The Theory of Contours,
and its Applications in Physical Science.

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PART I.

1. The word *contour* is largely used in ordinary language, but its meaning, when so used, is in general very different from its meaning as a scientific term. We speak of the *contour* of a hill, a cloud, a country, and so on; meaning usually a *profile* or an *outline*,—sometimes a particular outline only. Yet, even in this popular use of the word, we have an indication of its more exact significance. Thus, we see that the visible horizon, if we consider it to be a contour line, is the curve in which the earth's surface is met by its tangent-cone the vertex of which is at the observer's eye. The tangent-surface has a constant characteristic; and it is this possession of a distinctive property by all surfaces which give rise to contour lines, which furnishes the reason for the peculiar applicability of the method of contours to physical problems.

2. The word is used in a more special sense by ordnance surveyors. In their terminology a contour line is a line drawn on a map of a country through points which are at a constant height above mean sea-level. Here again we have a quantity which is constant all along the line, and also all over the surface which gives rise to the line by its intersection with the earth. This is a *level* surface, or surface of *equilibrium.* It is approximately true that the special characteristic is constant *height* above sea-level, so long as the area considered is so small that gravity has sensibly the same value all over it. Because of the variation of gravity from point to point of the earth's surface, and because of the earth's rotation, &c., it is not true that the mutual distance of two level surfaces is of uniform value. What is really constant is the kinetic energy acquired by a body in falling from the upper level to the lower by any path.

So, in extension of the third order, a contour may be the intersection of a surface by a surface over which some quantity is constant.

3. This idea may be extended to objects of other dimensions than the second, and extension of other order than the third. We may state generally that the *contour of an object of* n *dimensions, existing*

in extension of the $(n+1)^{th}$ *order, is the intersection of it with an object of* n *dimensions at every point of which some quantity is constant.* It is, therefore, of $(n-1)$ dimensions. But, since in extension of the $(n+1)^{th}$ order, we have objects of all positive dimensions up the $(n+1)^{th}$ inclusive; in such extension, we may have contours of all positive dimensions up to the nth inclusive. The contours of a curve are points; of a surface, curves; of a solid, surfaces; of a four-dimensional object, solids; and so on. Hence, in ordinary extension we may have point, curve, and surface, contours.

4. Since space is of three dimensions, a point in space may have three degrees of translational freedom,—it may be free to move in any three mutually perpendicular directions successively or simultaneously. We are said to introduce a degree of constraint when we take away one of these degrees of freedom. If we prevent the exercise of one kind of freedom, unless a second is simultaneously exercised, we introduce a degree of constraint in the most general way possible. The moving point is then forced to remain on some definite surface, and the nature of the constraint is best shown by writing down the equation to the surface, the most general being of the form $f(x, y, z) = 0$. If another such constraint be imposed, the moving point must remain in positions common to both the corresponding surfaces. Hence, it must move in a definite curve. Three such equations determine the position of a point. They may degenerate into $f_1(x) = 0$, $f_1(y) = 0$, $f_1(z) = 0$, where $f_1(x)$ is some function of x of the first degree, &c.; that is to say, the position of the point is determined by the intersection of three planes parallel to the co-ordinate planes.

5. A *curve* being of one dimension, its contours are *points*,—the points of intersection of it with curves along each of which some quantity is constant, although it differs in magnitude from one curve to another. These points may be projected upon any line in the plane (x, y).

We shall consider first a plane curve, and let its plane be taken as that of (x, y). We have the equations $f_n(x, y) = 0$ and $f_1(z) = 0$, where the suffix denotes the degree of the equation. Instead of the second equation, we may put $z = 0$, as this simply makes the plane of the curve that of (x, y). The equations to the curves, along which some quantity, say c, is constant, may be written in the general form $\phi_n(x, y, c) = 0$. In the different curves of the system, c has different values. As a particular example, the curves might be

circles of different radii. Again, the equation might be $\phi_1(y, c) = 0$.
This represents a series of lines parallel to the axis of x. It
furnishes the simplest case for consideration, and, at the same time,
the most useful.

From the reference already made (§ 2) to contour maps, it is
evident that the mutual variation of three quantities can be repre-
sented upon a plane surface by means of lines. So, in the case
considered, the mutual variation of two quantities can be represented
upon a right line. The curve in fig. 20 is intersected by lines
parallel to the axis of x, and the points of intersection are projected
upon the axis, and designated by numbers giving the various values
of y corresponding to the values of x. If the curve be continuous
a *maximum* or *minimum* value of y exists between two equal values.
It is a maximum if, as x increases from its least value corresponding
to the given value of y, y first increases and then diminishes. It is a
minimum if y first diminishes and then increases. The *steepness of slope*
is shown by the closeness of the contours for equal increments of y, and
its *direction* is shown by the order in which the values of y occur as
regards numerical magnitude when x increases. Similarly, the exist-
ence of a *maximum-minimum* value is indicated, and so on.

The contours of a closed oval and of a figure-of-eight curve, are
shown in figs. 21 and 23.

6. Hitherto we have considered only plane curves. In the case
of tortuous curves, it will be most convenient to obtain the contours
by cutting the curve by surfaces over which some quantity is con-
stant. As a particular case, these surfaces may be planes perpendi-
cular to the z-axis, the equations to which may be written in the
general form $f_1(z, c) = 0$. Obviously, the position of a moving point
in space is representable by a tortuous curve. Its position at any
time can be got from the curve, if the value of the time in terms any
one of the co-ordinates, say z is known, for we should then only
have to cut the curve by the plane $_1f_n(z, t) = 0$, where t represents the
time, and $_1f_n$ is a functional symbol showing that the equation is of
the first degree in z but may be of any degree in t. This condition
is rendered necessary by the fact that the point must be in one
definite position at a given time, but may occupy the same position
at different times. If we consider a number of such curves traced
out in space by moving material points, we can, by cutting the curves
by planes corresponding to a definite time and projecting the points
so got upon any co-ordinate plane, obtain the *diagram of configuration*

of the material system at that time. It is evident that, in general, the plane corresponding to a definite time will be different for each curve. The disadvantage so entailed may be got rid of by using trilinear co-ordinates to indicate the position of the point when the time is given. If the curve be cut by any plane, the distances of the point of intersection from three intersecting lines in that plane give the x, y, and z co-ordinates at an instant. This instant may be determined by the distance of the plane from a given fixed plane parallel to it. The curves, which are now taken to represent the positions of the points, are not the actual curves traced out by the points in their motion through space. But the diagram of configuration obtained from them has the advantage of showing at once the relative values of all the co-ordinates of all the points, whereas, in the Cartesian system this could not be done without projecting on all three co-ordinate planes. The triangles of reference will be similar whatever be the plane by which the curves are cut, but they will not necessarily be of the same magnitude. The similarity is necessary in order to give the proper *ratios* of the co-ordinates, the different magnitude is necessary in order to give the proper *values* of the co-ordinates. Fig. 22 shows the contours of curves a, b, by planes corresponding to times t_1, t_2.

By the aid of this diagram, the *diagram of total displacements* in a given time may be constructed. And by taking the displacements in $\frac{1}{n}th$ of the unit of time and magnifying them n times, the *diagram of velocities* can be got. Similarly, the diagrams of *acceleration, forces*, and so on, may be represented as the contours of curves. The curves, from which the diagrams of velocities, &c., are obtained, are, of course, different from the original curves representing the position of the moving points. In the case of velocities, for example, they are the hodographs of the original curves on this trilinear system of reference.