A Proof of the Addition Theorem in Trigonometry.

I. Prove $\sin\theta = -\sin(-\theta)$, $\cos\theta = \cos(-\theta)$.

Let $X'OX$, $Y'OY$ be rectangular axes, and let OP, OQ, starting from the position OX, describe angles $\theta, -\theta$. Then OP, OQ are in every case symmetrically placed with respect to OX. (Illustrate by taking positive and negative values of θ). Hence $x_P = x_Q$, $y_P = -y_Q$, and the results follow from the definitions of sine and cosine.

Cor. $\cos\theta = \cos(2n\pi \pm \theta)$, where n is any integer.

II. Prove $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$, $\sin\theta = -\cos\left(\theta + \frac{\pi}{2}\right)$.

If OP, OQ are radii of a circle such that $X\widehat{O}P = \theta$ and $X\widehat{O}Q = \frac{\pi}{2} + \theta$, Q is always a positive quadrant along the circumference from P. (Illustrate by taking positive and negative numerical values of θ). If M, N are the projections of PQ on OX, the triangles OMP, ONQ are congruent, hence $x_P = y_Q$ and $y_P = x_Q$ *numerically.*

If x_P is positive, P is on the semi-circle $Y'XY$.

$\therefore$ Q ,, ,, ,, XYX'.

$\therefore$ y_Q is positive.

Similarly if x_P is negative, y_Q is negative.

$\therefore$ $x_P = y_Q$, and $\cos\theta = \sin\left(\frac{\pi}{2} + \theta\right)$.

Again, if y_P is positive, P is on the semi-circle XYX'.

$\therefore$ Q ,, ,, ,, $YX'Y'$.

$\therefore$ x_Q is negative.

Similarly if y_P is negative, x_Q is positive.

$$\therefore \quad y_P = -x_Q, \text{ and } \sin\theta = -\cos\left(\frac{\pi}{2} + \theta\right).$$

III. Prove that in any triangle ABC, $a^2 = b^2 + c^2 - 2bc\cos A$.

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IV. Prove that if P, Q are the points (x_1, y_1), (x_2, y_2), then $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

(Illustrate by numerical examples the fact that the projections of PQ on the axes are in all cases $x_2 - x_1$, $y_2 - y_1$).

V. Prove that $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$, where α, β are any positive angles less than 360°.

If OP, OQ, each of length r, make positive angles α, β with OX, the angle POQ of the triangle POQ is equal to $\alpha - \beta$ or $\beta - \alpha$ or $360° - (\alpha - \beta)$ or $360° - (\beta - \alpha)$. (Illustrate by diagrams.)

In every case, by I, $\cos POQ = \cos(\alpha - \beta)$.

$$\therefore \quad PQ^2 = OP^2 + OQ^2 - 2OP \,.\, OQ \cos(\alpha - \beta)$$

$$\therefore \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = 2r^2 - 2r^2 \cos(\alpha - \beta).$$

But $x_1^2 + y_1^2 = r^2 = x_2^2 + y_2^2$.

$$\therefore \quad \cos(\alpha - \beta) = \frac{x_1 x_2}{r^2} + \frac{y_1 y_2}{r^2}$$

$$= \cos\alpha \cos\beta + \sin\alpha \sin\beta.$$

VI. Extend the above result to any angles.

If A, B are any angles whatever, coterminal with α, β, then $A = 2m\pi + \alpha$, $B = 2n\pi + \beta$, where m and n are integers, positive or negative.

$\therefore \quad A - B = 2k\pi + (\alpha - \beta)$, where k is an integer.

$$\therefore \quad \cos(A - B) = \cos(\alpha - \beta)$$

$$= \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$= \cos A \cos B + \sin A \sin B \quad \text{(by I, Cor.)}.$$

VII. By means of I and II extend the result to $\cos(A + B)$, $\sin(A - B)$, $\sin(A + B)$ in the usual way.

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