau'||\tau|. \tag{2}$$

Following [BSTY19], a morphism $\tau : \mathcal{A}^* \to \mathcal{B}^*$ is *left permutative* (respectively, *right permutative*) if the first (respectively, last) letters of $\tau(a)$ and $\tau(b)$ are different, for all distinct letters $a, b \in \mathcal{A}$. Two morphisms $\tau, \widetilde{\tau} : \mathcal{A}^* \to \mathcal{B}^*$ are said to be *rotationally conjugate* if there is a word $w \in \mathcal{B}^*$ such that $\tau(a)w = w\widetilde{\tau}(a)$ for all $a \in \mathcal{A}$ or $\widetilde{\tau}(a)w = w\tau(a)$ for all $a \in \mathcal{A}$.

2.2.3. *$\mathcal{S}$-adic subshifts.* We recall the definition of *$\mathcal{S}$-adic subshifts* as stated in [BSTY19]. A *directive sequence* $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq 0}$ is a sequence of non-erasing morphisms. A slightly more general definition is given in [DP22] including the case of erasing morphisms. When all morphisms τ_n for $n \geq 0$ are proper, we say that $\boldsymbol{\tau}$ is *proper*. For $0 \leq n \leq N$, we denote by $\tau_{[n,N)}$ the morphism $\tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_{N-1}$, where $\tau_{[n,n)} : \mathcal{A}_n^* \to \mathcal{A}_n^*$ is the identity map for each $n \geq 0$. We say $\boldsymbol{\tau}$ is *everywhere growing* if $\langle \tau_{[0,n)} \rangle \to +\infty$ as $n \to +\infty$ and say that it is *primitive* if for any $n \in \mathbb{N}$ there exists $N > n$ such that $M_{\tau_{[n,N)}}$ has positive entries, that is, for every $a \in \mathcal{A}_N$ the word $\tau_{[n,N)}(a)$ contains all letters in $\mathcal{A}_n$. Observe that primitivity implies everywhere growing. If $\boldsymbol{\tau}$ is primitive, then the subshift $(X_{\boldsymbol{\tau}}, S)$ is minimal (see, for instance, [DP22, Proposition 6.4.5]). However, there are minimal subshifts that are generated by non-everywhere-growing directive sequences, as for the Chacon subshift generated by a constant directive sequence given by the morphism $0 \mapsto 0010$, $1 \mapsto 1$.

For $n \in \mathbb{N}$, the *language* $\mathcal{L}^{(n)}(\boldsymbol{\tau})$ *of level n associated with* $\boldsymbol{\tau}$ is defined by

$$\mathcal{L}^{(n)}(\boldsymbol{\tau}) = \{w \in \mathcal{A}_n^* : w \text{ occurs in } \tau_{[n,N)}(a) \text{ for some } a \in \mathcal{A}_N \text{ and } N > n\}$$

and let $X_{\boldsymbol{\tau}}^{(n)}$ be the set of points $x \in \mathcal{A}_n^{\mathbb{Z}}$ such that $\mathcal{L}(x) \subseteq \mathcal{L}^{(n)}(\boldsymbol{\tau})$. This set clearly defines a subshift that we call the *subshift generated by* $\mathcal{L}^{(n)}(\boldsymbol{\tau})$. We set $X_{\boldsymbol{\tau}} = X_{\boldsymbol{\tau}}^{(0)}$ and call $(X_{\boldsymbol{\tau}}, S)$ or $X_{\boldsymbol{\tau}}$ the $\mathcal{S}$*-adic subshift* generated by the directive sequence $\boldsymbol{\tau}$.

A *contraction* of $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ is a directive sequence of the form

$$\widetilde{\boldsymbol{\tau}} = (\widetilde{\tau}_k = \tau_{[n_k,n_{k+1})} : \mathcal{A}_{n_{k+1}}^* \to \mathcal{A}_{n_k}^*)_{k\geq 0},$$

where the sequence $(n_k)_{k\geq 0}$ is such that $n_0 = 0$ and $n_k < n_{k+1}$ for all $k \geq 0$. Observe that any contraction of $\boldsymbol{\tau}$ generates the same $\mathcal{S}$-adic subshift $X_{\boldsymbol{\tau}}$.

We say that a directive sequence $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ is *invertible* if the linear map $M_{\tau_n} : \mathbb{R}^{\mathcal{A}_n} \to \mathbb{R}^{\mathcal{A}_{n+1}}$ (acting on row vectors) is invertible for all $n \geq 0$. Observe that this implies that the sequence $(|\mathcal{A}_n|)_{n\geq 0}$ is constant.

The following proposition generalizes [**BCBD+21**, Lemma 3.3]. The proof is similar and we include it here for the sake of completeness.

PROPOSITION 2.1. *Let* $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ *be a primitive and invertible directive sequence. Then* $(X_{\boldsymbol{\tau}}, S)$ *is minimal and aperiodic.*

Proof. It is enough to show that $(X_{\boldsymbol{\tau}}, S)$ is aperiodic. By contradiction, define $p \in \mathbb{N}$ to be the smallest possible period among all periodic points in $X_{\boldsymbol{\tau}}$.

Let $y = \ldots uu.uu \ldots$ be a periodic point in $X_{\boldsymbol{\tau}}$, where $|u| = p$. Since $\boldsymbol{\tau}$ is primitive, there exists $n \in \mathbb{N}$ such that $\langle \tau_{[0,n)} \rangle \geq p$. Without loss of generality, there exists $x \in \mathcal{A}_n^{\mathbb{Z}}$ such that $y = \tau_{[0,n)}(x)$. Furthermore, because $\boldsymbol{\tau}$ is primitive we can assume that every letter of $\mathcal{A}_n$ occurs in x.

If the word $\tau_{[0,n)}(x_0)$ is not a power of u, then there exists a non-empty prefix v (respectively, non-empty suffix w) of u such that $u = vw$, $\tau_{[0,n)}(x_0)$ ends with v and $\tau_{[0,n)}(x_1)$ starts with w. The word $\tau_{[0,n)}(x_1)$ starts with u, so there exists a suffix v' of u such that $u = wv'$. However, because $y = \ldots uu.uu \ldots$, the word v' is also a prefix of u with $|v'| = |v|$, so $v = v'$. The Fine–Wilf theorem then implies that v and w are powers of a same word, contradicting the definition of p.

This shows that $\tau_{[0,n)}(x_0) = u^{p_0}$ for some $p_0 \in \mathbb{N}$ and, inductively, for each $m \in \mathbb{Z}$ there exists $p_m \in \mathbb{N}$ such that $\tau_{[0,n)}(x_m) = u^{p_m}$. In particular, for each $a \in \mathcal{A}_n$ there exists $p_a \in \mathbb{N}$ such that $\tau_{[0,n)}(a) = u^{p_a}$. Therefore, the columns of $M_{\tau_{[0,n)}}$ are multiples of the column vector $(|u|_a)_{a\in\mathcal{A}_0}$. This contradicts the fact that the linear map given by $M_{\tau_{[0,n)}}$ is invertible and finishes the proof. □

2.2.4. *Recognizability.* Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism and $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift. For $x \in X$ and $k \in \mathbb{N}$ with $0 \leq k < |\tau(x_0)|$, the *cutting points* of the pair (k, x) are defined as follows. If $\ell \geq 0$, we define the ℓth cutting point of (k, x) as

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$$C^{\ell}_{\tau}(k, x) = |\tau(x_{[0,\ell)})| - k.$$

Similarly, if $\ell < 0$ the ℓth cutting point of (k, x) is $C^{\ell}_{\tau}(k, x) = -|\tau(x_{[\ell,0)})| - k$. Define $\mathcal{C}^{+}_{\tau}(k, x) = \{C^{\ell}_{\tau}(k, x) : \ell > 0\}$.

If $y = S^k\tau(x)$ with $x \in X$ and $k \in \mathbb{N}$, $0 \leq k < |\tau(x_0)|$, we say that (k, x) is a *centered τ-representation* of y. The centered τ-representation (k, x) *is in X* if x belongs to X. The morphism τ is *recognizable in X* (respectively, *recognizable in X for aperiodic points*) if any point $y \in \mathcal{B}^{\mathbb{Z}}$ (respectively, any aperiodic point $y \in \mathcal{B}^{\mathbb{Z}}$) has at most one centered τ-representation in X. If τ is recognizable in $\mathcal{A}^{\mathbb{Z}}$ (for aperiodic points), we say that τ is *fully recognizable* (for aperiodic points).

In what follows, we use the following results [**BSTY19**, Theorem 3.1, Lemma 3.5].

PROPOSITION 2.2. *Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism. Assume that τ is (rotationally conjugate to) a left or right permutative morphism. Then τ is fully recognizable for aperiodic points.*

PROPOSITION 2.3. *Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and $\tau : \mathcal{B}^* \to \mathcal{C}^*$ be two non-erasing morphisms, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $Y = \bigcup_{k\in\mathbb{Z}} S^k\sigma(X)$. If σ is recognizable in X for aperiodic points and τ is recognizable in Y for aperiodic points, then $\tau \circ \sigma$ is recognizable in X for aperiodic points.*

We also need the following straightforward lemma [**DP22**, Proposition 1.4.30].

LEMMA 2.4. *Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism and $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a minimal and aperiodic subshift. Suppose that τ is recognizable in X and let $Y = \bigcup_{k\in\mathbb{Z}} S^k\tau(X)$. Then (X, S) is topologically conjugate to the induced system $(\tau(X), S_{\tau(X)})$ of (Y, S) on $\tau(X)$.*

2.2.5. *Recognizability for sequences of morphisms.* Following [**BSTY19**], a directive sequence $\boldsymbol{\tau} = (\tau_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)_{n\geq 0}$ is said to be *recognizable at level n* if the morphism τ_n is recognizable in $X^{(n+1)}_{\boldsymbol{\tau}}$. We say that the directive sequence $\boldsymbol{\tau}$ is *recognizable* if it is recognizable at level n for each $n \geq 0$.

We have that $\boldsymbol{\tau}$ is recognizable if and only if for all $0 \leq n < N$ and any point $y \in X^{(n)}_{\boldsymbol{\tau}}$ there is a unique couple (k, x) with $x \in X^{(N)}_{\boldsymbol{\tau}}$ and $0 \leq k < |\tau_{[n,N)}(x_0)|$ such that $y = S^k\tau_{[n,N)}(x)$. This is the content of [**BSTY19**, Lemmas 3.5 and 4.2]. Indeed, $\boldsymbol{\tau}$ is recognizable if and only if for all $n \geq 0$ and any point $y \in X_{\boldsymbol{\tau}}$ there is a unique couple (k, x) with $x \in X^{(n)}_{\boldsymbol{\tau}}$ and $0 \leq k < |\tau_{[0,n)}(x_0)|$ such that $y = S^k\tau_{[0,n)}(x)$.

Lemma 2.4 implies the following.

COROLLARY 2.5. *Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)_{n\geq 0}$ be a recognizable directive sequence and let $\boldsymbol{\tau}' = (\tau_{n+1} : \mathcal{A}^*_{n+2} \to \mathcal{A}^*_{n+1})_{n\geq 0}$ be the shifted directive sequence. Suppose that the subshift $(X_{\boldsymbol{\tau}}, S)$ is minimal and aperiodic. Then $(X_{\boldsymbol{\tau}'}, S)$ is topologically conjugate to the induced system $(\tau_0(X_{\boldsymbol{\tau}'}), S_{\tau_0(X_{\boldsymbol{\tau}'})})$ of $(X_{\boldsymbol{\tau}}, S)$ on $\tau_0(X_{\boldsymbol{\tau}'})$.*

2.3. *Kakutani–Rokhlin partitions.* Let (X, T) be a minimal Cantor system.

2.3.1. *CKR partitions of minimal Cantor systems.* A *clopen Kakutani–Rokhlin (CKR) partition* $\mathcal{T}$ of (X, T) is a partition of X of the form

$$\mathcal{T} = \{T^k B(a) : a \in \mathcal{A}(\mathcal{T}), 0 \leq k < h(a)\},$$

where $\mathcal{A}(\mathcal{T})$ is a non-empty finite alphabet, the value $h(a)$ is a positive integer and $B(a)$ is a clopen set for all $a \in \mathcal{A}(\mathcal{T})$. Observe that

$$\bigcup_{a\in\mathcal{A}(\mathcal{T})} T^{h(a)} B(a) = \bigcup_{a\in\mathcal{A}(\mathcal{T})} B(a).$$

The *base* of $\mathcal{T}$ is the set $B(\mathcal{T}) = \bigcup_{a\in\mathcal{A}(\mathcal{T})} B(a)$. The set $\mathcal{T}(a) = \bigcup_{0\leq k<h(a)} T^k B(a)$ is called the *tower* indexed by $a \in \mathcal{A}(\mathcal{T})$ of $\mathcal{T}$ with *base* $B(a)$ and *height* $h(a)$.

Let

$$\mathcal{T}_n = \{T^k B_n(a) : a \in \mathcal{A}(\mathcal{T}_n), 0 \leq k < h_n(a)\}, \quad n \geq 0$$

be a sequence of CKR partitions of (X, T). It is *nested* if for any $n \geq 0$:

(KR1) $B(\mathcal{T}_{n+1}) \subseteq B(\mathcal{T}_n)$;

(KR2) $\mathcal{T}_n \preceq \mathcal{T}_{n+1}$, that is, for every $A \in \mathcal{T}_{n+1}$ there exists $B \in \mathcal{T}_n$ such that $A \subseteq B$;

(KR3) $\bigcap_{n\geq 0} B(\mathcal{T}_n) = \{x\}$ for some point $x \in X$; and

(KR4) the atoms of $\bigcup_{n\geq 0} \mathcal{T}_n$ generate the topology of X.

We remark that nested sequences always exist [**HPS92**, Theorem 4.2].

For each $n \geq 0$, the *incidence matrix* M_n between the partitions $\mathcal{T}_{n+1}$ and $\mathcal{T}_n$ is given for each $a \in \mathcal{A}(\mathcal{T}_n)$ and $b \in \mathcal{A}(\mathcal{T}_{n+1})$ by

$$M_n(a, b) = \#\{0 \leq k < h_{n+1}(b) : T^k B_{n+1}(b) \subseteq B_n(a)\}. \tag{3}$$

For $n \geq 0$ let h_n be the row vector called *height vector* and defined by

$$h_n = (h_n(a))_{a\in\mathcal{A}(\mathcal{T}_n)}.$$

We define $P_{m,n} = M_m M_{m+1} \ldots M_{n-1}$ for $0 \leq m < n$. Observe that $P_{n,n+1} = M_n$. By means of a simple induction argument, we have $h_n = h_m P_{m,n}$ and

$$P_{m,n}(a, b) = \#\{0 \leq k < h_n(b) : T^k B_n(b) \subseteq B_m(a)\}, \tag{4}$$

$a \in \mathcal{A}(\mathcal{T}_m), b \in \mathcal{A}(\mathcal{T}_n), 0 \leq m < n$.

The *topological rank* of (X, T) is the value

$$\operatorname{rank}(X, T) = \inf_{\substack{\text{nested sequence } (\mathcal{T}_n)_{n\geq 0} \\ \text{of CKR partitions of } (X,T)}} \liminf_{n\to+\infty} |\mathcal{A}(\mathcal{T}_n)|. \tag{5}$$

Roughly speaking, the topological rank of (X, T) is the smallest number of CKR towers needed to describe (X, T). The topological rank is invariant under topological conjugacy. See [**BDM10**, **DM08**] for more details.

2.3.2. *Invariant measures through CKR partitions.* Let $(\mathcal{T}_n)_{n\geq 0}$ be a nested sequence of CKR partitions. Any T-invariant probability measure μ of (X, T) is uniquely determined

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by the values it assigns to atoms of the partitions, hence to the bases $B_n(a)$, $a \in \mathcal{A}(\mathcal{T}_n)$ and $n \geq 0$.

For $n \geq 0$ let μ_n be the column vector called *measure vector* and defined by

$$\mu_n = (\mu_n(a))_{a \in \mathcal{A}(\mathcal{T}_n)}, \quad \text{where } \mu_n(a) = \mu(B_n(a)).$$

Therefore, the measure μ is completely determined by the sequence of measure vectors $(\mu_n)_{n \geq 0}$. As μ is a probability measure, we have

$$\mu(\mathcal{T}_n(a)) = h_n(a)\mu_n(a) \quad \text{and} \quad \sum_{a \in \mathcal{A}(\mathcal{T}_n)} \mu(\mathcal{T}_n(a)) = 1. \tag{6}$$

In addition, by (4) we have

$$\mu_m = P_{m,n}\mu_n, \quad 0 \leq m < n. \tag{7}$$

2.3.3. *CKR partitions of $\mathcal{S}$-adic subshifts.* Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq 0}$ be a primitive, proper and recognizable directive sequence which generates the $\mathcal{S}$-adic subshift $(X_{\boldsymbol{\tau}}, S)$. Define the sequence $(\mathcal{T}_n)_{n \geq 0}$ as follows:

$$\mathcal{T}_n = \{S^k \tau_{[0,n)}([a]) : a \in \mathcal{A}_n, 0 \leq k < |\tau_{[0,n)}(a)|\}, \quad n \geq 0. \tag{8}$$

The following result proved in [**DL12**, Proposition 2.2] shows that $(\mathcal{T}_n)_{n \geq 0}$ defines a nested sequence of CKR partitions. We include a proof for the sake of completeness.

PROPOSITION 2.6. *The sequence $(\mathcal{T}_n)_{n \geq 0}$ is a nested sequence of CKR partitions of $(X_{\boldsymbol{\tau}}, S)$. Moreover, for each $n \geq 0$ the incidence matrix M_n between the partitions $\mathcal{T}_{n+1}$ and $\mathcal{T}_n$ coincides with the composition matrix M_{τ_n} of the morphism τ_n:*

$$M_n = M_{\tau_n}.$$

Proof. As $\boldsymbol{\tau}$ is recognizable, $\mathcal{T}_n$ is a CKR partition of $X_{\boldsymbol{\tau}}$ for each $n \geq 0$. Observe that the tower $\mathcal{T}_n(a)$ has base $B_n(a) = \tau_{[0,n)}([a])$ for $a \in \mathcal{A}_n$. Clearly we have $B(\mathcal{T}_{n+1}) \subseteq B(\mathcal{T}_n)$ for $n \geq 0$.

Claim 2.6.1. We claim that $\mathcal{T}_n \preceq \mathcal{T}_{n+1}$.

Indeed, let $S^k\tau_{[0,n+1)}([a])$ be an atom of $\mathcal{T}_{n+1}$, $a \in \mathcal{A}_{n+1}$, $0 \leq k < |\tau_{[0,n+1)}(a)|$. Let $\tau_n(a) = b_0 b_1 \ldots b_{i-1}$ with $b_j \in \mathcal{A}_n$, $0 \leq j < i$. Then, there exists $j \in [0, i-1)$ satisfying

$$|\tau_{[0,n)}(b_0 b_1 \ldots b_j)| \leq k < |\tau_{[0,n)}(b_0 b_1 \ldots b_{j+1})|.$$

We deduce that if $k' = |\tau_{[0,n)}(b_0 b_1 \ldots b_j)|$, then $S^k\tau_{[0,n+1)}([a]) \subseteq S^{k-k'}\tau_{[0,n)}([b_{j+1}])$ with $0 \leq k - k' < |\tau_{[0,n)}(b_{j+1})|$. This proves the claim.

Claim 2.6.2. The atoms of $\bigcup_{n \geq 0} \mathcal{T}_n$ generate the topology of $X_{\boldsymbol{\tau}}$.

Indeed, let $n \geq 1$, $a \in \mathcal{A}_n$, $0 \leq k < |\tau_{[0,n)}(a)|$ and ℓ be a non-negative integer. As τ_n is proper, there exist two letters p_n and s_n in $\mathcal{A}_n$ such that $\tau_n(a)$ starts with p_n and ends with s_n for all $a \in \mathcal{A}_{n+1}$ and $n \geq 0$. As $\boldsymbol{\tau}$ is primitive, there exists $N \in \mathbb{N}$

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such that if $n \geq N$ then $\langle \tau_{[0,n-1)} \rangle \geq \ell$. Let $x', y' \in \tau_{[0,n)}([a])$, $u_n = \tau_{[0,n-1)}(s_{n-1})$ and $v_n = \tau_{[0,n)}(a)\tau_{[0,n-1)}(p_{n-1})$. We have

$$x'_{[-|u_n|,|v_n|)} = y'_{[-|u_n|,|v_n|)} = u_n v_n,$$

so that $x'_{[-\ell,\ell+k]} = y'_{[-\ell,\ell+k]}$. If x, y belong to $S^k\tau_{[0,n)}([a])$, $n \geq N$, then $x_{[-\ell-k,\ell]} = y_{[-\ell-k,\ell]}$ and, in particular, $x_{[-\ell,\ell]} = y_{[-\ell,\ell]}$. Therefore, $\mathrm{diam}(S^k\tau_{[0,n)}([a])) \to 0$ as $n \to +\infty$. This proves the claim. As the bases $(B(\mathcal{T}_n))_{n\geq 0}$ are nested, they converge to some point. This finishes the proof of the first statement.

The second statement follows easily from the recognizability of $\boldsymbol{\tau}$. □

We remark that the height vectors $(h_n)_{n\geq 0}$ of $(\mathcal{T}_n)_{n\geq 0}$ defined by (8) satisfy

$$h_n(a) = |\tau_{[0,n)}(a)|, \quad a \in \mathcal{A}_n, \quad n \geq 0. \tag{9}$$

2.4. *Dimension groups.* In this section we recall the basic on dimension groups and state the main results that we use throughout this article. We refer to [**DP22**, **GPS95**] for more complete references.

2.4.1. *Direct limits.* Let $(G_n)_{n\geq 0}$ be a sequence of abelian groups and let $i_{n+1,n} : G_n \to G_{n+1}$ for each $n \geq 0$ be a morphism. Define the subgroups Δ and Δ^0 of the direct product $\prod_{n\geq 0} G_n$ by

$$\Delta = \{(g_n)_{n\geq 0} \in \textstyle\prod_{n\geq 0} G_n : g_{n+1} = i_{n+1,n}(g_n) \text{ for every large enough } n\}$$

and

$$\Delta^0 = \{(g_n)_{n\geq 0} \in \textstyle\prod_{n\geq 0} G_n : g_n = 0 \text{ for every large enough } n\}.$$

Let $G = \Delta/\Delta^0$ be the quotient group and $\pi : \Delta \to G$ be the natural projection. The group G is called the *direct limit* of $(G_n)_{n\geq 0}$ and we write $G = \varinjlim G_n$. If $g \in G_n$, then all sequences $(g_k)_{k\geq 0}$ such that $g_n = g$ and $g_{k+1} = i_{k+1,k}(g_k)$ for all $k \geq n$ belong to Δ and have the same projection in G, denoted by $i_n(g)$. This defines a group morphism $i_n : G_n \to G$, which we call the *natural morphism* from G_n to G. For $0 \leq m < n$ define

$$i_{n,m} = i_{n-1,n} \circ i_{n,n+1} \circ \cdots \circ i_{m+1,m}.$$

We have $i_m = i_n \circ i_{n,m}$ and $G = \bigcup_{n\geq 0} \mathrm{Im}\, i_n$.

We can also define direct limits of vector spaces. Let $\mathbb{K}$ be a field. For each $n \geq 0$, let V_n be a vector space over $\mathbb{K}$ and $i_{n+1,n} : V_n \to V_{n+1}$ be a linear map. The *direct limit* $V = \varinjlim V_n$ is the vector space over $\mathbb{K}$, where the group structure on V is that given by the direct limit of the abelian groups V_n and the scalar multiplication is given by pointwise scalar multiplication on each coordinate.

2.4.2. *Orbit equivalence.* Two minimal Cantor systems (X, T) and (X', T') are *orbit equivalent* if there exists a homeomorphism $\Phi : X \to X'$ which sends orbits onto orbits, that is,

$$\Phi(\{T^n x : n \in \mathbb{Z}\}) = \{(T')^n \circ \Phi(x) : n \in \mathbb{Z}\}, \quad x \in X.$$

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This implies that there exist two maps $\alpha : X \to \mathbb{Z}$ and $\beta : X' \to \mathbb{Z}$, uniquely defined by aperiodicity, such that

$$\Phi \circ T(x) = (T')^{\alpha(x)} \circ \Phi(x) \quad \text{and} \quad \Phi \circ T^{\beta(x)}(x) = T' \circ \Phi(x), \quad x \in X.$$

The minimal Cantor systems (X, T) and (X', T') are *strongly orbit equivalent* if α and β both have at most one point of discontinuity.

2.4.3. *Dimension groups of minimal Cantor systems.* Denote by $C(X, \mathbb{Z})$ (respectively, $C(X, \mathbb{N})$) the group (respectively, monoid) of continuous functions from X to $\mathbb{Z}$ (respectively, $\mathbb{N}$) with the addition operation. Consider the map $\partial : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ defined by $\partial f = f \circ T - f$.

A map f is called a *coboundary* if there exists $g \in C(X, \mathbb{Z})$ such that $f = \partial g$. Two maps $f, f' \in C(X, \mathbb{Z})$ are said to be *cohomologous* if $f - f'$ is a coboundary.

Define the quotient group $H(X, T) = C(X, \mathbb{Z})/\partial C(X, \mathbb{Z})$. Let $[f]$ be the class of $f \in C(X, \mathbb{Z})$ in $H(X, T)$ and $\pi : C(X, \mathbb{Z}) \to H(X, T)$ be the projection map. Define $H^+(X, T) = \pi(C(X, \mathbb{N}))$ and denote by $\mathbf{1}_X$ the constant one valued function.

Consider the triple

$$K^0(X, T) = (H(X, T), H^+(X, T), [\mathbf{1}_X]).$$

It is an *ordered group* with *order unit* $[\mathbf{1}_X]$. As (X, T) is minimal, it is a *dimension group*. See [DP22, GPS95] for the definitions and more details. We call it *the dimension group of* (X, T).

It is classical to observe that if (X, T) is topologically conjugate to (X', T'), then the ordered groups with order units $K^0(X, T)$ and $K^0(X', T')$ are *unital order isomorphic*, i.e., there exists a group morphism $\delta : H(X, T) \to H(X', T')$ such that $\delta(H^+(X, T)) = H^+(X', T')$ and $\delta([\mathbf{1}_X]) = [\mathbf{1}_{X'}]$.

Denote by $\mathcal{M}(X, T)$ the set of invariant probability measures of (X, T). We define the set of *infinitesimals* of $H(X, T)$ as

$$\operatorname{Inf} H(X, T) = \left\{[f] \in H(X, T) : \int f \, d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, T)\right\}.$$

We have that $H(X, T)/\operatorname{Inf} H(X, T)$ with the induced order is also a dimension group. We denote it by $K^0(X, T)/\operatorname{Inf} K^0(X, T)$.

The dimension groups $K^0(X, T)$ and $K^0(X, T)/\operatorname{Inf} K^0(X, T)$ characterize *strong orbit equivalence* and *orbit equivalence*, respectively [GPS95].

Another description of the dimension group $K^0(X, T)$ is as follows. Let $(\mathcal{T}_n)_{n\geq 0}$ be a nested sequence of CKR partitions of (X, T) as defined in §2.3. Let $(\mathcal{A}(\mathcal{T}_n))_{n\geq 0}$, $(h_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ be the associated sequences of alphabets, height vectors and incidence matrices, respectively.

For $n \geq 0$ we consider $\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)}$ as an ordered group of row vectors with the usual order. Define the sequence of ordered groups with order units

$$\mathcal{G}_n = (\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)}, \mathbb{Z}_+^{\mathcal{A}(\mathcal{T}_n)}, h_n), \quad n \geq 0.$$

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Let $\mathcal{G} = \varinjlim \mathcal{G}_n$ be the direct limit of the groups $\mathcal{G}_n$ with respect to the morphisms $M_n : \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \to \mathbb{Z}^{\mathcal{A}(\mathcal{T}_{n+1})}$ given by the incidence matrix M_n (acting on row vectors). Let $\mathcal{G}^+$ be the projection in $\mathcal{G}$ of the set of points $(x_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)}$ for which there exists $N \in \mathbb{N}$ such that $x_N \in \mathbb{Z}_+^{\mathcal{A}(\mathcal{T}_N)}$ and $x_{k+1} = x_k M_k$, $k \geq N$. Denote by u the projection in $\mathcal{G}$ of the sequence $(h_n)_{n\geq 0}$.

The tuple $\mathcal{K} = (\mathcal{G}, \mathcal{G}^+, u)$ is a dimension group. The introduction of this dimension group is motivated by the following proposition [**DP22**, Theorem 5.3.6].

PROPOSITION 2.7. *Let (X, T) be a minimal Cantor system and let $\mathcal{K}$ be the dimension group associated to a nested sequence of CKR partitions of (X, T). Then, the dimension group $K^0(X, T)$ is unital order isomorphic to $\mathcal{K}$.*

2.4.4. *Dimension groups of $\mathcal{S}$-adic subshifts.* Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be a primitive, proper and recognizable directive sequence. Suppose that $(X_{\boldsymbol{\tau}}, S)$ is aperiodic. Let $(\mathcal{T}_n)_{n\geq 0}$ be the sequence of CKR partitions given in (8) and $\mathcal{K}$ be the dimension group associated to it. Recall that, by Proposition 2.6, the incidence matrix M_n between the partitions $\mathcal{T}_{n+1}$ and $\mathcal{T}_n$ coincide with the composition matrix M_{τ_n} of the morphism τ_n.

We call $\mathcal{K}$ *the dimension group of* $\boldsymbol{\tau}$. By Proposition 2.7, the dimension group of $(X_{\boldsymbol{\tau}}, S)$ is unital order isomorphic to $\mathcal{K}$.

In the case where all the linear maps M_{τ_n}, $n \geq 1$ are invertible, it is easy to check from the definition that the dimension group $K^0(X_{\boldsymbol{\tau}}, S)$ is unital order isomorphic to $(\mathcal{G}, \mathcal{G}^+, u)$, where

$$\mathcal{G} = \{x \in \mathbb{R}^{\mathcal{A}_1} : xM_{\tau_1}M_{\tau_2}\dots M_{\tau_n} \in \mathbb{Z}^{\mathcal{A}_{n+1}} \text{ for large enough } n\},$$
$$\mathcal{G}^+ = \{x \in \mathbb{R}^{\mathcal{A}_1} : xM_{\tau_1}M_{\tau_2}\dots M_{\tau_n} \in \mathbb{Z}_+^{\mathcal{A}_{n+1}} \text{ for large enough } n\},$$

and $u = (|\tau_0(a)|)_{a\in\mathcal{A}_1} \in \mathbb{R}^{\mathcal{A}_1}$.

3. *$\mathcal{S}$-adic representation of minimal Ferenczi subshifts*

3.1. *Ferenczi subshifts.* Following [**Fer96**, **Fer97**], we consider sequences of non-negative integers $(q_n)_{n\geq 0}$ and $(a_{n,i} : n \geq 0,\ 0 \leq i < q_n)$, which we call *cutting* and *spacers* parameters, respectively. These parameters define a sequence of *generating words* $\mathcal{W} = (w_n)_{n\geq 0}$ over the alphabet $\{0, 1\}$ inductively by

$$w_0 = 0 \quad \text{and} \quad w_{n+1} = w_n 1^{a_{n,0}} w_n 1^{a_{n,1}} \dots w_n 1^{a_{n,q_n-1}} w_n, \quad n \geq 0. \tag{10}$$

Observe that

$$|w_{n+1}| = (q_n + 1)|w_n| + \sum_{i=0}^{q_n-1} a_{n,i}, \quad n \geq 0. \tag{11}$$

The sequence $\mathcal{W}$ allows the construction of the subspace of $\{0, 1\}^{\mathbb{Z}}$ given by

$$X_{\mathcal{W}} = \{x \in \{0, 1\}^{\mathbb{Z}} : \text{every factor of } x \text{ is a factor of } w_n \text{ for some } n \geq 0\}$$

and a one-sided sequence $x \in \{0, 1\}^{\mathbb{N}}$ by

$$x_{[0,|w_n|)} = w_n, \quad n \geq 0. \tag{12}$$

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We define

$$Q_{m,n} = \prod_{j=m}^{n-1} (q_j + 1), \quad 0 \leq m < n. \tag{13}$$

A *contraction* of $\mathcal{W}$ is a sequence of generating words of the form $\widetilde{\mathcal{W}} = (w_{n_k})_{k\geq 0}$, where the sequence $(n_k)_{k\geq 0}$ is such that $n_0 = 0$ and $n_k < n_{k+1}$ for all $k \geq 0$. Observe that if $\widetilde{\mathcal{W}}$ is a contraction of $\mathcal{W}$, then the generating words of $\widetilde{\mathcal{W}}$ satisfy a relation of type (10) with new parameters $(\widetilde{q}_k : k \geq 0)$ such that

$$\widetilde{q}_k + 1 = Q_{n_k, n_{k+1}}, \quad k \geq 0. \tag{14}$$

Moreover, it is easy to check that $X_{\widetilde{\mathcal{W}}} = X_{\mathcal{W}}$.

The pair $(X_{\mathcal{W}}, S)$ is a subshift, which we call the *Ferenczi subshift* associated to $\mathcal{W}$. It is minimal if the sequence $(a_{n,i} : n \geq 0,\ 0 \leq i < q_n)$ is bounded. If such a sequence is otherwise unbounded, then the two-sided sequence 1^∞ given by $1^\infty_n = 1$ for all $n \in \mathbb{Z}$ belongs to $X_{\mathcal{W}}$ and $X_{\mathcal{W}}$ contains at least two points, in particular the subshift $(X_{\mathcal{W}}, S)$ is not minimal. Moreover, in the minimal case, $X_{\mathcal{W}}$ is finite if and only if the sequence x given by (12) is periodic. See [**GH16a**, §2].

In the next section we prove that minimal Ferenczi subshifts are $\mathcal{S}$-adic subshifts. This is summarized in Proposition 3.3.

3.2. *Minimal Ferenczi subshifts are $\mathcal{S}$-adic.* From now on, we assume that $(X_{\mathcal{W}}, S)$ is a minimal and aperiodic Ferenczi subshift. Let $\{a_1, a_2, \ldots, a_\ell\}$ be the set of values of the sequence $(a_{n,i} : n \geq 0,\ 0 \leq i < q_n)$ with $a_1 < a_2 < \cdots < a_\ell$.

We begin by constructing a sequence of alphabets $(\mathcal{A}_n)_{n\geq 0}$ as follows. Define $\mathcal{A}_0 = \{0, 1\}$ and for $n \geq 1$ we set

$$\mathcal{A}_n = \{a : a = a_{N,i} \text{ for some } N \geq n-1 \text{ and } 0 \leq i < q_N\}.$$

In particular, we have $\mathcal{A}_1 = \{a_1, a_2, \ldots, a_\ell\}$ and $\mathcal{A}_n$ is included in $\mathcal{A}_m$ if $1 \leq m \leq n$. Consequently, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}_n = \mathcal{A}_{n_0}$ for all $n \geq n_0$. We define

$$\mathcal{A}_{\mathcal{W}} = \mathcal{A}_{n_0} \quad \text{and} \quad d_{\mathcal{W}} = |\mathcal{A}_{\mathcal{W}}|. \tag{15}$$

It is easy to see that $\mathcal{A}_{\mathcal{W}}$ is well-defined and that if $\mathcal{W}'$ is a contraction of $\mathcal{W}$, then $\mathcal{A}_{\mathcal{W}} = \mathcal{A}_{\mathcal{W}'}$. Moreover, because $(X_{\mathcal{W}}, S)$ is aperiodic, we have $d_{\mathcal{W}} \geq 2$. Indeed, suppose that $\mathcal{A}_{\mathcal{W}} = \{a\}$ for some a. Then, one has that $w_n = w_{n_0} 1^a w_{n_0} 1^a \ldots w_{n_0} 1^a w_{n_0}$ for $n \geq n_0$, contradicting the aperiodicity.

Define the morphism $\tau_0 : \mathcal{A}_1^* \to \mathcal{A}_0^*$ by $\tau_0(a) = 01^a$ for $a \in \mathcal{A}_1$ and the morphism $\widetilde{\tau}_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ by

$$\widetilde{\tau}_n(a) = a_{n-1,0} a_{n-1,1} \ldots a_{n-1,q_{n-1}-1} a, \quad a \in \mathcal{A}_{n+1}, \quad n \geq 1. \tag{16}$$

Each morphism $\widetilde{\tau}_n$ for $n \geq 1$ is well-defined, of constant length and right permutative. Indeed, the images of letters under $\widetilde{\tau}_n$ differ only at the last letter.

We define the directive sequence $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}} = (\widetilde{\tau}_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$, where $\widetilde{\tau}_0 = \tau_0$.

LEMMA 3.1. *We have $\widetilde{\tau}_{[0,n+1)}(a) = w_n 1^a$ for all $n \geq 0$ and $a \in \mathcal{A}_{n+1}$.*

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Proof. By induction, the property holds if $n = 0$ because $w_0 = 0$. Now if the property holds for $n \geq 0$, then for $a \in \mathcal{A}_{n+2}$ we have

$$\widetilde{\tau}_{[0,n+2)}(a) = \widetilde{\tau}_{[0,n+1)}(a_{n,0}a_{n,1} \ldots a_{n,q_n-1}a) = w_n 1^{a_{n,0}} w_n 1^{a_{n,1}} \ldots w_n 1^{a_{n,q_n-1}} w_n 1^a,$$

which is precisely $w_{n+1}1^a$ by (10), proving the property by induction. □

LEMMA 3.2. *The directive sequence $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ is primitive. Moreover, for all $n \geq 0$ and $a \in \mathcal{A}_{n+1}$, there exists $N > n$ such that $w_n 1^a$ is a factor of the word w_N.*

Proof. For the first assumption let $n \geq 0$. One has to find $N > n$ such that $M_{\widetilde{\tau}_{[n,N)}}$ has positive entries. If $n = 0$ this is given by Lemma 3.1. Suppose $n \geq 1$. If a belongs $\mathcal{A}_n$, then, by definition, there exists $N \geq n - 1$ such that $a_{N,i} = a$ for some $0 \leq i < q_N$. This implies that a has an occurrence in $\widetilde{\tau}_{N+1}(b)$ for all $b \in \mathcal{A}_{N+2}$, and hence, by (16), it also has an occurrence in $\widetilde{\tau}_{[n,N+2)}(b)$. This proves the first claim.

For the second claim, let $n \geq 0$ and $a \in \mathcal{A}_{n+1}$. As $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ is primitive, there exist $N > n$ and $b \in \mathcal{A}_{N+1}$ such that $\widetilde{\tau}_{[n+1,N+1)}(b) = uavb$ for some words u, v. Hence, by Lemma 3.1 we obtain

$$w_N 1^b = \widetilde{\tau}_{[0,N+1)}(b) = \widetilde{\tau}_{[0,n+1)}(uavb) = u' w_n 1^a v' w_n 1^b,$$

for some words u', v', and thus $w_n 1^a$ is a factor of the word w_N. □

Now we prove that the directive sequence $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ generates the subshift $X_{\mathcal{W}}$.

PROPOSITION 3.3. *We have $X_{\mathcal{W}} = X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$.*

Proof. If x belongs to $X_{\mathcal{W}}$, then every factor of x is a factor of some generating word w_n for some $n \geq 0$ and, hence, also a factor of $w_n 1^a = \widetilde{\tau}_{[0,n+1)}(a)$ for some $a \in \mathcal{A}_{n+1}$ by Lemma 3.1. Thus, x belongs to $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$ and $X_{\mathcal{W}}$ is included in $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$.

If now x belongs to $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$, then every factor of x is a factor of $\widetilde{\tau}_{[0,n)}(a) = w_{n-1}1^a$ for some $n \geq 1$ and $a \in \mathcal{A}_n$, thus also a factor of w_N for some $N \geq n$ by Lemma 3.2. We conclude that x belongs to $X_{\mathcal{W}}$ and $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$ is included in $X_{\mathcal{W}}$. □

3.3. *Recognizable directive sequences for minimal Ferenczi subshifts.* In this section, by a slight modification of the directive sequence $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$, we describe a primitive, proper and recognizable directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ generating the minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$. This is summarized in Theorem 3.7.

We say that the sequence of generating words $\mathcal{W}$ is *standard* if the sequence $(q_n)_{n\geq 0}$ given by (10) satisfies $q_n \geq 2$ for each $n \geq 0$. Observe that we can assume without loss of generality that each sequence $\mathcal{W}$ is standard. Indeed, this follows directly from Equation (14). From now on assume that $\mathcal{W}$ is standard.

In order to apply Proposition 2.6 and obtain sequences of CKR partitions for the subshift $(X_{\mathcal{W}}, S)$, we need each morphism $\widetilde{\tau}_n$ for $n \geq 1$ to be proper, which is not the case. We define a morphism $\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ which is proper and rotationally conjugate (as defined in §2.2.2) to $\widetilde{\tau}_n$ by

$$\tau_n(a) = a_{n-1,1}a_{n-1,2} \ldots a_{n-1,q_{n-1}-1}\, a\, a_{n-1,0}, \quad a \in \mathcal{A}_{n+1}, \quad n \geq 1. \tag{17}$$

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As $\mathcal{W}$ is standard, this is a well-defined proper morphism of constant length which is rotationally conjugate to $\widetilde{\tau}_n$:

$$a_{n-1,0}\tau_n(a) = \widetilde{\tau}_n(a)a_{n-1,0}, \quad a \in \mathcal{A}_{n+1}, \quad n \geq 1. \tag{18}$$

We define the directive sequence $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$.

LEMMA 3.4. *For $a \in \mathcal{A}_{m+1}$ and $m \geq 1$ we have*

$$a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,m)}(a_{m-1,0})\tau_{[1,m+1)}(a)$$
$$= \widetilde{\tau}_{[1,m+1)}(a)\widetilde{\tau}_{[1,m)}(a_{m-1,0}) \ldots \widetilde{\tau}_{[1,2)}(a_{1,0})a_{0,0}.$$

Proof. We begin by proving the following.

Claim 3.4.1. For $1 \leq n \leq N$ and $a \in \mathcal{A}_{N+1}$, we have

$$\tau_{[n,N)}(a_{N-1,0})\tau_{[n,N+1)}(a) = \tau_{[n,N)}(\widetilde{\tau}_N(a))\tau_{[n,N)}(a_{N-1,0}).$$

Indeed, by means of a simple computation

$$\begin{aligned}
&\tau_{[n,N)}(a_{N-1,0})\tau_{[n,N+1)}(a)\\
&\quad = \tau_{[n,N)}(a_{N-1,0})\tau_{[n,N)}(\tau_N(a))\\
&\quad = \tau_{[n,N)}(a_{N-1,0})\tau_{[n,N)}(a_{N-1,1} \ldots a_{N-1,q_{N-1}-1}aa_{N-1,0})\\
&\quad = \tau_{[n,N)}(a_{N-1,0} \ldots a_{N-1,q_{N-1}-1}a)\tau_{[n,N)}(a_{N-1,0})\\
&\quad = \tau_{[n,N)}(\widetilde{\tau}_N(a))\tau_{[n,N)}(a_{N-1,0}),
\end{aligned}$$

proving the claim. This implies that for $n \in [1, N]$ and $w \in \mathcal{A}_{N+1}^*$, then

$$\tau_{[n,N)}(a_{N-1,0})\tau_{[n,N+1)}(w) = \tau_{[n,N)}(\widetilde{\tau}_N(w))\tau_{[n,N)}(a_{N-1,0}). \tag{19}$$

By induction, the statement in the lemma is true if $m = 1$ (see (18)). Assume that the statement holds for $m \geq 1$. By the claim, for $a \in \mathcal{A}_{m+2}$ we obtain

$$a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,m)}(a_{m-1,0})\tau_{[1,m+1)}(a_{m,0})\tau_{[1,m+2)}(a)$$
$$= a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,m)}(a_{m-1,0})\tau_{[1,m+1)}(\widetilde{\tau}_{m+1}(a))\tau_{[1,m+1)}(a_{m,0}).$$

By using (19) with $w = \widetilde{\tau}_{[k,m+2)}(a)$ for $k = m+1, m, \ldots, 2$ and the induction hypothesis, the last term is equal to

$$\widetilde{\tau}_{[1,m+2)}(a)a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,m+1)}(a_{m,0})$$
$$= \widetilde{\tau}_{[1,m+2)}(a)\widetilde{\tau}_{[1,m+1)}(a_{m,0}) \ldots \widetilde{\tau}_{[1,2)}(a_{1,0})a_{0,0},$$

finishing the proof by induction. □

We now prove that the sequences $\boldsymbol{\tau}_{\mathcal{W}}$ and $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ generate the same subshift.

PROPOSITION 3.5. *We have $X_{\mathcal{W}} = X_{\boldsymbol{\tau}_{\mathcal{W}}}$.*

Proof. By Proposition 3.3, it is enough to show that $X_{\boldsymbol{\tau}_{\mathcal{W}}} = X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$.

Claim 3.5.1. The word $a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,n)}(a_{n-1,0})$ is a suffix of $\tau_{[1,n+1)}(a)$ for all $a \in \mathcal{A}_{n+1}$ and $n \geq 1$.

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Indeed, this is true for $n = 1$. Assume that the claim holds for $n \geq 1$. If a belongs to $\mathcal{A}_{n+2}$, then the word $a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,n)}(a_{n-1,0})\tau_{[1,n+1)}(a_{n,0})$ is a suffix of the word

$$\begin{aligned}\tau_{[1,n+1)}(a_{n,1}) \ldots \tau_{[1,n+1)}(a_{n,q_n-1})\tau_{[1,n+1)}(a)\tau_{[1,n+1)}(a_{n,0})\\ = \tau_{[1,n+1)}(a_{n,1} \ldots a_{n,q_n-1}aa_{n,0}) = \tau_{[1,n+2)}(a),\end{aligned}$$

proving the claim by induction.

Let $x \in X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$ and w be a factor of x. Then w is a factor of $\tau_0 \circ \widetilde{\tau}_{[1,n+1)}(a)$ for some $n \geq 1$ and $a \in \mathcal{A}_{n+1}$. By Lemma 3.4, we deduce that w is a factor of the word

$$\tau_0(a_{0,0}\tau_{[1,2)}(a_{1,0}) \ldots \tau_{[1,n)}(a_{n-1,0}))\tau_0(\tau_{[1,n+1)}(a)).$$

By using the previous claim with $a = a_{n,q_n-1}$, the word w is a factor of

$$\tau_0(\tau_{[1,n+1)}(a_{n,q_n-1}))\tau_0(\tau_{[1,n+1)}(a)),$$

and by (17) also a factor of $\tau_0 \circ \tau_{[1,n+2)}(a)$. Thus, x belongs to $X_{\boldsymbol{\tau}_{\mathcal{W}}}$ and $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$ is included in $X_{\boldsymbol{\tau}_{\mathcal{W}}}$. Proving a similar claim reversing the roles of $X_{\boldsymbol{\tau}_{\mathcal{W}}}$ and $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$, we obtain that $X_{\boldsymbol{\tau}_{\mathcal{W}}}$ is included in $X_{\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}}$. □

We observe that the directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ is primitive. Indeed, this follows directly from Lemma 3.2 because τ_n is rotationally conjugate to $\widetilde{\tau}_n$ for each $n \geq 1$.

LEMMA 3.6. *The directive sequences $\boldsymbol{\tau}_{\mathcal{W}}$ and $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ are recognizable.*

Proof. Let $y \in X_{\mathcal{W}}$ be any aperiodic point. We prove the uniqueness of a couple (k, x) with $x \in \mathcal{A}_1^{\mathbb{Z}}$, $0 \leq k < |\tau_0(x_0)|$ such that $y = S^k\tau_0(x)$. Indeed, y can be decomposed uniquely into words from the set $\{01^a : a \in \mathcal{A}_1\}$, and so there exists a unique such couple (k, x) (the zero coordinate of x corresponds to the symbol $a \in \mathcal{A}_1$ such that the word 01^a covers the coordinate y_0). Hence, $\boldsymbol{\tau}_{\mathcal{W}}$ and $\widetilde{\boldsymbol{\tau}}_{\mathcal{W}}$ are recognizable at level zero.

For $n \geq 1$ the morphism τ_n is rotationally conjugate to the right permutative morphism $\widetilde{\tau}_n$ (see §3.2). Hence, the morphisms τ_n and $\widetilde{\tau}_n$ are fully recognizable for aperiodic points by Proposition 2.2. We conclude the proof using Proposition 2.3. □

By combining Proposition 3.5, Lemma 3.6 and the previous discussion, we deduce the following.

THEOREM 3.7. *A subshift (X, S) is a minimal Ferenczi subshift if and only if it is an $\mathcal{S}$-adic subshift generated by a directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ as in (17) where the sequence $(a_{n,i} : n \geq 0,\ 0 \leq i < q_n)$ is bounded.*

3.4. *Some useful computations for Ferenczi subshifts.*

In this section we show some useful relations between the parameters defining a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ defined by a sequence of generating words $\mathcal{W}$ given by (10).

Define

$$f_n(a) = \#\{0 \leq i < q_{n-1} : a_{n-1,i} = a\}, \quad a \in \mathcal{A}_n, \quad n \geq 1 \tag{20}$$

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and let f_n be the column vector $f_n = (f_n(a))_{a\in\mathcal{A}_n}$. For a vector f in $\mathbb{R}^{\mathcal{A}}$ we use the notation $|f| = \sum_{a\in\mathcal{A}} f(a)$. Observe that for $n \geq 1$,

$$|f_n| = q_{n-1} \tag{21}$$

$$\sum_{b\in\mathcal{A}_n} f_n(b)\cdot b = \sum_{i=0}^{q_{n-1}-1} a_{n-1,i}. \tag{22}$$

We now compute the height vectors associated with $\boldsymbol{\tau}_{\mathcal{W}}$ and give some estimates. We recall Equation (9):

$$h_n(a) = |\tau_{[0,n)}(a)|, \quad a \in \mathcal{A}_n, \quad n \geq 0.$$

LEMMA 3.8. *Let $\mathcal{W} = (w_n)_{n\geq 0}$ be a sequence of generating words and $\boldsymbol{\tau}_{\mathcal{W}}$ be the associated directive sequence given by (17). Then, the height vectors $(h_n)_{n\geq 0}$ associated with $\boldsymbol{\tau}_{\mathcal{W}}$ satisfy*

$$h_n(a) = a + |w_{n-1}|, \quad a \in \mathcal{A}_n, \quad n \geq 1. \tag{23}$$

In particular, there exists $K \geq 1$ such that

$$K^{-1}h_n(b) \leq h_n(a) \leq K h_n(b), \quad a, b \in \mathcal{A}_n, \quad n \geq 0. \tag{24}$$

Moreover, there exists a constant $L \geq 1$ such that

$$L^{-1}Q_{0,n-1} \leq h_n(a) \leq L Q_{0,n-1}, \quad a \in \mathcal{A}_n, \quad n \geq 1 \tag{25}$$

Proof. The computation of h_1 is clear from the definition. Assume that (23) holds for $n \geq 1$. For $a \in \mathcal{A}_{n+1}$, by using (21), (22) and (11), we obtain

$$\begin{aligned}
h_{n+1}(a) &= \sum_{b\in\mathcal{A}_n} h_n(b)M_{\tau_n}(b,a) = h_n(a)(1 + f_n(a)) + \sum_{b\in\mathcal{A}_n,\, b\neq a} h_n(b)f_n(b)\\
&= (a + |w_{n-1}|)(1 + f_n(a)) + \sum_{b\in\mathcal{A}_n,\, b\neq a} (b + |w_{n-1}|)f_n(b)\\
&= a + |w_{n-1}| + \sum_{b\in\mathcal{A}_n} f_n(b)\cdot|w_{n-1}| + \sum_{b\in\mathcal{A}_n} f_n(b)\cdot b\\
&= a + (q_{n-1}+1)|w_{n-1}| + \sum_{i=0}^{q_{n-1}-1} a_{n-1,i}\\
&= a + |w_n|,
\end{aligned}$$

proving (23) by induction. The estimate (24) follows directly from (23).

By the definition of the morphism τ_0, there exists a constant $L \geq 1$ such that

$$L^{-1}|w| \leq |\tau_0(w)| \leq L|w|, \quad w \in \mathcal{A}_1^*.$$

Let $a \in \mathcal{A}_n$, $n \geq 1$. By (2), we have

$$h_n(a) = |\tau_0 \circ \tau_{[1,n)}(a)| \leq L|\tau_{[1,n)}(a)| = L \prod_{i=1}^{n-1} |\tau_i| = L Q_{0,n-1}.$$

Analogously, we obtain $L^{-1} Q_{0,n-1} \leq h_n(a)$, thus obtaining (25). □

The composition matrices of the directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ can be computed as

$$M_{\tau_0} = \begin{pmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_\ell \end{pmatrix}, \quad M_{\tau_n} = I_{n,n+1} + f_n \cdot \boldsymbol{u}_n, \quad n \geq 1, \tag{26}$$

where the matrix $I_{n,n+1}$ is given for each $a \in \mathcal{A}_n$ and $b \in \mathcal{A}_{n+1}$ by $I_{n,n+1}(a, b) = 1$ if $a = b$ and 0 otherwise and $\boldsymbol{u}_n$ is the row vector of ones in $\mathbb{R}^{\mathcal{A}_{n+1}}$.

Let $n_0 \in \mathbb{N}$ be such that $\mathcal{A}_n = \mathcal{A}_{\mathcal{W}}$ for all $n \geq n_0$, let I be the identity matrix in $\mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$ and let $\boldsymbol{u}$ be the row vector of ones in $\mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$.

LEMMA 3.9. *Let $g_1, g_2, \dots, g_n$ be column vectors indexed by a finite alphabet $\mathcal{A}$. Let*

$$A_i = I + g_i \cdot \boldsymbol{u}, \quad 1 \leq i \leq n,$$

where I is the identity in $\mathbb{R}^{\mathcal{A}}$ and $\boldsymbol{u}$ is the row vector of ones in $\mathbb{R}^{\mathcal{A}}$. Then

$$A_1 A_2 \dots A_n = I + \left(\sum_{k=1}^{n} \prod_{j=k+1}^{n} (1 + |g_j|) g_k \right) \cdot \boldsymbol{u}, \quad n \geq 1,$$

and

$$A_i^{-1} = I - \frac{g_i}{|g_i| + 1} \cdot \boldsymbol{u}, \quad 1 \leq i \leq n.$$

Proof. It is easy to check that the inverse of A_i is as given. The formula for the product $A_1 A_2 \dots A_n$ is clearly true for $n = 1$. Suppose that it is true for n and let us show that it is true for $n + 1$. In fact,

$$\begin{aligned} A_1 \dots A_n A_{n+1} &= \left(I + \left(\sum_{k=1}^{n} \prod_{j=k+1}^{n} (1 + |g_j|) g_k \right) \cdot \boldsymbol{u} \right) (I + g_{n+1} \cdot \boldsymbol{u}) \\ &= I + (1 + |g_{n+1}|) \left(\sum_{k=1}^{n} \prod_{j=k+1}^{n} (1 + |g_j|) g_k \right) \cdot \boldsymbol{u} + g_{n+1} \cdot \boldsymbol{u} \\ &= I + \left(\sum_{k=1}^{n} \prod_{j=k+1}^{n+1} (1 + |g_j|) g_k \right) \cdot \boldsymbol{u} + g_{n+1} \cdot \boldsymbol{u} \\ &= I + \left(\sum_{k=1}^{n+1} \prod_{j=k+1}^{n+1} (1 + |g_j|) g_k \right) \cdot \boldsymbol{u}. \end{aligned}$$

□

By Lemma 3.9 and (13), we have

$$M_{\tau_m} M_{\tau_{m+1}} \dots M_{\tau_{n-1}} = I + f_{m,n} \cdot \boldsymbol{u}, \quad n_0 \leq m < n, \tag{27}$$

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where

$$f_{m,n} = \sum_{k=m}^{n-1} Q_{k,n-1} f_k. \tag{28}$$

Observe that

$$|f_{m,n}| + 1 = Q_{m-1,n-1}. \tag{29}$$

Thus, Lemma 3.9 implies

$$(M_{\tau_m} M_{\tau_{m+1}} \ldots M_{\tau_{n-1}})^{-1} = I - \frac{f_{m,n}}{Q_{m-1,n-1}} \cdot \boldsymbol{u}, \quad n_0 \le m < n. \tag{30}$$

Example 3.10. Let $a < b < c < d$ be positive integers. Define a sequence $\mathcal{W} = (w_n)_{n\ge 0}$ of generating words such that for infinitely many values of n

$$w_{n+1} = w_n 1^a w_n 1^b w_n \quad \text{and} \quad w_{n+1} = w_n 1^c w_n 1^d w_n.$$

Hence, $\mathcal{A}_{\mathcal{W}} = \{a, b, c, d\}$. The directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ consists of two morphisms $\tau_{a,b}$ and $\tau_{c,d}$, each one occurring infinitely many times in $\boldsymbol{\tau}_{\mathcal{W}}$, defined, for $u \in \mathcal{A}_{\mathcal{W}}$, by

$$\tau_{a,b}(u) = bua,$$
$$\tau_{c,d}(u) = duc.$$

The composition matrices indexed by $\mathcal{A}_{\mathcal{W}}$ are

$$M_{\tau_{a,b}} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\tau_{c,d}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

4. *Topological dynamical properties of minimal Ferenczi subshifts*

In what follows, $\mathcal{W}$ is a standard sequence as given by (10) generating a minimal Ferenczi subshift. Let $(X_{\mathcal{W}}, S)$ be the subshift it generates and $\boldsymbol{\tau}_{\mathcal{W}}$ the associated directive sequence given by (17).

4.1. *Unique ergodicity.*

The unique ergodicity of Ferenczi subshifts is a folklore result [**Fer97**, §1.1.4]. We provide a short proof.

PROPOSITION 4.1. *The system $(X_{\mathcal{W}}, S)$ is uniquely ergodic.*

Proof. The directive sequence $\boldsymbol{\tau}_{\mathcal{W}}$ defines a sequence of measure vectors $(\mu_n)_{n\ge 0}$ given in §2.3.2. By (7), to prove unique ergodicity of $(X_{\mathcal{W}}, S)$ it is sufficient to prove that the vector μ_n is uniquely determined for infinitely many values of n.

Recall the definition of $Q_{0,m}$ for $m \ge 1$ in (13). Let us consider the vectors $(t_m)_{m\ge 1}$ defined by $t_m = Q_{0,m-1}\mu_m$, $m \ge 1$. By (7) we have

$$t_m = Q_{0,m-1}\mu_m = \frac{Q_{0,m}}{q_{m-1}+1}(M_m \mu_{m+1}) = \frac{1}{q_{m-1}+1} M_m t_{m+1}, \quad m \ge 1.$$

Consequently, from equations (27) and (28),

$$t_m = \frac{1}{Q_{m-1,n-1}} M_m M_{m+1} \cdots M_{n-1} t_n = \frac{1}{Q_{m-1,n-1}} \bigg(I + \bigg(\sum_{k=m}^{n-1} Q_{k,n-1} f_k \bigg) \cdot \boldsymbol{u} \bigg) t_n$$

$$= Q_{0,m-1}\mu_n + \bigg(\sum_{k=m}^{n-1} \frac{f_k}{Q_{m-1,k}} \bigg) \cdot |t_n|, \quad n_0 \le m < n.$$

It can be checked that $\sum_k (f_k/Q_{m-1,k})$ converges, we define $v_m = \sum_{k=m}^{\infty} (f_k/Q_{m-1,k})$.

We deduce $L^{-1} \le |t_n| \le L$ from (25) and, because $\mu_n \to 0$ as $n \to +\infty$, there exists a sequence of non-negative numbers $(\alpha_m)_{m \ge n_0}$ such that

$$\mu_m = \alpha_m v_m, \quad m \ge n_0.$$

We deduce $\alpha_{n_0} = 1/|P_{0,n_0} v_{n_0}|$ from (7). Again, from (7) we obtain

$$\alpha_m = \frac{|v_{n_0}|}{|P_{0,n_0} v_{n_0}||P_{n_0,m} v_m|}, \quad m \ge n_0$$

and, finally,

$$\mu_m = \bigg(\frac{|v_{n_0}|}{|P_{0,n_0} v_{n_0}||P_{n_0,m} v_m|} \bigg) v_m, \quad m \ge n_0.$$

This completes the proof. □

4.2. *Clean directive sequences.* To go further in the study of Ferenczi subshifts we need the following notion inspired by the definition of *clean Bratteli diagram* given in [**BDM10**, §5], see also [**BKMS13**, Theorem 3.3].

Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \ge 0}$ be a recognizable directive sequence and let μ be an ergodic invariant probability measure of $(X_{\boldsymbol{\tau}}, S)$.

We say that $\boldsymbol{\tau}$ is *clean* with respect to μ if:

(1) there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}_n = \mathcal{A}_{n_0}$ for all $n \ge n_0$; put $\mathcal{A} = \mathcal{A}_{n_0}$;

(2) there exist a constant $c > 0$ and $\mathcal{A}_\mu \subseteq \mathcal{A}$ such that

$$\mu(\mathcal{T}_n(a)) \ge c, \quad n \ge n_0, \quad a \in \mathcal{A}_\mu, \quad \text{and} \tag{31}$$

$$\lim_{n \to +\infty} \mu(\mathcal{T}_n(a)) = 0, \quad a \in \mathcal{A} \setminus \mathcal{A}_\mu.$$

We remark that we can always contract the directive sequence $\boldsymbol{\tau}$ so that it becomes clean with respect to μ. If $\mathcal{A}_\mu = \mathcal{A}$, we say that $\boldsymbol{\tau}$ is of *exact finite rank*.

It is proven in [**BKMS13**] that exact finite rank of $\boldsymbol{\tau}$ implies that $(X_{\boldsymbol{\tau}}, S)$ is uniquely ergodic. However, the converse is not true, even for Ferenczi subshifts.

Example 4.2. (Ferenczi subshift with non-exact rank) Consider a sequence of generating words $\mathcal{W}$ with associated cutting parameters $(q_n)_{n \ge 0}$, as defined in (10). Suppose that $q_n \to +\infty$ as $n \to +\infty$ and that there exists a letter a^* in $\mathcal{A}_{\mathcal{W}}$ such that $f_n(a^*) + 1 \le C$ for all large enough values of n and some value $C > 0$.

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If μ is the unique invariant probability measure of $(X_{\mathcal{W}}, S)$, from (6) we obtain

$$\mu(\mathcal{T}_n(a^*)) = h_n(a^*)\mu_n(a^*) = h_n(a^*) \sum_{b\in\mathcal{A}_{\mathcal{W}}} M_n(a^*, b)\frac{\mu(\mathcal{T}_{n+1}(b))}{h_{n+1}(b)}$$
$$\leq \frac{Ch_n(a^*)}{\min_{b\in\mathcal{A}_{\mathcal{W}}} h_{n+1}(b)} \leq \frac{CK}{q_{n-1}+1},$$

where we used

$$h_{n+1}(b) = \sum_{c\in\mathcal{A}_{\mathcal{W}}} h_n(c)M_n(c, b) \geq K^{-1}h_n(a^*) \sum_{c\in\mathcal{A}_{\mathcal{W}}} M_n(c, b)$$
$$= K^{-1}h_n(a^*)(q_{n-1}+1), \quad b \in \mathcal{A}_{\mathcal{W}}.$$

Therefore, $\mu(\mathcal{T}_n(a^*)) \to 0$ as $n \to +\infty$ and $\boldsymbol{\tau}_{\mathcal{W}}$ is not of exact finite rank.

A subshift (X, S) is *linearly recurrent* if it is minimal and there exists a constant $K > 0$ such that if $u \in \mathcal{L}(X)$ and w is a right return word to u in X, then

$$|w| \leq K|u|.$$

We refer to [**DP22**, **Dur00**] for more details on linearly recurrent shifts. In [**BKMS13**] it is shown that linearly recurrent subshifts have exact finite rank and that the converse is not true. The following example shows that the converse is not true, even in the family of Ferenczi subshifts.

Example 4.3. (Ferenczi subshift with exact finite rank that is not linearly recurrent) Consider a sequence of generating words $\mathcal{W}$ such that $d_{\mathcal{W}} = 2$. Let $\mathcal{A}_{\mathcal{W}} = \{a, b\}$ and define the morphism τ_n by

$$\tau_n(a) = a^n b^{2n-1}ab \quad \text{and} \quad \tau_n(b) = a^n b^{2n-1}bb, \quad n \geq 1.$$

The composition matrix of τ_n indexed by $\mathcal{A}_{\mathcal{W}}$ is $M_{\tau_n} = \left(\begin{smallmatrix} n+1 & n \\ 2n & 2n+1 \end{smallmatrix}\right)$ and, hence, $\boldsymbol{\tau}_{\mathcal{W}}$ is of exact finite rank [**BKMS13**, Proposition 5.7]. Observe that for all n the word $\tau_{[0,n)}(a)^{n+1}$ belongs to the language of $X_{\mathcal{W}}$. Hence, the subshift $(X_{\mathcal{W}}, S)$ is not linearly recurrent, see [**DHS99**, Theorem 24].

In the following, we characterize exact finite rank of $\boldsymbol{\tau}_{\mathcal{W}}$.

PROPOSITION 4.4. *For $a \in \mathcal{A}_{\mathcal{W}}$, we have* $\liminf_{m\to+\infty} \mu(\mathcal{T}_m(a)) > 0$ *if and only if*

$$\liminf_{m\to+\infty} \sum_{k=m}^{\infty} \frac{f_k(a)}{Q_{m-1,k}} > 0. \tag{32}$$

In particular, $\boldsymbol{\tau}_{\mathcal{W}}$ is of exact finite rank if and only if (32) holds for all $a \in \mathcal{A}_{\mathcal{W}}$.

Proof. Let $n_0 \in \mathbb{N}$ be such that $\mathcal{A}_{n_0} = \mathcal{A}_{\mathcal{W}}$. Consider $m \geq n_0$ and $a \in \mathcal{A}_{\mathcal{W}}$. As $d_{\mathcal{W}} \geq 2$, there exists $b \in \mathcal{A}_{\mathcal{W}}$ with $b \neq a$. By [**BKMS13**, Proposition 5.1] one has

$$\mu_m(a) = \lim_{n\to+\infty} \frac{|\tau_{[m,n)}(b)|_a}{h_n(b)}.$$

By (27) and (28), because $b \neq a$ one obtains

$$\frac{|\tau_{[m,n)}(b)|_a}{h_n(b)} = \frac{\sum_{k=m}^{n-1} Q_{k,n-1} f_k(a)}{h_n(b)}.$$

Hence, by (6)

$$\mu(\mathcal{T}_m(a)) = h_m(a)\mu_m(a) = \lim_{n\to+\infty} \frac{h_m(a)}{h_n(b)} \sum_{k=m}^{n-1} Q_{k,n-1} f_k(a)$$

$$= \lim_{n\to+\infty} \frac{h_m(a)}{h_n(b)} Q_{m-1,n-1} \sum_{k=m}^{n-1} \frac{f_k(a)}{Q_{m-1,k}}.$$

Using (25), there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{h_m(a)}{h_n(b)} Q_{m-1,n-1} \leq C, \quad a, b \in \mathcal{A}_{\mathcal{W}}, \quad n_0 \leq m < n.$$

Therefore, $\liminf_{m\to+\infty} \mu(\mathcal{T}_m(a)) > 0$ if and only if

$$\liminf_{m\to+\infty} \sum_{k=m}^{\infty} \frac{f_k(a)}{Q_{m-1,k}} > 0,$$

where it can be checked that $\sum_k f_k(a)/Q_{m-1,k}$ converges. □

4.3. *Toeplitz induced systems.* Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift and $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be the directive sequence given by (17). Denote by $\boldsymbol{\tau}'_{\mathcal{W}} = (\tau_{n+1} : \mathcal{A}_{n+2}^* \to \mathcal{A}_{n+1}^*)_{n\geq 0}$ the shifted directive sequence of $\boldsymbol{\tau}_{\mathcal{W}}$ and let $U_{\mathcal{W}} = \tau_0(X_{\boldsymbol{\tau}'_{\mathcal{W}}})$. From Corollary 2.5 the induced system $(U_{\mathcal{W}}, S_{U_{\mathcal{W}}})$ of $(X_{\mathcal{W}}, S)$ on $U_{\mathcal{W}}$ is topologically conjugate to the $\mathcal{S}$-adic subshift $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$.

Recall from §§3.2 and 3.3 that each morphism τ_n has constant length and is rotationally conjugate to the right permutative morphism $\widetilde{\tau}_n$ for $n \geq 1$. Therefore, $\boldsymbol{\tau}'_{\mathcal{W}}$ is recognizable and the subshift $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$ is minimal and aperiodic. Moreover, the associated sequence of incidence matrices $(M_n)_{n\geq 0}$ coincides with the sequence of composition matrices $(M_{\tau_{n+1}})_{n\geq 0}$ by Proposition 2.6.

We deduce that the latter has the *equal path number property*, that is, for each $n \geq 0$ the sum of each column of M_n is constant. This implies that $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$ is topologically conjugate to a minimal Toeplitz subshift [**GJ00**, Theorem 8]. We recall that a subshift (X, S) with $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is Toeplitz if X is the closure of the orbit $\{S^n x : n \in \mathbb{Z}\}$ for some sequence $x = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$ there exists $p \in \mathbb{N}$ with $x_n = x_{n+kp}$ for all $k \in \mathbb{Z}$.

We prove that this Toeplitz subshift is *mean equicontinuous*. We recall that a topological dynamical system (X, T) with a metric d on X is mean equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$, then $\rho_b(x, y) \leq \varepsilon$. Here, ρ_b denotes the Besicovitch pseudo-metric given by

$$\rho_b(x, y) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^k y), \quad x, y \in X.$$

Let (X, T) be a minimal system. Denote by $(X_{\mathrm{eq}}, T_{\mathrm{eq}})$ to the maximal equicontinuous factor of (X, T), by ν to its unique invariant probability measure and let $\pi_{\mathrm{eq}} : X \to X_{\mathrm{eq}}$ be the corresponding factor map.

The system (X, T) is mean equicontinuous if and only if it is uniquely ergodic (with unique invariant probability measure μ) and π_{eq} is a measurable isomorphism between the systems (X, T, μ) and $(X_{\mathrm{eq}}, T_{\mathrm{eq}}, \nu)$ [**DG16**, **LTY15**]. In particular, this implies that the system (X, T, μ) has discrete spectrum, that is, there exists an orthonormal basis of $L^2(X, \mu)$ consisting of measurable eigenfunctions of (X, T). We refer to [**GRJY21**] for more details about mean equicontinuity.

In what follows, we need the following definitions. For a sequence of positive integers $(p_n)_{n\geq 0}$ such that p_n divides p_{n+1} for $n \geq 0$, the *odometer* given by this sequence is the system $(\mathbb{Z}_{(p_n)_{n\geq 0}}, T)$, where

$$\mathbb{Z}_{(p_n)_{n\geq 0}} = \varprojlim \mathbb{Z}/p_n\mathbb{Z} = \Big\{(x_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}/p_n\mathbb{Z} : x_{n+1} \equiv x_n (\text{mod } p_n), n \geq 0\Big\}$$

and the map $T : \mathbb{Z}_{(p_n)_{n\geq 0}} \to \mathbb{Z}_{(p_n)_{n\geq 0}}$ is given by

$$T((x_n)_{n\geq 0}) = (x_n + 1 (\text{mod } p_n))_{n\geq 0}.$$

Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be a primitive, proper and recognizable directive sequence such that the morphism τ_n has constant length for each $n \geq 0$. It is classical to show that the maximal equicontinuous factor of $(X_{\boldsymbol{\tau}}, S)$ corresponds to the odometer $(\mathbb{Z}_{(|\tau_{[0,n)}|)_{n\geq 0}}, T)$ [**GJ00**]. The factor map $\pi_{\mathrm{eq}} : X_{\boldsymbol{\tau}} \to \mathbb{Z}_{(|\tau_{[0,n)}|)_{n\geq 0}}$ can be described as follows. Let $x \in X_{\boldsymbol{\tau}}$. By recognizability of $\boldsymbol{\tau}$, for every $n \geq 0$ there exists a letter $a_n(x)$ in $\mathcal{A}_n$ and $k_n(x)$ with $0 \leq k_n(x) < |\tau_{[0,n)}|$, uniquely determined, such that

$$x \in S^{k_n(x)} \tau_{[0,n)}([a_n(x)]).$$

Then we define

$$\pi_{\mathrm{eq}}(x) = (k_n(x))_{n\geq 0}. \tag{33}$$

It can be observed that $k_{n+1}(x) \equiv k_n(x) (\text{mod } |\tau_{[0,n)}|)$ for $x \in X_{\boldsymbol{\tau}}$.

For a morphism $\tau : \mathcal{A}^* \to \mathcal{B}^*$ of constant length $|\tau|$, we say that it has a *coincidence* at index $0 \leq i < |\tau|$ if $\tau(a)_i = \tau(a')_i$ for every $a, a' \in \mathcal{A}$. The notion of coincidence has been used in [**Dek78**] to characterize the discrete spectrum of constant length substitution systems. See also [**Que87**].

PROPOSITION 4.5. *The system $(U_{\mathcal{W}}, S_{U_{\mathcal{W}}})$ is mean equicontinuous.*

Proof. As was observed previously, the system $(U_{\mathcal{W}}, S_{U_{\mathcal{W}}})$ is topologically conjugate to the $\mathcal{S}$-adic subshift $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$. Moreover, as in the proof of Proposition 4.1, this subshift is uniquely ergodic. Denote by μ its unique invariant probability measure.

The directive sequence $\boldsymbol{\tau}'_{\mathcal{W}}$ is primitive, proper, recognizable and consists of morphisms of constant length. Indeed, from (17) we have $|\tau_{n+1}| = q_n + 1, n \geq 0$. Hence, the maximal equicontinuous factor of $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$ is the odometer $(\mathbb{Z}_{(Q_{0,n})_{n\geq 0}}, T)$. Denote by ν the unique invariant probability measure of this odometer and let $\pi_{\mathrm{eq}} : X_{\boldsymbol{\tau}'_{\mathcal{W}}} \to \mathbb{Z}_{(Q_{0,n})_{n\geq 0}}$ be the factor map given by (33). Denote by $(\mathcal{T}'_n)_{n\geq 0}$ the nested sequence of CKR partitions of $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S)$ given by (8).

For each $z = (z_n)_{n\geq 0}$ in $\mathbb{Z}_{(Q_{0,n})_{n\geq 0}}$, we write

$$z_n = Q_{0,n-1}t_n(z) + r_n(z), \quad 0 \leq r_n(z) < Q_{0,n-1}, \quad 0 \leq t_n(z) < q_{n-1} + 1, \quad n \geq 1.$$

We define

$$C_n = \{0 \leq i < (q_{n-1} + 1) : \tau_n \text{ has a coincidence at index } i\}$$
$$D_n = \{z \in \mathbb{Z}_{(Q_{0,n})_{n\geq 0}} : t_n(z) \notin C_n\}, \quad n \geq 1.$$

Claim 4.5.1. If a point $z = (z_n)_{n\geq 0}$ in $\mathbb{Z}_{(Q_{0,n})_{n\geq 0}}$ is such that $t_n(z)$ belongs to C_n for infinitely many values of n, then $|\pi_{\mathrm{eq}}^{-1}(\{z\})| = 1$.

Indeed, let $x, y \in X_{\boldsymbol{\tau}'_{\mathcal{W}}}$ be such that $k_n(x) = k_n(y) = z_n$ for $n \geq 0$. If $t_n(z)$ belongs to C_n, then there exists a letter $\ell(z)$ in $\mathcal{A}_n$ such that $\tau_n(a_{n+1})_{t_n(z)} = \ell(z)$ for every a_{n+1} in $\mathcal{A}_{n+1}$. Consequently, we have

$$S^{z_n}\tau_{[1,n+1)}([a_{n+1}]) \subseteq S^{r_n(z)}\tau_{[1,n)}([\ell(z)]), \quad a_{n+1} \in \mathcal{A}_{n+1}.$$

This implies that there exists infinitely many values of n for which x and y belong to $S^{r_n(z)}\tau_{[1,n)}([\ell(z)])$. As $(\mathcal{T}'_n)_{n\geq 0}$ is a nested sequence, we deduce that $\mathrm{diam}(S^{r_n(z)}\tau_{[1,n)}([\ell(z)])) \to 0$ as $n \to +\infty$ and, therefore, $x = y$, proving the claim.

From the claim, it follows that if we denote by $\mathcal{Z}$ the set of points in $\mathbb{Z}_{(Q_{0,n})_{n\geq 0}}$ that are not invertible under π_{eq}, then

$$\mathcal{Z} \subseteq \bigcup_{n\geq 0} \bigcap_{m\geq n} D_m.$$

Observe that, from (17), we obtain $\nu(D_m) = 1 - |C_m|/(q_m + 1) = 1/(q_m + 1)$ for $m \geq 0$. Thus,

$$\nu\bigg(\bigcap_{m\geq n} D_m\bigg) = \prod_{m\geq n} \frac{1}{q_m + 1} \leq \prod_{m\geq n} \frac{1}{2} = 0,$$

and, hence, $\nu(\mathcal{Z}) = 0$. This proves that π_{eq} is a measurable isomorphism between $(X_{\boldsymbol{\tau}'_{\mathcal{W}}}, S, \mu)$ and $(\mathbb{Z}_{(Q_{0,n})_{n\geq 0}}, T, \nu)$, and concludes the proof. □

4.4. *Computation of the topological rank.* In this section we compute explicitly the topological rank of a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$. We refer to §§2.3 and 2.4 for the definitions.

For an abelian group G we denote by rank G the *rational rank* of G, that is,

$$\mathrm{rank}_{\mathbb{Q}}\, G = \dim_{\mathbb{Q}} G \otimes \mathbb{Q}.$$

4.4.1. *Basics on tensor products.* We need very classical facts on tensor products between abelian groups and $\mathbb{Q}$. We recall what is needed to follow our arguments and refer to [**Bou62**] for more details.

Let G be an abelian group. Then, it has a $\mathbb{Z}$-module structure and we can define the tensor product $G \otimes \mathbb{Q}$. Moreover, this product has the structure of a vector space over $\mathbb{Q}$.

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Elements in $G \otimes \mathbb{Q}$ are linear combinations of the form

$$\sum_{k=0}^{n} g_k \otimes q_k, \quad n \in \mathbb{N}, \quad g_k \in G, \quad q_k \in \mathbb{Q}, \quad 0 \leq k \leq n.$$

An element of the form $g \otimes q$ with $g \in G$ and $q \in \mathbb{Q}$ is said to be a *pure tensor*.

PROPOSITION 4.6. *Let G be an abelian group and $(G_n)_{n\geq 0}$ be a sequence of abelian groups. We have the following:*

(a) *$(\varinjlim G_n) \otimes \mathbb{Q}$ and $\varinjlim (G_n \otimes \mathbb{Q})$ are isomorphic as vector spaces over $\mathbb{Q}$;*
(a) *if $g \otimes q = 0$ in $G \otimes \mathbb{Q}$, then $q = 0$ or g is a torsion element in G.*

4.4.2. *Back to the computation of the topological rank.* The following lemma gives a lower bound for the topological rank of a minimal Cantor system.

LEMMA 4.7. *Let (X, T) be a minimal Cantor system. Then*

$$\operatorname{rank}_{\mathbb{Q}} H(X, T) \leq \operatorname{rank}(X, T). \tag{34}$$

Proof. We can assume $\operatorname{rank}(X, T) < +\infty$. Let $(\mathcal{T}_n)_{n\geq 0}$ be any nested sequence of CKR partitions of (X, T) such that $\liminf_{n\to+\infty} |\mathcal{A}(\mathcal{T}_n)| < +\infty$. Denote by $(M_n)_{n\geq 0}$ the sequence of incidence matrices of $(\mathcal{T}_n)_{n\geq 0}$. We can assume that $|\mathcal{A}(\mathcal{T}_n)| = p$, $n \geq 0$ for some $p \in \mathbb{N}$.

By Proposition 2.7, the dimension group $H(X, T)$ can be seen as the direct limit $\varinjlim \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)}$ with linear maps $M_n : \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \to \mathbb{Z}^{\mathcal{A}(\mathcal{T}_{n+1})}$, $n \geq 0$. Define the linear maps $j_{n+1,n} : \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q} \to \mathbb{Z}^{\mathcal{A}(\mathcal{T}_{n+1})} \otimes \mathbb{Q}$ on pure tensors by

$$j_{n+1,n}(v \otimes q) = vM_n \otimes q, \quad v \in \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)}, \quad q \in \mathbb{Q}, \quad n \geq 0$$

and extend them by linearity to $\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q}$. We consider $\varinjlim (\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q})$ with linear maps $(j_{n+1,n})_{n\geq 0}$.

Proposition 4.6 implies that $H(X, T) \otimes \mathbb{Q}$ and $\varinjlim (\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q})$ are isomorphic vector spaces over $\mathbb{Q}$. Each morphism $j_n : \mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q} \to \varinjlim (\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q})$ is linear and we have

$$\dim_{\mathbb{Q}} \operatorname{Im} j_n \leq \dim_{\mathbb{Q}} \operatorname{Im} j_n + \dim_{\mathbb{Q}} \ker j_n = p.$$

As $\operatorname{Im} j_m \subseteq \operatorname{Im} j_n$ for $m < n$, there exists $N \in \mathbb{N}$ such that $\operatorname{Im} j_m = \operatorname{Im} j_n$ for all $m, n \geq N$. This, together with the fact that $\varinjlim (\mathbb{Z}^{\mathcal{A}(\mathcal{T}_n)} \otimes \mathbb{Q}) = \bigcup_{n\geq 0} \operatorname{Im} j_n$, implies that $\operatorname{rank}_{\mathbb{Q}} H(X, T) \leq p$.

As the choice of the sequence $(\mathcal{T}_n)_{n\geq 0}$ is arbitrary, we deduce (34). □

The proof of the next lemma is essentially given in [**BCBD+21**, Theorem 4.1].

PROPOSITION 4.8. *Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be a primitive, proper and invertible directive sequence. Let $d = |\mathcal{A}_0|$. Then*

$$\operatorname{rank}_{\mathbb{Q}} H(X_{\boldsymbol{\tau}}, S) = \operatorname{rank}(X_{\boldsymbol{\tau}}, S) = d.$$

Proof. By Proposition 2.1 we have that $(X_{\boldsymbol{\tau}}, S)$ is an aperiodic subshift. Let $(\mathcal{T}_n)_{n\geq 0}$ be the sequence given by (8). As $\boldsymbol{\tau}$ is recognizable [**BSTY19**, Theorem 3.1], by Proposition 2.6 we have that $(\mathcal{T}_n)_{n\geq 0}$ is a nested sequence of CKR partitions. Hence, by (5) we have $\mathrm{rank}(X_{\boldsymbol{\tau}}, S) \leq d$.

Now we show that $H(X_{\boldsymbol{\tau}}, S) \otimes \mathbb{Q}$ is finite-dimensional. Indeed, we prove that

$$B = \{[\chi_{[a]}] \otimes 1 : a \in \mathcal{A}_0\}$$

is a basis of $H(X_{\boldsymbol{\tau}}, S) \otimes \mathbb{Q}$, where $[\chi_{[a]}]$ denotes the class of the characteristic function of the cylinder $[a]$ in $H(X_{\boldsymbol{\tau}}, S)$. This will finish the proof because (34) implies

$$d = \mathrm{rank}_{\mathbb{Q}}\, H(X_{\boldsymbol{\tau}}, S) \leq \mathrm{rank}(X_{\boldsymbol{\tau}}, S) \leq d.$$

Following the same steps as in the proof of [**BCBD+21**, Theorem 4.1] and because the matrix $M^{-1}_{\tau_{[0,n)}}$ has rational entries, we deduce that B spans $H(X_{\boldsymbol{\tau}}, S) \otimes \mathbb{Q}$. Suppose that $\alpha = (\alpha_a)_{a\in\mathcal{A}_0} \in \mathbb{Z}^{\mathcal{A}_0}$ is such that

$$\sum_{a\in\mathcal{A}_0} \alpha_a[\chi_{[a]}] \otimes 1 = 0.$$

The fact that $H(X_{\boldsymbol{\tau}}, S)$ is a torsion-free abelian group [**DP22**, Proposition 2.1.13] and Proposition 4.6 imply that $\sum_{a\in\mathcal{A}_0} \alpha_a[\chi_{[a]}] = 0$ in $H(X_{\boldsymbol{\tau}}, S)$. Then, as in the proof of [**BCBD+21**, Theorem 4.1], we obtain $\alpha = 0$ and B is a basis. □

Proposition 4.8 and (30) directly imply the following.

COROLLARY 4.9. *Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift. Then*

$$\mathrm{rank}(X_{\mathcal{W}}, S) = d_{\mathcal{W}}.$$

4.5. *Computation of the dimension group.* We now compute the dimension group of a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$. Let us explain how we proceed.

Let $\boldsymbol{\tau}_{\mathcal{W}}$ be the directive sequence associated with $(X_{\mathcal{W}}, S)$ and $n_0 \in \mathbb{N}$ be such that $\mathcal{A}_n = \mathcal{A}_{\mathcal{W}}$ for $n \geq n_0$. Define the directive sequence

$$\widehat{\boldsymbol{\tau}}_{\mathcal{W}} = (\tau_{n+n_0} : \mathcal{A}^*_{\mathcal{W}} \to \mathcal{A}^*_{\mathcal{W}})_{n\geq 0}.$$

By Lemma 3.9, $\widehat{\boldsymbol{\tau}}_{\mathcal{W}}$ is invertible. By Lemma 2.4, the $\mathcal{S}$-adic subshift $(X_{\widehat{\boldsymbol{\tau}}_{\mathcal{W}}}, S)$ is topologically conjugate to an induced system of $(X_{\boldsymbol{\tau}_{\mathcal{W}}}, S)$ on some clopen set and, from §2.4.4, the dimension group of $\widehat{\boldsymbol{\tau}}_{\mathcal{W}}$ is unital order isomorphic to $(\mathcal{H}_{\mathcal{W}}, \mathcal{H}^+_{\mathcal{W}}, \mathbf{1})$, where

$$\mathcal{H}_{\mathcal{W}} = \{y \in \mathbb{R}^{\mathcal{A}_{\mathcal{W}}} : yM_{\tau_{n_0}}M_{\tau_{n_0+1}} \ldots M_{\tau_{n_0+n-1}} \in \mathbb{Z}^{\mathcal{A}_{\mathcal{W}}} \text{ for all } n \text{large enough}\},$$
$$\mathcal{H}^+_{\mathcal{W}} = \{y \in \mathbb{R}^{\mathcal{A}_{\mathcal{W}}} : yM_{\tau_{n_0}}M_{\tau_{n_0+1}} \ldots M_{\tau_{n_0+n-1}} \in \mathbb{Z}^{\mathcal{A}_{\mathcal{W}}}_+ \text{ for all } n \text{ large enough}\},$$

and $\mathbf{1}(a) = 1$ for $a \in \mathcal{A}_{\mathcal{W}}$. Moreover, the dimension group of $(X_{\mathcal{W}}, S)$ is unital order isomorphic to $(\mathcal{H}_{\mathcal{W}}, \mathcal{H}^+_{\mathcal{W}}, v_{\mathcal{W}})$, where $v_{\mathcal{W}} = (|\tau_{[0,n_0)}(a)|)_{a\in\mathcal{A}_{\mathcal{W}}}$.

Recall the definition of the sequence $(q_n)_{n\geq 0}$ given in (10) and of $Q_{m,n}$ in (13).

By Proposition 4.1 and Lemma 2.4, the system $(X_{\widehat{\boldsymbol{\tau}}_{\mathcal{W}}}, S)$ is uniquely ergodic. Denote by $\widehat{\mu}$ its unique invariant probability measure and define the column probability vector

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$\widehat{\boldsymbol{\mu}} \in \mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$ by

$$\widehat{\boldsymbol{\mu}}(a) = \widehat{\mu}([a]), \quad a \in \mathcal{A}_{\mathcal{W}}.$$

As in the proof of Proposition 4.4, by using [**BKMS13**, Proposition 5.1] we have

$$\widehat{\boldsymbol{\mu}}(a) = \sum_{k=n_0}^{\infty} \frac{f_k(a)}{Q_{n_0-1,k}}, \quad a \in \mathcal{A}_{\mathcal{W}}. \tag{35}$$

In order to describe the dimension group of $(X_{\mathcal{W}}, S)$, we need to define, for a sequence of positive integers $(a_n)_{n\geq N}$, the following additive group

$$\mathbb{Z}[(a_n)_{n\geq N}] = \left\{ \frac{m}{a_N a_{N+1} \cdots a_n} : m \in \mathbb{Z},\ n \geq N \right\}.$$

If $a_n = a$ for all $n \geq N$, we write $\mathbb{Z}[1/a] = \mathbb{Z}[(a_n)_{n\geq N}]$.

Let $a' \in \mathcal{A}_{\mathcal{W}}$ be such that $a' = \min_{a\in\mathcal{A}_{\mathcal{W}}} a$. Define $\mathcal{B}_{\mathcal{W}} = \mathcal{A}_{\mathcal{W}} \setminus \{a'\}$.

We see elements in $\mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$ as vectors in $\mathbb{R}^{\mathcal{B}_{\mathcal{W}}} \times \mathbb{R}$. Define the column vector $\boldsymbol{z} \in \mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$ by

$$\boldsymbol{z}(b) = \widehat{\boldsymbol{\mu}}(b), \quad b \in \mathcal{B}_{\mathcal{W}} \quad \text{and} \quad \boldsymbol{z}(a') = 1.$$

PROPOSITION 4.10. *Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift and $(q_n)_{n\geq 0}$ be the sequence given in (10). The dimension group $K^0(X_{\mathcal{W}}, S)$ is unital order isomorphic to $(\mathcal{G}_{\mathcal{W}}, \mathcal{G}_{\mathcal{W}}^+, u_{\mathcal{W}})$, where*

$$\mathcal{G}_{\mathcal{W}} = \mathbb{Z}^{\mathcal{B}_{\mathcal{W}}} \times \mathbb{Z}[(q_n + 1)_{n\geq n_0-1}];$$
$$\mathcal{G}_{\mathcal{W}}^+ = \{x \in \mathcal{G}_{\mathcal{W}} : x \cdot \boldsymbol{z} > 0\} \cup \{0\};$$

and $u_{\mathcal{W}}$ is given by $u_{\mathcal{W}}(b) = b - a'$, $b \in \mathcal{B}_{\mathcal{W}}$ and $u_{\mathcal{W}}(a') = a' + |w_{n_0-1}|$.

Proof. We begin by proving the following.

Claim 4.10.1. We have

$$\mathcal{H}_{\mathcal{W}}^+ = \{y \in \mathcal{H}_{\mathcal{W}} : y \cdot \widehat{\boldsymbol{\mu}} > 0\} \cup \{0\}.$$

Indeed, because the directive sequence $\widehat{\boldsymbol{\tau}}$ is primitive, proper and recognizable, the measure $\widehat{\mu}$ is uniquely determined by the associated sequence of measure vectors $(\widehat{\mu}_n)_{n\geq 0}$ as defined in §2.3.2. For $a \in \mathcal{A}_{\mathcal{W}}$, denote by $e_a \in \mathbb{Z}^{\mathcal{A}}$ the vector such that $e_a(b) = 1$ if $a = b$ and 0 otherwise.

Let $\widehat{P}_n = M_{\tau_{n_0}} M_{\tau_{n_0+1}} \ldots M_{\tau_{n_0+n-1}}$ for $n > 0$. By (7), we have

$$\widehat{\boldsymbol{\mu}} = \widehat{P}_n \mu_n, \quad n > 0.$$

If y belongs to $\mathcal{H}_{\mathcal{W}}^+ \setminus \{0\}$ and $n > 0$ is such that $y\widehat{P}_n$ is in $\mathbb{Z}_+^{\mathcal{A}_{\mathcal{W}}}$, then

$$y \cdot \widehat{\boldsymbol{\mu}} = y \cdot \widehat{P}_n \mu_n = (y\widehat{P}_n) \cdot \mu_n > 0.$$

Now, let $y \in \mathcal{H}_{\mathcal{W}}$ with $y \cdot \widehat{\boldsymbol{\mu}} > 0$. By contradiction, if y is not in $\mathcal{H}_{\mathcal{W}}^+$, there exists $N \in \mathbb{N}$ and a sequence $(a_n)_{n\geq N}$ such that a_n belongs to $\mathcal{A}_{\mathcal{W}}$ and $(y\widehat{P}_n) \cdot e_{a_n} \leq -1$ for all $n \geq N$. Hence, there exists $a \in \mathcal{A}_{\mathcal{W}}$ and a sequence $(n_k)_{k\geq 0}$ such that $n_k \geq N$ and

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$(y\widehat{P}_{n_k})\cdot e_a \leq -1$ for $k\geq 0$. Let $\widehat{\boldsymbol{\nu}}$ be a limit point of the sequence of probability vectors $(\widehat{P}_{n_k}e_a/|\widehat{P}_{n_k}e_a|)_{k\geq 0}$.

By unique ergodicity of $(X_{\widehat{\tau}}, S)$, we deduce $\widehat{\boldsymbol{\nu}}=\widehat{\boldsymbol{\mu}}$. Finally, up to passing to a subsequence,

$$0 < y\cdot\widehat{\boldsymbol{\mu}} = \lim_{k\to+\infty}\frac{(y\widehat{P}_{n_k})\cdot e_a}{|\widehat{P}_{n_k}e_a|}\leq 0,$$

a contradiction. This proves the claim.

Let $y\in\mathcal{H}_{\mathcal{W}}$. There exists $n\geq n_0$ with $yM_{\tau_{n_0}}M_{\tau_{n_0+1}}\ldots M_{\tau_{n_0+n-1}}\in\mathbb{Z}^{\mathcal{A}_{\mathcal{W}}}$. Recall that $\boldsymbol{u}$ is the row vector of ones in $\mathbb{R}^{\mathcal{A}_{\mathcal{W}}}$. By (27), we see that

$$yM_{\tau_{n_0}}M_{\tau_{n_0+1}}\ldots M_{\tau_{n_0+n-1}} = y+(y\cdot f_{n_0,n_0+n})\boldsymbol{u}\in\mathbb{Z}^{\mathcal{A}_{\mathcal{W}}},$$

and, hence, $y(b)-y(a')$ belongs to $\mathbb{Z}$, $b\in\mathcal{B}_{\mathcal{W}}$. Observe that

$$\boldsymbol{u}M_{\tau_{n_0}}M_{\tau_{n_0+1}}\ldots M_{\tau_{n_0+n-1}} = (|f_{n_0,n_0+n}|+1)\boldsymbol{u}.$$

Moreover, from (13) and (29), we have

$$|f_{n_0,n_0+n}|+1 = (q_{n_0-1}+1)(q_{n_0}+1)\ldots(q_{n_0+n-2}+1),$$

so $y(a')$ belongs to $\mathbb{Z}[(q_n+1)_{n\geq n_0-1}]$.

These two observations allow us to define the following group isomorphism

$$\begin{aligned}\psi:\mathcal{H}_{\mathcal{W}}&\to\mathbb{Z}^{\mathcal{B}_{\mathcal{W}}}\times\mathbb{Z}[(q_n+1)_{n\geq n_0-1}]\\ y&\mapsto(y',y(a')),\end{aligned}$$

where $y'(b)=y(b)-y(a')$ for $b\in\mathcal{B}_{\mathcal{W}}$. Moreover, for $y\in\mathcal{H}_{\mathcal{W}}$ we have $y\cdot\widehat{\boldsymbol{\mu}}=\psi(y)\cdot\boldsymbol{z}$ and, hence,

$$\psi(\mathcal{H}^+_{\mathcal{W}}) = \{x\in\mathcal{G}_{\mathcal{W}} : x\cdot\boldsymbol{z}>0\}\cup\{0\}.$$

From Lemma 3.8, we obtain $\psi(v_{\mathcal{W}})=u_{\mathcal{W}}$. This completes the proof. □

4.6. *Zoology of dimension groups of Ferenczi type.* We now characterize the dimension groups that can be obtained from minimal Ferenczi subshifts. For this, we need to recall the following well-known fact about numeration systems.

4.6.1. *Facts about numeration systems.* Let $(p_k)_{k\geq 0}$ be a sequence of positive integers with $p_k\geq 2$, $k\geq 0$. Then, for every real number x with $0\leq x\leq 1$, there exists a sequence $(f_k)_{k\geq 1}$ such that $0\leq f_k\leq p_{k-1}$ for $k\geq 1$ and

$$x=\sum_{k=1}^{\infty}\frac{f_k}{p_0p_1\ldots p_{k-1}}.$$

We say that $(f_k)_{k\geq 1}$ is the *expansion* of x in the *base* $(p_k)_{k\geq 0}$.

4.6.2. *Ferenczi-type dimension groups.* Let $\mathcal{B}$ be a non-empty alphabet. We define

$$U^{\mathcal{B}} = \{u\in\mathbb{Z}^{\mathcal{B}}_{>0} : u(b)\neq u(b') \text{ for } b,b'\in\mathcal{B}\}.$$

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Observe that the unit $u_{\mathcal{W}}$ in Proposition 4.10 belongs to $U^{\mathcal{B}_{\mathcal{W}}} \times \mathbb{Z}_{>0}$ because all elements in $\mathcal{A}_{\mathcal{W}}$ are distinct. Define

$$\Delta^{\mathcal{B}} = \{z \in \mathbb{R}^{\mathcal{B}}_{>0} : \textstyle\sum_{b \in \mathcal{B}} z(b) < 1\}$$

and let $(r_n)_{n \geq 0}$ be a sequence of integers with $r_n \geq 2$.

We say that a dimension group $(\mathcal{G}, \mathcal{G}^+, u)$ is of *Ferenczi type* if there exist a non-empty alphabet $\mathcal{B}$, a sequence $(r_n)_{n \geq 0}$ as before and $\boldsymbol{z} \in \Delta^{\mathcal{B}} \times \{1\}$ such that

$$\begin{aligned} \mathcal{G} &= \mathbb{Z}^{\mathcal{B}} \times \mathbb{Z}[(r_n + 1)_{n \geq 0}]; \\ \mathcal{G}^+ &= \{x \in \mathcal{G} : x \cdot \boldsymbol{z} > 0\} \cup \{0\}; \quad \text{and} \\ u &\in U^{\mathcal{B}} \times \mathbb{Z}_{>0}. \end{aligned}$$

Proposition 4.10 shows that the dimension group of a minimal Ferenczi subshift is of Ferenczi type. Conversely, let $(\mathcal{G}, \mathcal{G}^+, u)$ be a dimension group of Ferenczi type given by $\mathcal{B}$, $(r_n)_{n \geq 0}$ and $\boldsymbol{z}$. Write $u = (v, w)$, where $v \in U^{\mathcal{B}}$ and $w \in \mathbb{Z}_{>0}$.

Let $a' = w - 1$ and $s(b) = a' + v(b)$ for $b \in \mathcal{B}$. Observe that $s(b) > a'$ and $s(b) \neq s(b')$ for $b, b' \in \mathcal{B}$. Define any sequence of generating words $\mathcal{W}$ such that:

(1) $n_0 = 1$ and $\mathcal{A}_{\mathcal{W}} = \{s(b) : b \in \mathcal{B}\} \cup \{a'\}$;

(2) $q_n = r_n$ for $n \geq 0$; and

(3) for $b \in \mathcal{B}_{\mathcal{W}}$, let $(f_k(s(b)))_{k \geq 1}$ be the expansion of $\boldsymbol{z}(b)$ in the base $(q_k + 1)_{k \geq 0}$, that is,

$$\boldsymbol{z}(b) = \sum_{k=1}^{\infty} \frac{f_k(s(b))}{(q_0 + 1)(q_1 + 1) \ldots (q_{k-1} + 1)}, \quad b \in \mathcal{B}.$$

From Equation (35), we have thus proved the following.

COROLLARY 4.11. *A dimension group $\mathcal{K} = (\mathcal{G}, \mathcal{G}^+, u)$ is of Ferenczi type if and only if there exists a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ such that $\mathcal{K}$ is unital order isomorphic to $K^0(X_{\mathcal{W}}, S)$.*

Example 4.12

(1) The *Chacon subshift* is defined by the sequence of generating words $\mathcal{W}$ which satisfies

$$w_{n+1} = w_n w_n 1 w_n, \quad n \geq 0.$$

Proposition 4.10 shows that the dimension group of the Chacon subshift is

$$(\mathbb{Z} \times \mathbb{Z}[1/3], \quad \{(x, y) \in \mathbb{Z} \times \mathbb{Z}[1/3] : x + 2y > 0\} \cup \{(0, 0)\}, \quad (1, 1)).$$

This dimension group is unital order isomorphic to

$$(\mathbb{Z} \times \mathbb{Z}[1/3], \quad \mathbb{Z} \times \mathbb{Z}_+[1/3], \quad (1, 1)).$$

(2) The *Thue–Morse subshift* is the subshift generated by the constant directive sequence $\boldsymbol{\tau} = (\tau, \tau, \ldots)$, where the morphism $\tau : \{a, b\}^* \to \{a, b\}^*$ is given by $\tau(a) = ab$ and $\tau(b) = ba$. Its dimension group is

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$$(\mathbb{Z} \times \mathbb{Z}[1/2], \quad \{(x, y) \in \mathbb{Z} \times \mathbb{Z}[1/2] : -x + 3y > 0\} \cup \{(0, 0)\}, \quad (0, 1)),$$

see [**DP22**, Example 4.6.11].

We claim that the Thue–Morse subshift is not strongly orbit equivalent to a minimal Ferenczi subshift. Indeed, by Corollary 4.11, suppose that there exists a non-empty alphabet $\mathcal{B}$, a sequence $(r_n)_{n\geq 0}$, an order unit $u \in U^{\mathcal{B}} \times \mathbb{Z}_{>0}$ and a isomorphism

$$\psi : \mathbb{Z} \times \mathbb{Z}[1/2] \to \mathbb{Z}^{\mathcal{B}} \times \mathbb{Z}[(r_n + 1)_{n\geq 0}]$$

such that $\psi(0, 1) = u$.

The existence of ψ ensures the existence of an isomorphism between $(\mathbb{Z} \times \mathbb{Z}[1/2]) \otimes \mathbb{Q}$ and $(\mathbb{Z}^{\mathcal{B}} \times \mathbb{Z}[(r_n + 1)_{n\geq 0}]) \otimes \mathbb{Q}$ and, thus, $|\mathcal{B}| = 1$.

For a prime number p, denote by $v_p(\cdot)$ the p-adic valuation. For an integer sequence $(a_n)_{n\geq 0}$, the sequence $(v_p(a_0 \ldots a_n))_{n\geq 0}$ is increasing and, hence, it is eventually constant or tends to $+\infty$. We denote by $v_p((a_n)_{n\geq 0})$ the eventually constant value of it (either finite or $+\infty$).

It is easy to show that $v_2((r_n + 1)_{n\geq 0}) = +\infty$ and $v_p((r_n + 1)_{n\geq 0}) = 0$ for $p \neq 2$, so we can suppose $\mathbb{Z}[(r_n + 1)_{n\geq 0}] = \mathbb{Z}[1/2]$.

Write $\psi(0, 1) = (m, w)$, $m \in \mathbb{Z}$, $w \in \mathbb{Z}[1/2]$ and $\psi(0, 1/2^n) = (m_n, w_n)$, $m_n \in \mathbb{Z}$, $w_n \in \mathbb{Z}[1/2]$. Then

$$(2^n m_n, 2^n w_n) = (m, w), \quad n \in \mathbb{N}.$$

In particular, 2^n divides m for all $n \geq 0$, hence $m = 0$. If $\psi(1, 0) = (d, v)$ for $d \in \mathbb{Z}$ and $v \in \mathbb{Z}[1/2]$, we obtain

$$\psi(s, t) = (ds, sv + tw), \quad s \in \mathbb{Z}, \quad t \in \mathbb{Z}[1/2].$$

We have $u = \psi(0, 1) = (0, w)$, but 0 does not belong to $U^{\mathcal{B}}$. This shows that the Thue–Morse subshift is not strongly orbit equivalent to any minimal Ferenczi subshift.

4.7. *Comments on orbit equivalence.* In this section we characterize the orbit equivalence class of minimal Ferenczi subshifts. With this purpose, we compute explicitly the dimension group $K^0(X_{\mathcal{W}}, S)/\operatorname{Inf} K^0(X_{\mathcal{W}}, S)$ of a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$. Recall the definition of $\mathcal{G}_{\mathcal{W}}$, z and $u_{\mathcal{W}}$ given in Proposition 4.10.

PROPOSITION 4.13. *Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift. Let $\widetilde{z}$ be the unique vector collinear to z and such that $u_{\mathcal{W}} \cdot \widetilde{z} = 1$. Define*

$$\mathcal{J}_{\mathcal{W}} = \{x \cdot \widetilde{z} : x \in \mathcal{G}_{\mathcal{W}}\}.$$

Then, the dimension group $K^0(X_{\mathcal{W}}, S)/\operatorname{Inf} K^0(X_{\mathcal{W}}, S)$ is unital order isomorphic to

$$(\mathcal{J}_{\mathcal{W}}, \quad \{y \in \mathcal{J}_{\mathcal{W}} : y \geq 0\}, \quad 1).$$

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Proof. From Proposition 4.10, we see that $\mathrm{Inf}\,\mathcal{G}_{\mathcal{W}} = \{x \in \mathcal{G}_{\mathcal{W}} : x \cdot \widetilde{z} = 0\}$. It is straightforward to check that the map

$$\begin{aligned} \mathcal{G}_{\mathcal{W}} / \mathrm{Inf}\,\mathcal{G}_{\mathcal{W}} &\to \mathcal{J}_{\mathcal{W}} \\ [x] &\mapsto x \cdot \widetilde{z} \end{aligned}$$

is an isomorphism between the dimension groups $\mathcal{G}_{\mathcal{W}} / \mathrm{Inf}\,\mathcal{G}_{\mathcal{W}}$ and $\mathcal{J}_{\mathcal{W}}$. Moreover, this map sends the induced image of $\mathcal{G}_{\mathcal{W}}^{+}$ in $\mathcal{G}_{\mathcal{W}} / \mathrm{Inf}\,\mathcal{G}_{\mathcal{W}}$ to $\{x \in \mathcal{J}_{\mathcal{W}} : x \geq 0\}$ and $[u_{\mathcal{W}}]$ to 1 because $u_{\mathcal{W}} \cdot \widetilde{z} = 1$. □

In particular, observe that if z has rationally independent entries, then the strong orbit equivalence class of $(X_{\mathcal{W}}, S)$ coincides with the orbit equivalence class.

One can check that $\widetilde{z} = cz$, where

$$c = \lim_{n \to +\infty} \frac{Q_{n_0-1, n_0+n-1}}{|w_{n_0+n-1}|}.$$

4.8. *Continuous eigenvalues.* In this section we recall results in [GZ19] concerning continuous eigenvalues, topological weak mixing and topological mixing of minimal Ferenczi subshifts. We observe they can be deduced from the general framework provided by [DFM19] and Theorem 3.7. It can be easily observed that Ferenczi subshifts $(X_{\mathcal{W}}, S)$ has no continuous irrational eigenvalues and, moreover, the complex value $\lambda = \exp(2\pi i p/q)$ with p/q a rational number is a continuous eigenvalue of $(X_{\mathcal{W}}, S)$ if and only if there exists $n \geq 0$ such that q divides all the coordinates of the heights h_n. From Lemma 3.8, this condition translates into q divides $|w_n| + a_{m,i}$ for all $m \geq n$ and $0 \leq i < q_m$.

As a consequence the following results can be deduced.

PROPOSITION 4.14. [GZ19, Theorem 1.1] *A minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ is topologically weakly mixing if and only if for all integer $q > 1$ and all $n \geq 0$ there exists $m \geq n$ and $0 \leq i < q_m$ such that q does not divide $|w_n| + a_{m,i}$.*

PROPOSITION 4.15. [GZ19, Theorem 1.5] *Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift. Then, there exists a maximal integer q such that $(\mathbb{Z}/q\mathbb{Z}, +1 \pmod{q})$ is a topological factor of $(X_{\mathcal{W}}, S)$. Moreover, this factor corresponds to the maximal equicontinuous factor of $(X_{\mathcal{W}}, S)$.*

4.9. *Topological mixing.* We recall that a topological dynamical system (X, T) is said to be *topologically mixing* if for any non-empty open sets $U, V \subseteq X$, there exists $N \in \mathbb{N}$ such that

$$T^n U \cap V \neq \emptyset, \quad n \geq N.$$

We prove in the following a necessary condition for a minimal subshift to be topologically mixing that have its own interest beyond Ferenczi subshifts. As a direct consequence, we deduce that minimal Ferenczi subshifts are not topologically mixing.

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For a subshift (X, S) with $X \subseteq \{0, 1\}^{\mathbb{Z}}$, we define the quantities

$$a(n) = \min_{w \in \mathcal{L}_n(X)} |w|_0 \quad \text{and} \quad b(n) = \max_{w \in \mathcal{L}_n(X)} |w|_0, \quad n \geq 1.$$

LEMMA 4.16. *Suppose that (X, S) is minimal and topologically mixing. Then*

$$\lim_{n \to +\infty} b(n) - a(n) = +\infty.$$

Proof. Following [**KSS05**, Proposition 3.2], if (X, S) is topologically mixing we have

$$\liminf_{n \to +\infty} b(n) - a(n) = \sup_{n \geq 1} b(n) - a(n). \tag{36}$$

Claim 4.16.1. For any invariant ergodic probability measure μ of (X, S) we have

$$a(n) \leq n\mu([0]) \leq b(n), \quad n \geq 1.$$

Indeed, it is easy to see that the sequence $(a(n)/n)_{n \geq 1}$ is superadditive, that is, $a(m+n) \geq a(m) + a(n)$ for all $m, n \geq 1$ and that the sequence $(b(n)/n)_{n \geq 1}$ is subadditive, that is, $b(m+n) \leq b(m) + b(n)$ for $m, n \geq 1$. In particular, we deduce

$$\lim_{n \to +\infty} \frac{a(n)}{n} = \sup_{n \geq 1} \frac{a(n)}{n} \quad \text{and} \quad \lim_{n \to +\infty} \frac{b(n)}{n} = \inf_{n \geq 1} \frac{b(n)}{n}.$$

Observe that

$$\frac{a(n)}{n} \leq \frac{1}{n} \#\{0 \leq k < n : x_k = 0\} = \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[0]}(S^k x), \quad x \in X, \quad n \geq 1.$$

By Birkhoff's theorem, there exists $x \in X$ such that

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{k=0}^{m-1} \chi_{[0]}(S^k x) = \mu([0]).$$

Hence, because $a(n)/n \leq \sup_{m \geq 1} a(m)/m$, we obtain $a(n)/n \leq \mu([0])$. Analogously, we obtain $\mu([0]) \leq b(n)/n$. We obtain

$$\left| \sum_{k=0}^{n-1} \chi_{[0]}(S^k x) - n\mu([0]) \right| \leq b(n) - a(n), \quad x \in X, \quad n \geq 1. \tag{37}$$

Let $f = \chi_{[0]} - \mu([0])$. If $\limsup_{n \to +\infty} b(n) - a(n) \neq +\infty$, by (37) we deduce that there exists a constant $C > 0$ such that $|\sum_{k=0}^{n-1} f(S^k x)| \leq C$, for all $x \in X$, $n \geq 1$. The Gottschalk–Hedlund theorem then implies that $f = g - g \circ S$ for some continuous map $g : X \to \mathbb{R}$. In particular,

$$\exp(2\pi i g \circ S) = \exp(2\pi i \mu([0])) \exp(2\pi i g),$$

that is, $\exp(2\pi i \mu([0]))$ is a non-trivial continuous eigenvalue of (X, S). This contradicts the fact that (X, S) is topologically weakly mixing.

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Finally, we deduce $\limsup_{n\to+\infty} b(n) - a(n) = +\infty$ and, together with (36), we obtain $\lim_{n\to+\infty} b(n) - a(n) = +\infty$. □

As a consequence of Lemma 4.16 we deduce the following.

PROPOSITION 4.17. [**GZ19**, Theorem 1.3] *Minimal Ferenczi subshifts are not topologically mixing.*

4.10. *Asymptotic classes and automorphism group.* Let (X, T) be a topological dynamical system and $d : X \times X \to \mathbb{R}$ be a metric on X. We say that two points $x, y \in X$ are *asymptotic* if

$$\lim_{n\to+\infty} d(T^n x, T^n y) = 0.$$

Non-trivial asymptotic pairs of points may not exist in an arbitrary topological dynamical system, but they always exist in the context of non-empty aperiodic subshifts [**Aus88**, Ch. 1].

We define the relation $\sim$ in X as follows: $x \sim y$ if x is asymptotic to $T^k y$ for some $k \in \mathbb{Z}$. This defines an equivalence relation. An equivalence class for $\sim$ that is not the orbit of a single point is called an *asymptotic class*.

An *automorphism* of a topological dynamical system (X, T) is a homeomorphism $\phi : X \to X$ such that

$$\phi \circ T = T \circ \phi.$$

We denote by $\mathrm{Aut}(X, T)$ the group of automorphism of (X, T) and by $\langle T \rangle$ the subgroup of $\mathrm{Aut}(X, T)$ generated by integer powers of T.

For a minimal Ferenczi subshift we show that there exists a unique asymptotic class. We first need the following lemma, for which we recall the definition of cutting points given in §2.2.4.

LEMMA 4.18. *Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $Y = \bigcup_{k\in\mathbb{Z}} S^k \tau(X)$. Assume that τ is recognizable in X and that if a and b are two distinct letters in $\mathcal{A}$, then $\tau(a)$ is not a suffix of $\tau(b)$. Let y, y' in Y be such that $y_0 \neq y'_0$ and $y_{(0,+\infty)} = y'_{(0,+\infty)}$. Suppose that (k, x) and (k', x') are the unique centered τ-representations of y and y' in X, respectively. Then*

$$\mathcal{C}^+_\tau(k, x) = \mathcal{C}^+_\tau(k', x'), \quad x_0 \neq x'_0 \quad \textit{and} \quad x_{(0,+\infty)} = x'_{(0,+\infty)}.$$

Proof. We begin by proving the following.

Claim 4.18.1. There exist infinitely many pairs (ℓ, ℓ') with $\ell, \ell' \geq 0$ such that

$$C^\ell_\tau(k, x) = C^{\ell'}_\tau(k', x').$$

Indeed, by [**DDMP21**, Lemma 3.2] there exists a constant $R > 0$ such that if (k, x) and (k', x') are two centered τ-representations in X of points $y, y' \in Y$ and $y_{[-R,R)} = y'_{[-R,R)}$, then $k = k'$ and $x_0 = x'_0$.

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Arguing by contradiction, if the claim is not true and because $y_{(0,+\infty)} = y'_{(0,+\infty)}$ there exists $\ell_0 \geq 0$ such that if $\ell \geq \ell_0$, then $C^{\ell}_{\tau}(k, x) \notin \mathcal{C}^{+}_{\tau}(k', x')$ and $(S^j y)_{[-R,R)} = (S^j y')_{[-R,R)}$, where $j = C^{\ell_0}_{\tau}(k, x)$. However, then j belongs to $\mathcal{C}^{+}_{\tau}(k', x')$, a contradiction.

By the claim, there exists an increasing sequence $(\ell_n)_{n\geq 0}$ such that $\ell_n \geq 0$ and

$$C^{\ell_n}_{\tau}(k, x) = C^{\ell'_n}_{\tau}(k', x'), \quad \text{for some } \ell'_n \geq 0.$$

If $\ell_n \geq 2$, because $y_{(0,+\infty)} = y'_{(0,+\infty)}$ and $C^{\ell_n}_{\tau}(k, x) = C^{\ell'_n}_{\tau}(k', x')$, we deduce that $\tau(x_{\ell_n -1})$ is a suffix of $\tau(x'_{\ell'_n -1})$ or that $\tau(x'_{\ell'_n -1})$ is a suffix of $\tau(x_{\ell_n -1})$. By assumption, this implies that $x_{\ell_n -1} = x'_{\ell'_n -1}$. By repeating the argument, we see that $\ell_n = \ell'_n$ and $C^{\ell}_{\tau}(k, x) = C^{\ell}_{\tau}(k', x')$, $1 \leq \ell \leq \ell_n$, $n \geq 0$. Therefore, as $(\ell_n)_{n\geq 0}$ is increasing, we deduce that $\mathcal{C}^{+}_{\tau}(k, x) = \mathcal{C}^{+}_{\tau}(k', x')$, $x_0 \neq x'_0$ because $y_0 \neq y'_0$ and $x_{(0,+\infty)} = x'_{(0,+\infty)}$. This completes the proof. □

Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift and $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)_{n\geq 0}$ be the directive sequence given by (17). In order to study the asymptotic classes of $(X_{\mathcal{W}}, S)$ we need the following definitions.

Define the words $L_n = a_{n-1,1}a_{n-1,2} \ldots a_{n-1,q_{n-1}-1}$ and $R_n = a_{n-1,0}$. Observe that they satisfy

$$\tau_n(a) = L_n a R_n, \quad a \in \mathcal{A}_{n+1}, \quad n \geq 1.$$

Inductively, define $L_{1,1} = L_1$, $R_{1,1} = R_1$, and for $n \geq 1$ we let

$$L_{1,n+1} = \tau_{[1,n+1)}(L_{n+1})L_{1,n} \quad \text{and} \quad R_{1,n+1} = R_{1,n}\tau_{[1,n+1)}(R_{n+1}). \tag{38}$$

Hence, we have

$$\tau_{[1,n+1)}(a) = L_{1,n} a R_{1,n}, \quad a \in \mathcal{A}_{n+1}, \quad n \geq 1.$$

PROPOSITION 4.19. *A minimal Ferenczi subshift has a unique asymptotic class.*

Proof. Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift generated by the directive sequence $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)_{n\geq 0}$ given by (17). For $n \geq 0$ recall the definition of the subshift $X^{(n)}_{\boldsymbol{\tau}_{\mathcal{W}}}$ given in §2.2.3 and that $\mathcal{A}_1$ is the set of values of the sequence $(a_{n,i} : n \geq 0, 0 \leq i < q_n)$.

For $n \geq 1$ and $a \neq b$ in $\mathcal{A}_n$, we have that the word $\tau_{[0,n)}(a)$ is not a suffix of the word $\tau_{[0,n)}(b)$. Indeed, this is clear if $n = 1$. If $n \geq 1$, by (38) we have

$$\tau_{[0,n+1)}(c) = \tau_0(L_{1,n})01^c\tau_0(R_{1,n}), \quad c \in \mathcal{A}_{n+1}, \tag{39}$$

from which the claim follows easily.

As $(X_{\mathcal{W}}, S)$ is minimal and aperiodic, there exists at least one asymptotic class. Let z and z' be two points in this asymptotic class such that $z_0 \neq z'_0$ and $z_{(0,+\infty)} = z'_{(0,+\infty)}$. Without lost of generality, we assume that $z_0 = 0$ and $z'_0 = 1$. By Lemma 3.6, there exist pairs (k, y) and (k', y') with $y, y' \in X^{(1)}_{\boldsymbol{\tau}_{\mathcal{W}}}$, $0 \leq k < |\tau_0(y_0)|$, $0 \leq k' < |\tau_0(y'_0)|$ and

$$z = S^k\tau_0(y), \quad z' = S^{k'}\tau_0(y').$$

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As $z_0 = 0$ and $\tau_0(c) = 01^c$ for $c \in \mathcal{A}_1$, we deduce that $k = 0$ and from Lemma 4.18 we obtain $C^1_{\tau_0}(0, y) = C^1_{\tau_0}(k', y')$. Define $a = y_0$ and $b = y'_0$. The fact that $z_{(0,+\infty)} = z'_{(0,+\infty)}$ and $z'_0 = 1$ implies that $a < b$.

Now fix $n \geq 0$. By Lemma 3.6, there exist pairs (j, x) and (j', x') with $x, x' \in X^{(n+1)}_{\boldsymbol{\tau}_\mathcal{W}}$, $0 \leq j < |\tau_{[0,n+1)}(x_0)|$, $0 \leq j' < |\tau_{[0,n+1)}(x'_0)|$ and

$$z = S^j \tau_{[0,n+1)}(x), \quad z' = S^{j'} \tau_{[0,n+1)}(x').$$

From Lemma 4.18 we have $C^1_{\tau_{[0,n+1)}}(j, x) = C^1_{\tau_{[0,n+1)}}(j', x')$. Let $s = C^1_{\tau_{[0,n+1)}}(j, x)$, so that $z_{[1,s)} = 1^a \tau_0(y_{[1,m)}) = z'_{[1,s)}$ for some $m \in \mathbb{N}$. This, together with (39), implies that $x_0 = a$ and $x'_0 = b$. We conclude that

$$z_{[1,s)} = z'_{[1,s)} = 1^a \tau_0(R_{1,n}).$$

As $R_{1,n}$ is a prefix of $R_{1,n+1}$ for each $n \geq 1$ and $(|R_{1,n}|)_{n\geq 1}$ is increasing, there exists a one-sided sequence $u = (u_n)_{n\in\mathbb{N}}$ in $\{0, 1\}^\mathbb{N}$ such that

$$u_{[0,|\tau_0(R_{1,n})|)} = \tau_0(R_{1,n}), \quad n \geq 1.$$

We deduce that $z_{[a+1,+\infty)} = z'_{[a+1,+\infty)} = u$, which does not depend on the points z and z' but only on $\boldsymbol{\tau}_\mathcal{W}$. This proves the result. □

For a minimal topological dynamical system (X, T), the existence of a unique asymptotic class implies that the automorphism group $\mathrm{Aut}(X, T)$ is trivial, that is,

$$\mathrm{Aut}(X, T) = \langle T \rangle.$$

Indeed, let $x \in X$ be an element in the unique asymptotic class and ϕ be an element in $\mathrm{Aut}(X, T)$. As the map ϕ sends asymptotic classes to asymptotic classes, we deduce that $\phi(x)$ is asymptotic to $T^m x$ for some $m \in \mathbb{Z}$. From [**DDMP16**, Lemma 2.3] we have that $\phi = T^m$, and we conclude that $\mathrm{Aut}(X, T) = \langle T \rangle$.

Therefore, Proposition 4.19 implies the following.

COROLLARY 4.20. [**GH16b**, Theorem 1.2] *The automorphism group of a minimal Ferenczi subshift is trivial.*

5. *Measurable eigenvalues of minimal Ferenczi subshifts*

In this section we further develop the spectral study of minimal Ferenczi subshifts initiated in §4.8 for continuous eigenvalues by analyzing their measurable eigenvalues.

We first give a general necessary condition for a complex number to be a measurable eigenvalue of certain $\mathcal{S}$-adic subshifts. This is stated in Proposition 5.1. Then, we show that, under the hypothesis of exact finite rank, all measurable eigenvalues of minimal Ferenczi subshifts are continuous, thus improving previous known results [**GH16a**, Theorem 4.1]. This is stated in Corollary 5.4.

5.1. *The Veech criterion for $\mathcal{S}$-adic subshifts.* We now give a general necessary condition for a complex value to be a measurable eigenvalue with respect to an ergodic invariant probability measure of some $\mathcal{S}$-adic subshifts. Such a condition, originally due

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to Veech [**Vee84**] in the context of interval exchange transformations, was stated as the *Veech criterion* in several articles [**AD16**, **AF07**, **Vee84**] and was crucial in order to obtain generic weak mixing for interval exchange transformations and translation flows in certain Veech surfaces.

For convenience, we state and prove here the necessary condition in the context of $\mathcal{S}$-adic subshifts following the lines of the original proof of the Veech criterion. As we only consider minimal Cantor systems of finite topological rank, there is no loss in generality [**DM08**]. See [**DFM19**] for a finer analysis of measurable eigenvalues in the more general context of minimal Cantor systems.

PROPOSITION 5.1. *Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be an everywhere growing and recognizable directive sequence and μ be an ergodic invariant probability measure of $(X_{\boldsymbol{\tau}}, S)$. Assume that $\boldsymbol{\tau}$ is clean with respect to μ and let $\mathcal{A}_\mu$ be the set of letters such that (31) holds. Suppose that:*

(i) *there exists $K > 0$ such that $|\tau_{[0,n)}|/\langle\tau_{[0,n)}\rangle \leq K$ for all large enough n; and*
(ii) *there exists $\delta > 0$, and, for all large enough n, a non-empty word $u_n \in \mathcal{A}_0^*$ and indices c_n, d_n with $0 \leq c_n < d_n \leq \min_{a\in\mathcal{A}_\mu} |\tau_{[0,n)}(a)|$ which satisfy $|u_n| \geq \delta \min_{a\in\mathcal{A}_\mu} |\tau_{[0,n)}(a)|$ and*

$$\tau_{[0,n)}(a)_{[c_n,d_n)} = u_n, \quad a \in \mathcal{A}_\mu.$$

If $\lambda = \exp(2\pi i\alpha)$ is a measurable eigenvalue of $(X_{\boldsymbol{\tau}}, S)$ with respect to μ, then

$$\lim_{n\to+\infty} |||\alpha h_n(a)||| = 0, \quad a \in \mathcal{A}_\mu. \tag{40}$$

Proof. Let $f : X_{\boldsymbol{\tau}} \to \mathbb{C}$, $f \neq 0$ be a measurable eigenfunction of $(X_{\boldsymbol{\tau}}, S)$ with respect to μ with eigenvalue λ. We can assume $|f| = 1$ μ-almost everywhere by ergodicity. Remember the definition of the sets $B_n(a)$ for $a \in \mathcal{A}_n$ and B_n in (8).

Let $c > 0$ be such that (31) holds and $0 < \varepsilon < c/3K$. From now on, we choose n large enough such that items (i) and (ii) hold. We set

$$B_{n,\mu} = \bigcup_{a\in\mathcal{A}_\mu} B_n(a).$$

Define $t_n = c_n + \lceil (d_n - c_n)/4\rceil$, $\ell_n = \lceil (d_n - c_n)/2\rceil$ and $A_{n,\mu} = \bigcup_{k=t_n}^{t_n+\ell_n-1} S^k B_{n,\mu}$. See Figure 1.

Observe that the union which defines $A_{n,\mu}$ is disjoint and that

$$\mu(A_{n,\mu}) \geq (d_n - c_n)\mu(B_{n,\mu})/2 \geq \delta \min_{a\in\mathcal{A}_\mu} |\tau_{[0,n)}(a)|\mu(B_{n,\mu})/2 \geq \delta c/2.$$

Claim 5.1.1. For all large enough n there exists some value $k_n \in \mathbb{N}$ and some complex value $w_{k_n} \in \mathbb{C}$ such that $t_n \leq k_n < t_n + \ell_n$ and

$$\int_{S^{k_n} B_{n,\mu}} |f - w_{k_n}|\, d\mu \leq \varepsilon^2 \mu(B_{n,\mu}).$$

Indeed, by Lusin's theorem there exists a compact set $C \subseteq X_{\boldsymbol{\tau}}$ such that $f|_C$ is uniformly continuous and $\mu(C) \geq 1 - \chi$, where $\chi = \varepsilon^2\delta c/8$.

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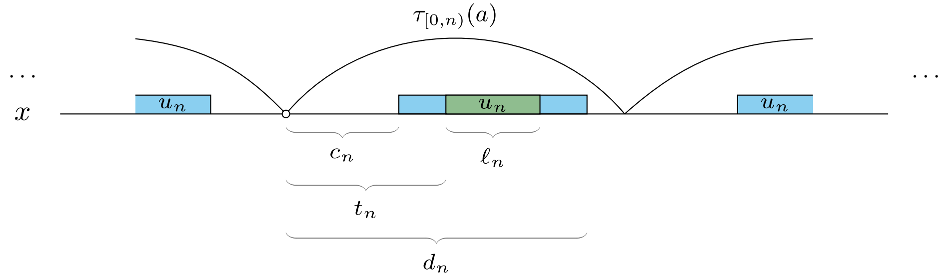

FIGURE 1. A centered $\tau_{[0,n)}$-representation of a point x in $B_n(a)$, $a \in \mathcal{A}_\mu$. The white point represents the zero coordinate of x. The green part represents the word $u_{n[t_n,t_n+\ell_n)}$.

Choose k_n such that $\mu(S^{k_n}B_{n,\mu} \cap C) = \max_{t_n \le k < t_n+\ell_n} \mu(S^k B_{n,\mu} \cap C)$. Then

$$\frac{\mu(S^{k_n}B_{n,\mu} \cap C)}{\mu(B_{n,\mu})} \ge \frac{\sum_{k=t_n}^{t_n+\ell_n-1} \mu(S^k B_{n,\mu} \cap C)}{\ell_n \mu(B_{n,\mu})} = \frac{\mu(A_{n,\mu} \cap C)}{\mu(A_{n,\mu})} \ge \frac{\mu(A_{n,\mu}) - \chi}{\mu(A_{n,\mu})},$$

so that

$$\mu(S^{k_n}B_{n,\mu} \cap (X_{\boldsymbol{\tau}} \setminus C))/\mu(B_{n,\mu}) \le \chi/\mu(A_{n,\mu}) \le \varepsilon^2/4.$$

On the other hand, by the choice of t_n, ℓ_n and because $\boldsymbol{\tau}$ is everywhere growing, we have $\operatorname{diam}(S^{k_n}B_{n,\mu}) \to 0$ as $n \to +\infty$. Hence, for all large enough n we have

$$\sup_{x,y \in S^{k_n}B_{n,\mu} \cap C} |f(x) - f(y)| < \varepsilon^2/2.$$

Put $w_{k_n} = f(y_n)$ for some point $y_n \in S^{k_n}B_{n,\mu} \cap C$, then

$$\begin{aligned}\int_{S^{k_n}B_{n,\mu}} |f - w_{k_n}|\, d\mu &= \int_{S^{k_n}B_{n,\mu} \cap C} |f - w_{k_n}|\, d\mu + \int_{S^{k_n}B_{n,\mu} \cap (X_{\boldsymbol{\tau}} \setminus C)} |f - w_{k_n}|\, d\mu \\ &\le \frac{\varepsilon^2}{2}\mu(B_{n,\mu}) + 2\mu(S^{k_n}B_{n,\mu} \cap (X_{\boldsymbol{\tau}} \setminus C)) \\ &\le \varepsilon^2 \mu(B_{n,\mu}),\end{aligned}$$

which proves the claim.

Put $w'_n = w_{k_n}\lambda^{-k_n}$. We deduce that

$$\int_{B_{n,\mu}} |f - w'_n|\, d\mu = \int_{S^{k_n}B_{n,\mu}} |f - w_{k_n}|\, d\mu \le \varepsilon^2 \mu(B_{n,\mu}). \tag{41}$$

The Markov inequality and (41) imply

$$\mu(\{x \in B_{n,\mu} : |f(x) - w'_n| \ge \varepsilon\}) \le \varepsilon \mu(B_{n,\mu}). \tag{42}$$

To show that (40) holds, it suffices to prove the following.

Claim 5.1.2. Let $a \in \mathcal{A}_\mu$. Then, for all large enough n, there exists $x \in B_n(a)$ such that

$$|f(x)| = 1, \quad |f(x) - w'_n| < \varepsilon \quad \text{and} \quad |f(S^{h_n(a)}x) - w'_n| < \varepsilon.$$

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Let $a \in \mathcal{A}_\mu$. We begin by observing that

$$\frac{\mu(B_n \setminus B_{n,\mu})}{\mu(B_{n,\mu})} = \frac{\sum_{b\in\mathcal{A}\setminus\mathcal{A}_\mu} \mu(\mathcal{T}_n(b))/|\tau_{[0,n)}(b)|}{\sum_{a\in\mathcal{A}_\mu} \mu(\mathcal{T}_n(a))/|\tau_{[0,n)}(a)|} \leq \left(\frac{K}{c|\mathcal{A}_\mu|}\right) \sum_{b\in\mathcal{A}\setminus\mathcal{A}_\mu} \mu(\mathcal{T}_n(b)).$$

Hence, by (31) for all large enough n we obtain

$$\mu(B_n \setminus B_{n,\mu}) < \varepsilon\mu(B_{n,\mu}). \tag{43}$$

Let n be large enough so that (42) and (43) hold. If the claim is not true for such n, after neglecting a set of measure zero we have

$$\begin{aligned} B_n(a) \subseteq \{x \in B_{n,\mu} : |f(x) - w_n'| \geq \varepsilon\} \cup S^{-h_n(a)}\{x \in B_{n,\mu} : |f(x) - w_n'| \geq \varepsilon\} \\ \cup S^{-h_n(a)}\{x \in B_n \setminus B_{n,\mu} : |f(x) - w_n'| \geq \varepsilon\}, \end{aligned}$$

and then

$$\mu_n(a) < 3\varepsilon\mu(B_{n,\mu}) < \frac{c}{K}\mu(B_{n,\mu}). \tag{44}$$

However, by (44) we obtain the following contradiction

$$\begin{aligned} \mu(B_{n,\mu}) &= \sum_{b\in\mathcal{A}_\mu} \mu_n(b) = \sum_{b\in\mathcal{A}_\mu} \mu_n(b) \frac{|\tau_{[0,n)}(b)||\tau_{[0,n)}(a)|}{|\tau_{[0,n)}(b)||\tau_{[0,n)}(a)|} \\ &\leq \frac{K}{|\tau_{[0,n)}(a)|} \sum_{b\in\mathcal{A}_\mu} \mu(\mathcal{T}_n(b)) \leq \frac{K}{c}\mu_n(a) < \mu(B_{n,\mu}), \end{aligned}$$

where we used $\mu(\mathcal{T}_n(a)) = \mu_n(a)|\tau_{[0,n)}(a)| \geq c$. This finishes the proof. □

Remark 5.2

(1) Proposition 5.1(ii) holds, in particular, if for each $n \geq 0$ and $a \in \mathcal{A}$ there exists a prefix p_n (or suffix s_n) of $\tau_{[0,n)}(a)$ and $\delta > 0$ such that the length $|p_n|$ (or $|s_n|$) is at least $\delta\langle\tau_{[0,n)}\rangle$.

(2) In [**DFM15**, Example 2] the authors describe an $\mathcal{S}$-adic subshift of Toeplitz type and of exact finite rank such that $\exp(2\pi i/6)$ is a measurable and non-continuous eigenvalue for the unique invariant probability measure. The associated height vectors $(h_n)_{n\geq 0}$ satisfy

$$h_n(a) \equiv 1 (\text{mod } 6), \quad a \in \mathcal{A}, \quad n \geq 0.$$

A simple computation shows that Proposition 5.1(ii) does not hold. Therefore, the condition given by (40) is not always necessary.

5.2. *Veech criterion applied to Ferenczi subshifts.* We apply Proposition 5.1 to the study of measurable eigenvalues of minimal Ferenczi subshifts. We first need the following lemma to fulfill item (ii).

LEMMA 5.3. *Let $(X_\mathcal{W}, S)$ be a minimal Ferenczi subshift and $\boldsymbol{\tau}_\mathcal{W} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be the directive sequence given in (17). Then, for all $n \geq 1$, there exists a common prefix p_n of the words $\tau_{[0,n)}(a)$, $a \in \mathcal{A}_n$, satisfying*

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$$|p_n| \geq \frac{\min_{1\leq i\leq \ell} a_i + 1}{3(\max_{1\leq i\leq \ell} a_i + 1)} \langle \tau_{[0,n)} \rangle.$$

Proof. Let $m = \min_{1\leq i\leq \ell} a_i + 1$ and $M = \max_{1\leq i\leq \ell} a_i + 1$. Define $p_1 = 01^{m-1}$ and $p_n = \tau_0(L_{1,n-1})$, $n \geq 2$ as in (38). By definition, p_n is a prefix of $\tau_{[0,n)}(a)$ for each $a \in \mathcal{A}_n$, $n \geq 1$. As each morphism τ_n, $n \geq 1$, is of constant length, we have $|L_{1,n+1}| = |L_{1,n}| + |L_{n+1}||\tau_{[1,n+1)}|$, $n \geq 1$. Observe that $|L_n| = q_{n-1} - 1$ for $n \geq 1$, so that $|L_{1,1}| = q_0 - 1 \geq (q_0 + 1)/3 = |\tau_{[1,2)}|/3$.

Inductively, if $|L_{1,n}| \geq |\tau_{[1,n+1)}|/3$, we obtain

$$\begin{aligned}|L_{1,n}| + |L_{n+1}||\tau_{[1,n+1)}| &\geq \frac{|\tau_{[1,n+1)}|}{3} + (q_n - 1)|\tau_{[1,n+1)}| \geq \frac{|\tau_{[1,n+1)}|}{3}(q_n + 1)\\ &= \frac{|\tau_{[1,n+2)}|}{3},\end{aligned}$$

where we used $q_n - 1 \geq q_n/3$. This shows that $|L_{1,n+1}| \geq |\tau_{[1,n+2)}|/3$, $n \geq 0$.

Finally, we deduce $|p_1| = m \geq m/3M\langle \tau_0 \rangle$ and

$$|p_n| \geq m|L_{1,n-1}| \geq m\frac{|\tau_{[1,n)}|}{3} \geq \frac{m}{3M}\langle \tau_{[0,n)} \rangle, \quad n \geq 2. \qquad \square$$

For a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ with unique invariant probability measure μ, we set $d_{\mathcal{W}_\mu} = |\mathcal{A}_\mu|$, where $\mathcal{A}_\mu$ is defined as in §4.2. A direct application of Lemmas 3.8, 5.3 and Proposition 5.1 allows us to obtain the following.

COROLLARY 5.4. *Let $(X_{\mathcal{W}}, S)$ be a minimal Ferenczi subshift and μ be the unique invariant probability measure.*

(1) *If $d_{\mathcal{W}_\mu} = d_{\mathcal{W}}$ (i.e., if $\tau_{\mathcal{W}}$ is of exact finite rank), then all measurable eigenvalues of $(X_{\mathcal{W}}, S)$ with respect to μ are continuous.*

(2) *If $d_{\mathcal{W}_\mu} \geq 2$, then the system $(X_{\mathcal{W}}, S)$ has no irrational measurable eigenvalues with respect to μ.*

Proof. Let $\lambda = \exp(2\pi i\alpha)$ be a measurable eigenvalue of $(X_{\mathcal{W}}, S)$ with respect to μ and $\boldsymbol{\tau}_{\mathcal{W}}$ be the directive sequence given by (17). We see that, by Lemmas 5.3 and 3.8, all hypotheses of Proposition 5.1 are verified.

(1) If $d_{\mathcal{W}_\mu} = d_{\mathcal{W}}$, Proposition 5.1 implies that

$$|||\alpha h_n||| \to 0 \tag{45}$$

as $n \to +\infty$. Then, from Lemma 3.8, there exist two distinct letters a, b in $\mathcal{A}_{\mathcal{W}}$ such that

$$|||\alpha(h_n(a) - h_n(b))||| = |||\alpha(a - b)||| \to 0$$

as $n \to +\infty$. We deduce that α must be rational. Moreover, if $\alpha = p/q$ is rational, then from (45) the integer q must divide all coordinates of h_n for all large enough n. We conclude that λ is a continuous eigenvalue from §4.8.

(2) If $d_{\mathcal{W}_\mu} \geq 2$, there exist two distinct elements a, b in $\mathcal{A}_{\mathcal{W}}$ which satisfy

$$|||\alpha(a - b)||| = 0.$$

In particular, α must be rational. $\square$

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Thus, we have showed that measurable and continuous eigenvalues coincide in the case where $d_{\mathcal{W}_\mu} = d_{\mathcal{W}}$. However, when $d_{\mathcal{W}_\mu} \neq d_{\mathcal{W}}$, we can obtain different behaviors that we comment in the following section.

5.3. *Various examples exhibiting different spectral behaviors.* In this section, we make precise the situation where $d_{\mathcal{W}_\mu} \neq d_{\mathcal{W}}$. In this case, we give explicit examples of minimal Ferenczi subshifts $(X_{\mathcal{W}}, S)$ having a prescribed topological rank $d_{\mathcal{W}}$, a prescribed quantity $d_{\mathcal{W}_\mu}$ with $d_{\mathcal{W}_\mu} \neq d_{\mathcal{W}}$, with no non-trivial continuous eigenvalue, but with any rational measurable eigenvalue $\lambda = \exp(2\pi i/p)$.

However, when $d_{\mathcal{W}_\mu} = 1$, we were not able to show that there are no irrational measurable eigenvalues. We leave this as an open question.

5.3.1. *A realization result on measurable eigenvalues with rank constraints.*

PROPOSITION 5.5. *Let p be a prime number and d, d' be such that $1 \leq d' < d$. Then, there exists a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ with unique invariant probability measure μ such that* $\operatorname{rank}(X_{\mathcal{W}}, S) = d$, *$d_{\mathcal{W}_\mu} = d'$, the system $(X_{\mathcal{W}}, S)$ is topologically weakly mixing and $\lambda =$* $\exp(2\pi i/p)$ *is a measurable eigenvalue of $(X_{\mathcal{W}}, S)$.*

Proof. Consider any set of non-negative numbers $\mathcal{A} = \{a_i : 1 \leq i \leq d\}$ such that p divides $a_i + 1$, $1 \leq i \leq d'$ and $a_d - a_{d'} = 1$. Put $\mathcal{A}_0 = \{0, 1\}$ and $\mathcal{A}_n = \mathcal{A}$ if $n \geq 1$. Let $v = a_{j^*}$ for some $d' < j^* \leq d$ and define the words

$$U = a_1 a_2 \ldots a_{d'}$$
$$W = a_{d'+1} \ldots a_{j^*-1} a_{j^*+1} \ldots a_d$$

in $\mathcal{A}^*$. Consider any increasing function $g : \mathbb{N} \to \mathbb{Z}_{>0}$ such that $\sum_{n=0}^{\infty}(1/g(n)) < +\infty$. Let $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be the directive sequence given by

$$\tau_n(a) = U^{pg(n)} W^p v^{p-1} a v, \quad a \in \mathcal{A}, \quad n \geq 1. \tag{46}$$

Theorem 3.7 implies that $\boldsymbol{\tau}_{\mathcal{W}}$ defines a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$. Moreover, Corollary 4.9 implies that $\operatorname{rank}(X_{\mathcal{W}}, S) = d$. Let $(\mathcal{T}_n)_{n\geq 0}$ be the nested sequence of CKR partitions of $X_{\mathcal{W}}$ given by (8) and μ be the unique invariant probability measure of $(X_{\mathcal{W}}, S)$.

Observe that the vector $(f_n(a))_{a\in\mathcal{A}}$ associated with the morphism τ_n, as defined by (20), is given by

$$f_n(a_i) = \begin{cases} pg(n) & \text{if } 1 \leq i \leq d', \\ p & \text{if } d' < i \leq d. \end{cases}$$

From (26) the composition matrix of the morphism τ_n is given by

$$M_{\tau_n} = I + f_n \cdot \boldsymbol{u},$$

where I is the identity matrix indexed by $\mathcal{A}$ and $\boldsymbol{u}$ is the row vector of ones in $\mathbb{R}^{\mathcal{A}}$.

CLAIM 5.5.1. *The system $(X_{\mathcal{W}}, S)$ is topologically weakly mixing and $d_{\mathcal{W}_\mu} = d'$.*

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Proof. As $a_d - a_{d'} = 1$, we have that the system $(X_{\mathcal{W}}, S)$ is topologically weakly mixing.

It remains to show that $\mathcal{A}_\mu = \{a_i : 1 \le i \le d'\}$. From (24), there exists a constant K such that for all $a, b \in \mathcal{A}$ and $n \ge 0$ we have

$$h_{n+1}(b) = \sum_{c \in \mathcal{A}} h_n(c) M_{\tau_n}(c, b) \le K h_n(a) \sum_{c \in \mathcal{A}} M_{\tau_n}(c, b) \le K|\mathcal{A}|(pg(n) + 1) h_n(a),$$

and

$$h_{n+1}(b) = \sum_{c \in \mathcal{A}} h_n(c) M_{\tau_n}(c, b) \ge K^{-1} h_n(a) \sum_{c \in \mathcal{A}} M_{\tau_n}(c, b) \ge K^{-1}|\mathcal{A}| pg(n) h_n(a).$$

Let $i \in \{1, \ldots, d'\}$, $j \in \{d' + 1, \ldots, d\}$, $b \in \mathcal{A}$ and $n \ge 0$. From the above, we obtain

$$\begin{aligned}\mu(\mathcal{T}_n(a_i)) &= h_n(a_i)\mu_n(a_i) = h_n(a_i) \sum_{b \in \mathcal{A}} M_{\tau_n}(a_i, b)\mu_{n+1}(b)\\ &\ge pg(n) h_n(a_i) \sum_{b \in \mathcal{A}} \frac{\mu(\mathcal{T}_{n+1}(b))}{h_{n+1}(b)} \ge pg(n) \min_{b \in \mathcal{A}} \frac{h_n(a_i)}{h_{n+1}(b)}\\ &\ge \frac{pg(n)}{K|\mathcal{A}|(pg(n) + 1)} \ge \frac{1}{2K|\mathcal{A}|}\end{aligned}$$

and

$$\begin{aligned}\mu(\mathcal{T}_n(a_j)) &= h_n(a_j)\mu_n(a_j) = h_n(a_j) \sum_{b \in \mathcal{A}} M_{\tau_n}(a_j, b)\mu_{n+1}(b)\\ &\le (p + 1) h_n(a_j) \sum_{b \in \mathcal{A}} \frac{\mu(\mathcal{T}_{n+1}(b))}{h_{n+1}(b)} \le (p + 1) \max_{b \in \mathcal{A}} \frac{h_n(a_j)}{h_{n+1}(b)}\\ &\le \frac{K(p + 1)}{|\mathcal{A}| pg(n)} \to 0\end{aligned}$$

as $n \to +\infty$. This shows that $\mathcal{A}_\mu = \{a_i : 1 \le i \le d'\}$ and proves the claim. □

Let $\lambda = \exp(2\pi i/p)$. We define a measurable eigenfunction f as a limit of the sequence $(f_n)_{n \ge 0}$ defined by

$$f_n(x) = \exp(2\pi i j/p) \quad \text{if } x \in S^j B_n(a), \quad 0 \le j < h_n(a), \quad a \in \mathcal{A}.$$

Define

$$A_n = \{x \in X_{\mathcal{W}} : f_{n+1}(x) \ne f_n(x)\}, \quad n \ge 0$$

and

$$t_n = |\tau_{[0,n)}(U^{pg(n+1)})|, \quad n \ge 0.$$

Observe, from (26), that p divides $h_n(a)$ for all $a \in \mathcal{A}_\mu$. Moreover, $f_{n+1}(x) = f_n(x)$ whenever x belongs to

$$\bigcup_{1 \le i \le d'} \bigcup_{0 \le j < t_n} S^j B_{n+1}(a_i).$$

Consequently, we deduce

$$A_n \subseteq \bigg(\bigcup_{1\le i\le d'} \bigcup_{t_n\le j<h_{n+1}(a_i)} S^j B_{n+1}(a_i) \bigg) \cup \bigg(\bigcup_{d'<j\le d} \mathcal{T}_{n+1}(a_j) \bigg).$$

Let $1 \le i \le d'$. From (24), we have

$$h_{n+1}(a_i) - t_n = h_n(a_i) + p \sum_{d'<j\le d} h_n(a_j) \le pK|\mathcal{A}| h_n(a_i).$$

From the previous computations, we obtain

$$\begin{aligned}
\mu(A_n) &\le pK|\mathcal{A}| \sum_{1\le i\le d'} \mu_{n+1}(a_i) h_n(a_i) + \sum_{d'<j\le d} \mu(\mathcal{T}_{n+1}(a_j)) \\
&\le pK|\mathcal{A}| \sum_{1\le i\le d'} \frac{h_n(a_i)}{h_{n+1}(a_i)} \mu(\mathcal{T}_{n+1}(a_i)) + \frac{K(p+1)}{pg(n+1)} \\
&\le \frac{K^2}{g(n)} + \frac{K(p+1)}{pg(n+1)} \le \frac{2K^2(p+1)}{g(n)}
\end{aligned}$$

and consequently $\sum \mu(A_n)$ converges. The Borel–Cantelli lemma implies that $\mu(\limsup_{n\to+\infty} A_n) = 0$. Hence, the sequence $(f_n)_{n\ge 0}$ converges μ-almost everywhere to some function f.

Moreover, if x is not in $\bigcup_{a\in\mathcal{A}} S^{h_n(a)-1} B_n(a)$, then $f_n(Sx) = \lambda f_n(x)$. As $\mu(\bigcup_{a\in\mathcal{A}} S^{h_n(a)-1} B_n(a)) \to 0$ as $n \to +\infty$, we conclude that f is a measurable eigenfunction with eigenvalue λ of $(X_{\mathcal{W}}, S)$ with respect to μ. □

5.3.2. *An example with $d_{\mathcal{W}_\mu} = 1$ with no non-continuous rational eigenvalue.* We now present an example in the situation where $d_{\mathcal{W}_\mu} = 1$ and every measurable rational eigenvalue is continuous. For this purpose, we use the following useful result of [**DFM19**, §4]. Note that we have adapted this result to fit the context of $\mathcal{S}$-adic subshifts.

LEMMA 5.6. [**DFM19**, Corollary 16] *Let $\boldsymbol{\tau} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\ge 0}$ be a proper, primitive and recognizable directive sequence and let μ be an ergodic invariant probability measure of $(X_{\boldsymbol{\tau}}, S)$. Assume that $\boldsymbol{\tau}$ is clean with respect to μ and let $\mathcal{A}_\mu$ be the set of letters such that (31) holds.*

Let λ be a complex number of modulus 1. If for all $a, b \in \mathcal{A}_\mu$,

$$\frac{\left| \sum_{w\in W_{m,n}(a,b)} \lambda^{\langle \ell(w), h_m\rangle} \right|}{|\tau_{[m,n)}(b)|_a} \to 1 \quad \text{as} \quad m \to +\infty \quad \text{uniformly for} \quad n > m, \tag{47}$$

then λ is an eigenvalue of $(X_{\boldsymbol{\tau}}, S)$ with respect to μ, where:

- *$h_m = (|\tau_{[0,m)}(a)| : a \in \mathcal{A}_m)$ for $m > 0$;*
- *the set $W_{m,n}(a, b)$ is defined by*

$$\{\tau_{[m,n)}(b)_{[i,|\tau_{[m,n)}(b)|)} : i \text{ occurrence of } a \text{ in } \tau_{[m,n)}(b)\};$$

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- *for a word $w \in \mathcal{A}_m^*$,*

$$\ell(w) = (|w|_a : a \in \mathcal{A}_m).$$

The converse is also true, up to a contraction of the directive sequence $\boldsymbol{\tau}$.

Let us be precise here that a sequence $(a_{m,n})_{m,n\geq 0}$ converges to ℓ as $m \to +\infty$ uniformly for $n > m$ if for every $\varepsilon > 0$ there exists $m_0 \geq 0$ such that for all $n > m \geq m_0$ we have

$$|a_{m,n} - \ell| < \varepsilon.$$

Let a, b be two positive integers with $a > b$ and let $\boldsymbol{\tau}_{\mathcal{W}} = (\tau_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ be the directive sequence given by

$$\tau_n(a) = a^n b a^n \quad \text{and} \quad \tau_n(b) = a^n b a^{n-2} b a, \quad n \geq 1. \tag{48}$$

Theorem 3.7 implies that $\boldsymbol{\tau}_{\mathcal{W}}$ defines a minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$.

PROPOSITION 5.7. *The minimal Ferenczi subshift $(X_{\mathcal{W}}, S)$ defined by (48) is such that $d_{\mathcal{W}_\mu} = 1$ and every rational measurable eigenvalue is continuous, for the unique ergodic measure μ of $(X_{\mathcal{W}}, S)$.*

Proof. We say that a word $w \in \{a, b\}^*$ consists of *a-blocks* if we can write

$$w = B_0(w) b B_1(w) b \ldots b B_{b(w)-1}(w), \tag{49}$$

where $b(w) \geq 2$ and $B_j(w)$ is a non-trivial power of a for $0 \leq j < b(w)$.

Define $w(m, n) = \tau_{[m,n)}(a)$ and $w'(m, n) = \tau_{[m,n)}(b)$ for $n > m \geq 0$. It is easy to check that $w(m, n)$ and $w'(m, n)$ consist of a-blocks. We use the notation introduced in (49) for these words.

We have $\mathcal{A}_\mu = \{a\}$. Indeed, the composition matrix M_{τ_n} of the morphism τ_n is

$$M_{\tau_n} = \begin{pmatrix} 2n & 2n-1 \\ 1 & 2 \end{pmatrix}.$$

Hence, from (6) and (7), we obtain

$$\begin{aligned} \mu(\mathcal{T}_n(b)) &= h_n(b)\mu_n(b) = h_n(b)(\mu_{n+1}(a) + 2\mu_{n+1}(b)) \\ &= h_n(b)\left(\frac{\mu(\mathcal{T}_{n+1}(a))}{h_{n+1}(a)} + 2\frac{\mu(\mathcal{T}_{n+1}(b))}{h_{n+1}(b)}\right) \leq \frac{2}{2n+1}, \end{aligned}$$

where we used $h_{n+1}(a) = 2nh_n(a) + h_n(b) \geq (2n+1)h_n(b)$ by (23) and, analogously, $h_{n+1}(b) \geq (2n+1)h_n(b)$. Thus, $d_{\mathcal{W}_\mu} = 1$ and, from [**BKMS13**, Proposition 5.1], we deduce

$$|w(m, n)|_a / |w(m, n)| \to 1 \quad \text{as} \quad m \to +\infty \quad \text{uniformly for} \quad n > m. \tag{50}$$

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Suppose that p is a prime number and that $\lambda = \exp(2\pi i/p)$ is a rational eigenvalue of $(X_{\mathcal{W}}, S)$ with respect to μ that is not continuous. From (47) and (50), we should have

$$\frac{\left|\sum_{w\in W_{m,m+2}(a,a)} \lambda^{\langle \ell(w),h_m\rangle}\right|}{|w(m,m+2)|} \to 1 \quad \text{as} \quad m \to +\infty. \tag{51}$$

Let us show this is not the case. Let $m \geq 0$. From the definition of the set $W_{m,m+2}(a,a)$ and (49), we have

$$\sum_{w\in W_{m,m+2}(a,a)} \lambda^{\langle \ell(w),h_m\rangle} = \sum_{j=0}^{b(w(m,m+2))-1} \sum_{i=0}^{|B_j(w(m,m+2))|-1} \lambda^{\langle \ell(u_{j,i}),h_m\rangle},$$

where $u_{j,i} = w(m,m+2)_{[(\sum_{k=0}^{j-1}|B_k(w(m,m+2))|)+j+i,|w(m,m+2))|}$.

On the other hand, from (23) we have $h_m(b) = h_m(a) + (b-a)$, therefore

$$\begin{aligned}\langle \ell(u_{j,i}), h_m\rangle &= h_m(a)\left(\sum_{k=0}^{j-1}|B_k(w(m,m+2))| + i\right) + h_m(b)j \\ &= h_m(a)\left(\sum_{k=0}^{j-1}|B_k(w(m,m+2))| + i + j\right) + (b-a)j.\end{aligned}$$

As $\mathcal{A}_\mu = \{a\}$, the Veech criterion (40) implies that p divides $h_m(a)$ for all large enough m. Moreover, as λ is a non-continuous eigenvalue, p does not divide $h_m(b)$ for any $m \geq 0$ and, hence, p does not divide $b-a$. Denote by $(b-a)^{-1}$ the inverse of $b-a (\text{mod } p)$. For each $0 \leq \ell < p$, let r_ℓ be such that $0 \leq r_\ell < p$ and $r_\ell \equiv \ell \cdot (b-a)^{-1} (\text{mod } p)$.

For all large enough m we have

$$\begin{aligned}\sum_{w\in W_{m,m+2}(a,a)} \lambda^{\langle \ell(w),h_m\rangle} &= \sum_{j=0}^{b(w(m,m+2))-1} |B_j(w(m,m+2))| \exp(2\pi i(b-a)j/p) \\ &= \sum_{\ell=0}^{p-1} a_\ell \exp(2\pi i\ell/p),\end{aligned} \tag{52}$$

with

$$a_\ell = \sum_{j=0}^{\lfloor b(w(m,m+2))/p\rfloor-1} |B(w(m,m+2))_{jp+r_\ell}|. \tag{53}$$

From the shape of the images of the morphism τ_m (48) and the fact that $\tau_{[m,m+2)}(a)$ belongs to the free monoid $\{\tau_m(a), \tau_m(b)\}^*$, we have

$$|B_j(w(m,m+2)| - |B_{j'}v(w(m,m+2))| \leq m+2, \quad 0 \leq j, j' < b(w(m,m+2)). \tag{54}$$

This, together with (53) implies that

$$a_\ell - \min_{0\leq \ell' < p} a_{\ell'} \leq (m+2)\frac{b(w(m,m+2))}{p}, \quad 0 \leq \ell < p. \tag{55}$$

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From (52) and (55), we have

$$\frac{\left|\sum_{w\in W_{m,m+2}(a,a)} \lambda^{\langle \ell(w),h_m\rangle}\right|}{|w(m,m+2)|} \leq (m+2)\frac{b(w(m,m+2))}{|w(m,m+2)|}, \tag{56}$$

where we used the fact that $\sum_{\ell=0}^{p-1}\exp(2\pi i\ell/p)=0$.

By computing $\tau_m \circ \tau_{m+1}(a)$, one can easily check that the number of a-blocks of $\tau_{[m,m+2)}(a)$ is $b(w(m,m+2)) = 2m+5$. Then we deduce

$$(m+2)\frac{b(w(m,m+2))}{|w(m,m+2)|} = \frac{(m+2)(2m+5)}{(2m+1)(2m+3)} \to \frac{1}{2} \quad \text{as} \quad m\to+\infty.$$

This contradicts (51) and finishes the proof. □

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