

## A Geometrical Proof of certain Trigonometrical Formulæ.

By J. W. BUTTERS, M.A., B.Sc.

§ 1. Direct geometrical proofs of the addition theorem, and others allied to it, are often valid for only a limited range of values, and the constructions used are applicable to only one set of formulæ. In the following paper a method is given which is valid for angles of all values and which is applicable to all the usual formulæ involving sines, cosines, and their simple products. One of each set is proved: the others may be obtained either directly or by substitution in the usual manner.

§ 2. The following construction for  $2\alpha$ , although not *obviously* valid for all values of  $\alpha$ , is inserted as having led to the general construction. It should be compared with that in § 4.

FIGURE 19.

$$\begin{aligned}
 \sin 2\alpha &= \frac{MA}{r} \\
 &= \frac{MA}{XA} \cdot \frac{XA}{r} \\
 &= \frac{MA}{XA} \cdot \frac{2BA}{r} \\
 &= \cos \alpha \cdot 2\sin \alpha \\
 &= 2\sin \alpha \cdot \cos \alpha \\
 \cos 2\alpha &= \frac{OM}{r} = \frac{OX}{r} - \frac{MX}{r} \\
 &= 1 - \frac{MX}{XA} \cdot \frac{XA}{r} \\
 &= 1 - \sin \alpha \cdot 2\sin \alpha \\
 &= 1 - 2\sin^2 \alpha
 \end{aligned}$$

FIGURE 20.

§ 3. If  $AB$  be a chord subtending an angle  $\theta$  at the centre of a circle of radius  $r$ , and  $\phi$  be the angle between  $OX$  and the perpendicular  $OC$  to  $AB$ , then, since  $OY$ ,  $AB$  are respectively perpendicular to  $OX$ ,  $OC$ ,  $\phi$  is the angle between  $OY$  and  $AB$ .

Hence the projection of AB on OY is  $AB\cos\phi$

$$= 2CB\cos\phi$$

$$= 2r\sin\frac{1}{2}\theta \cdot \cos\phi \quad - \quad - \quad (1).$$

Also angle between OX and AB is  $\left(\frac{\pi}{2} + \phi\right)$

$$\therefore \text{projection of AB on OX is } 2r\sin\frac{1}{2}\theta\cos\left(\frac{\pi}{2} + \phi\right)$$

$$= -2r\sin\frac{1}{2}\theta\sin\phi \quad - \quad - \quad (2).$$

FIGURE 21.

§ 4. Since the projection of OA = the projection of OX plus the projection of XA, if we project these on OX we get from (2)

$$r\cos 2\alpha = r - 2r\sin\alpha \cdot \sin\alpha$$

$$\text{or } \cos 2\alpha = 1 - 2\sin^2\alpha.$$

For this operation we shall in future write

$$\text{Project (OA = OX + XA) on OX}$$

Thus to get  $\sin 2\alpha$ : project (OA = OX + XA) on OY.

FIGURE 22.

§ 5. Project (OB = OA + AB) on OY

$$r\sin 3\alpha = r\sin\alpha + 2r\sin\alpha\cos 2\alpha \quad \text{from (1)}$$

$$\text{or } \sin 3\alpha = \sin\alpha + 2\sin\alpha(1 - 2\sin^2\alpha)$$

$$= 3\sin\alpha - 4\sin^3\alpha.$$

Similarly  $\cos 3\alpha$  by projecting on OX.

The sines and cosines of higher multiples may be obtained in the same manner—the reductions are usually somewhat shorter than by the ordinary methods.

FIGURE 23.

§ 6. Projecting (AB = AO + OB)

or say (OB - OA = AB)

on OY, and dividing by  $r$

$$\sin\alpha - \sin\beta = 2\sin\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\alpha + \beta).$$

For the other three related formulæ

$$\text{put } -\beta \text{ for } \beta; \text{ project on OX; put } \pi + \beta \text{ for } \beta.$$

FIGURE 24.

§ 7. Project  $(AB = AX + XB)$  on  $OX$   
 $- 2rsin(\alpha - \beta)sin(\alpha + \beta) = 2rsin^2\beta - 2rsin^2\alpha$   
 $\therefore sin(\alpha - \beta)sin(\alpha + \beta) = sin^2\alpha - sin^2\beta.$

For related formulæ

put  $\pi/2 - \alpha$  for  $\alpha$  ; project on  $OY$  ; put  $-\beta$  for  $\beta$ .

FIGURE 25.

§ 8. Project  $(AC = AB + BC)$  on  $OY$   
 $2rsin(\alpha + \beta)cos(\alpha + \beta + \chi) = 2rsin\alpha cos(\alpha + \chi) + 2rsin\beta cos(2\alpha + \beta + \chi) - (A).$   
 Put  $\alpha + \beta + \chi = 0$   
 Then  $sin(\alpha + \beta) = sin\alpha cos(-\beta) + sin\beta cos\alpha$   
 $= sin\alpha cos\beta + sin\beta cos\alpha.$

Related formulæ are obtained in the usual manner.

From the general formula (A) we may get an endless variety of results: Thus putting almost at random

$A + B$  for  $\alpha + \beta$ ,  $C - D$  for  $\alpha + \beta + \chi$ , and  $A + D$  for  $\alpha$   
 we have  $\alpha = A + D$ ,  $\beta = B - D$ ,  $\chi = C - A - B - D$   
 whence (A) becomes

$$sin(A + B)cos(C - D) = sin(A + D)cos(C - B) + sin(B - D)cos(A + C).$$

The general formula (A) must evidently be expressible in a form involving the three angles symmetrically: to get this, put

$$\begin{aligned} \alpha &= Y - Z \\ \beta &= Z - X \\ \chi &= X - Y + Z \end{aligned}$$

and we have

$$sin(Y - Z)cosX + sin(Z - X)cosY + sin(X - Y)cosZ = 0$$

From this as in Fig. 26 we get the well-known theorems for a pencil of rays

$$sinbc cosad + sincacosbd + sinabcoscd = 0$$

$$\text{and } sinbcsinad + sincasinbd + sinabsincd = 0$$

(since the cosines become sines when the projection is on  $OX$ ).

We may also obtain Ptolemy's Theorem and hence Euler's relation  $BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$  for points on a range.

FIGURE 27.

## § 9. Project

$$(A_0A_1 + A_1A_2 + \dots + A_{n-1}A_n = A_0A_n) \text{ on } OY.$$

We have

$$2r \sin \frac{\beta}{2} \{ \cos \alpha + \cos(\alpha + \beta) + \dots + \cos(\alpha + n - 1)\beta \} = 2r \sin \frac{n\beta}{2} \cos \left( \alpha + \frac{n-1}{2}\beta \right)$$

$$\therefore \cos \alpha + \cos(\alpha + \beta) + \dots + \cos(\alpha + n - 1)\beta = \frac{\sin \frac{n\beta}{2} \cos \left( \alpha + \frac{n-1}{2}\beta \right)}{\sin \frac{\beta}{2}}.$$

By projection on OX the cosines become sines.

From these or by projection of a regular (crossed)  $n$ -gon we have

$$\cos \frac{m\pi}{n} + \cos \frac{3m\pi}{n} + \dots + \cos \frac{(2n-1)m\pi}{n} = 0$$

where  $m$  and  $n$  are integers; and similarly for the sines.

