# ON THE DISTRIBUTION OF $n\theta$ MODULO 1

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**Introduction.** In recent work of E. Arthurs and L. A. Shepp on a problem of H. Dym concerning the existence of an ergodic stationary stochastic process with zero entropy (cf. **1**), the function $d_\theta(n)$ was introduced as follows:

For an irrational number $\theta$, let

$$0 = a_0 < a_1 < a_2 < \ldots < a_n < a_{n+1} = 1$$

be the sequence of points $\{l\theta\}$, $1 \leqq l \leqq n$, (where $\{x\}$ denotes $x - [x]$, the fractional part of $x$) and define*

$$d_\theta(n) = \max(a_i - a_{i-1}), \qquad 1 \leqq i \leqq n + 1.$$

It is our purpose in this paper to establish several asymptotic results for $d_\theta(n)$. In particular, we prove that

$$\sup_\theta \liminf_{n\to\infty} nd_\theta(n) = \frac{1+\sqrt{2}}{2}$$

and

$$\inf_\theta \limsup_{n\to\infty} nd_\theta(n) = 1 + \frac{2\sqrt{5}}{5}$$

(cf. Theorems 1 and 2).

**Notation.** We consider an irrational number $\theta$. $[b_0, b_1, b_2, \ldots]$ is the simple continued fraction expansion of $\theta$, i.e.,

$$\theta = b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots .$$

The convergents $h_n/k_n$ of $\theta$ satisfy (cf. **3**)

$$h_{-1} = 1, \quad h_0 = b_0, \quad h_i = b_i h_{i-1} + h_{i-2}, \qquad i \geqq 1,$$

$$k_{-1} = 0, \quad k_0 = 1, \quad k_i = b_i k_{i-1} + k_{i-2}, \qquad i \geqq 1.$$

We define $\theta_n$ by

$$\theta_0 = \theta, \quad \theta_{i+1} = 1/(\theta_i - [\theta_i]), \qquad i \geqq 0.$$

Received July 15, 1966 and in revised form, December 13, 1966.

*It should be noted that the related function $d'_\theta(n) = \min_{1\leqslant i\leqslant n+1}(a_i - a_{i-1})$ has been extensively studied by Sós, Halton, and others (cf. **2**; **4**; **5**; **6**; **7**; **8**; and **9**).

We have (cf. **3**)

$$b_n = [\theta_n]$$

and

$$\theta - \frac{h_n}{k_n} = \frac{(-1)^n}{k_n(k_n\theta_{n+1} + k_{n-1})}.$$

Finally, we define $x_n$ and $y_n$ by

$$x_n = k_{n+1}/k_n, \qquad y_n = 1/\theta_{n+2}.$$

We then have

$$x_{n+1} = [1/y_n] + 1/x_n, \qquad y_{n+1} = -[1/y_n] + 1/y_n, \qquad n \geqq 0.$$

It follows easily from the definitions that

(1) $$x_n = [b_{n+1}, b_n, \ldots, b_1]$$

and

(2) $$y_n = 1/[b_{n+2}, b_{n+3}, b_{n+4}, \ldots].$$

We shall use the following basic lemma.

LEMMA 1.
$$d_\theta(m) = |k_n\theta - h_n + \alpha(k_{n+1}\theta - h_{n+1})|$$
*if*
$$k_n + (\alpha + 1)k_{n+1} - 1 \leqq m \leqq k_n + (\alpha + 2)k_{n+1} - 2$$
*and* $0 \leqq \alpha \leqq b_{n+2} - 1$.

The proof of this result appears implicitly in (**5**) and (**7**) and will not be given here. It depends upon the somewhat surprising and apparently little-known fact that the set of numbers $\{a_{i+1} - a_i: 0 \leqq i \leqq n\}$ (using the notation in § 1) always consists of at most three elements.

We are now prepared to prove the statements given in the introduction.

**The main results.**

THEOREM 1.
$$\sup_\theta \liminf_{n\to\infty} nd_\theta(n) = \frac{1+\sqrt{2}}{2}.$$

*Proof.* We observe that, for $n \to \infty$,

$$\liminf nd_\theta(n) \leqq \liminf(k_n + k_{n+1})|k_n\theta - h_n|$$
$$= \liminf(1 + x_n)(y_n + x_n)^{-1}.$$

We first show that

(3) $$\liminf(1 + x_n)(y_n + x_n)^{-1} \leqq \tfrac{1}{2}(1 + \sqrt{2}), \qquad n \to \infty.$$

Equivalently, we must show that

(4) $\limsup(y_n + x_n)(1 + x_n)^{-1} \geqq 2(1 + \sqrt{2})^{-1} = 2(\sqrt{2} - 1), \quad n \to \infty.$

We prove (4) by establishing

LEMMA 2. *Let* $\theta = [b_0, b_1, b_2, \ldots]$, *where* $b_n \leqq M$ *for all* $n$. *Then*

$$\limsup x_n y_n \geqq 1, \qquad n \to \infty,$$

*with equality if and only if the* $b_n$ *are eventually constant.*

*Proof.* By (1) and (2) we have

$$\begin{aligned} x_n y_n &= [b_{n+1}, b_n, \ldots, b_1]/[b_{n+2}, b_{n+3}, \ldots] \\ &> (b_{n+1} + (M+1)^{-1})(b_{n+2} + 1)^{-1}. \end{aligned}$$

(i) If $b_{n+1} > b_{n+2}$ infinitely often, then $b_{n+1} \geqq 1 + b_{n+2}$ infinitely often and hence, for $n \to \infty$,

$$\begin{aligned} \limsup x_n y_n &\geqq \limsup(b_{n+1} + (M+1)^{-1})(b_{n+2} + 1)^{-1} \\ &\geqq \limsup(b_{n+2} + 1 + (M+1)^{-1})(b_{n+2} + 1)^{-1}. \\ &\geqq 1 + (M+1)^{-2} > 1. \end{aligned}$$

(ii) If $b_{n+1} > b_{n+2}$ for just a finite number of values of $n$, then there is an $N$ such that $b_m = N$ for all sufficiently large $m$. Hence, if $\alpha = [N, N, N, \ldots]$ then

$$\lim x_n y_n = (\lim x_n)(\lim y_n) = \alpha \cdot (1/\alpha) = 1, \qquad n \to \infty.$$

This proves Lemma 2.

It follows that, for any $\epsilon > 0$, infinitely many of the pairs $(x_n, y_n)$ lie in the hyperbolic region given by $x \geqq 0$ and $xy \geqq 1 - \epsilon$. We observe that this region is contained in that defined by $x \geqq 0$ and $(y + x)/(1 + x) \geqq 2(\sqrt{2} - 1) - \epsilon$, since the last boundary line passes below the hyperbola, for all sufficiently small $\epsilon$; and (4) now follows. Thus (4) holds in case the $b_n$ are bounded. On the other hand, if the $b_n$ are unbounded, then the $x_n$ are unbounded and

$$\limsup(y_n + x_n)(1 + x_n)^{-1} \geqq \limsup x_n(1 + x_n)^{-1} = 1 > 2(\sqrt{2} - 1),$$

$n \to \infty$. This proves (4).

Finally, suppose that $\theta = 1 + \sqrt{2}$. Then $b_n = 2$ for $n = 0, 1, 2, \ldots$. The relations for $h_n$ and $k_n$ can be solved to give

$$\begin{aligned} h_n &= (2\sqrt{2})^{-1}[(1 + \sqrt{2})^{n+2} - (1 - \sqrt{2})^{n+2}], \\ k_n &= (2\sqrt{2})^{-1}[(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}], \end{aligned}$$

and all $\theta_n = 1 + \sqrt{2}$. Hence, we have

$$k_n\theta - h_n = (-1)^n(\sqrt{2} - 1)^{n+1}.$$

By Lemma 1,

$$d_\theta(m) = (\sqrt{2} - 1)^{n+1}[1 - \alpha(\sqrt{2} - 1)]$$

if $m = k_n + (\alpha + 1)k_{n+1} + c$, where $0 \leqq \alpha \leqq b_{n+2} - 1$ and $-1 \leqq c \leqq k_{n+1} - 2$; that is, if, for large $n$,

$$m \sim (2\sqrt{2})^{-1}(1 + \sqrt{2})^{n+1}[1 + (\alpha + 1)(1 + \sqrt{2})],$$

where $\alpha = 0$ or $1$ and $-1 \leqq c < (2\sqrt{2})^{-1}(1 + \sqrt{2})^{n+1} - 1$. When $m \to \infty$, $n \to \infty$; therefore

$$\begin{aligned}\liminf_{m\to\infty} md_\theta(m) &= \inf_\alpha (2\sqrt{2})^{-1}[1 - \alpha(\sqrt{2} - 1)][1 + (\alpha + 1)(1 + \sqrt{2})] \\ &= \inf_\alpha (2\sqrt{2})^{-1}[2 + \sqrt{2} + \alpha - \alpha^2] = \frac{1 + \sqrt{2}}{2}.\end{aligned}$$

This completes the proof of the theorem.

THEOREM 2.

$$\inf_\theta \limsup_{n\to\infty} nd_\theta(n) = 1 + \frac{2\sqrt{5}}{5}.$$

*Proof.* By Lemma 1, it is sufficient to prove

$$(5)\quad \begin{aligned}\limsup (k_n + 2k_{n+1})|k_n\theta - h_n| &= \limsup (1 + 2x_n)(y_n + x_n)^{-1} \\ &\geqq 1 + 2\sqrt{5}/5, \qquad n \to \infty,\end{aligned}$$

in order to show that

$$\limsup nd_\theta(n) \geqq 1 + 2\sqrt{5}/5, \qquad n \to \infty.$$

If $y_n \leqq \frac{1}{2}$ infinitely often, then

$$(1 + 2x_n)(y_n + x_n)^{-1} \geqq 2$$

infinitely often and we have

$$\limsup (1 + 2x_n)(y_n + x_n)^{-1} \geqq 2, \qquad n \to \infty.$$

If $y_n > \frac{1}{2}$ for all sufficiently large $n$, then $b_n = 1$ for all sufficiently large $n$. Hence, as $n \to \infty$,

$$\lim x_n = [1, 1, 1, \ldots] = (1 + \sqrt{5})/2, \qquad \lim y_n = (-1 + \sqrt{5})/2$$

and

$$\lim (1 + 2x_n)(y_n + x_n)^{-1} = 1 + 2\sqrt{5}/5.$$

This proves (5). An easy calculation shows that

$$\lim nd_\theta(n) = 1 + 2\sqrt{5}/5, \qquad n \to \infty,$$

for $\theta = (1 + \sqrt{5})/2$, and Theorem 2 is proved.

We note that if

$$k_n + k_{n+1} - 1 \leqq m \leqq k_n + (\alpha + 2)k_{n+1} - 2,$$

where $\alpha = b_{n+2} - 1$, we have

$$\max_m md_\theta(m) = \max_{0 \leqq \mu \leqq b_{n+2} - 1} (1 + (\mu + 2)x_n)(1 - \mu y_n)(x_n + y_n)^{-1}.$$

We conclude with

THEOREM 3.

$$\limsup nd_\theta(n) = \infty \Leftrightarrow \limsup b_n = \infty, \qquad n \to \infty.$$

*Proof.* (i) If $\limsup_{n\to\infty} b_n = \infty$, then $\liminf_{n\to\infty} y_n = 0$. If $y_n$ is sufficiently small, then we can take $\mu = [1/2y_n] - 1$ (since this is less than $b_{n+2} - 1$) and we find

$$(1 + (\mu + 2)x_n)(1 - \mu y_n)(x_n + y_n)^{-1} \geqq x_n(2y_n)^{-1}(\tfrac{1}{2})(x_n + 1)^{-1} \to \infty$$

for a subsequence of $y_n$ which tends to 0.

(ii) If $\limsup_{n\to\infty} nd_\theta(n) = \infty$, then certainly

$$\limsup(1 + (\mu^* + 2)x_n)(1 - \mu^* y_n)(x_n + y_n)^{-1} = \infty, \qquad n \to \infty,$$

where

$$\mu^* = (2y_n)^{-1} - (2x_n)^{-1} - 1$$

(the expression considered is a quadratic form in $\mu$ with a maximum for $\mu = \mu^*$). Hence, as $n \to \infty$,

$$\limsup 2^{-1}[(x_n + y_n)(2x_n y_n)^{-1} + 1][1 + 2x_n y_n(x_n + y_n)^{-1}] = \infty$$

and this implies $\liminf_{n\to\infty} y_n = 0$, i.e., $\limsup_{n\to\infty} b_n = \infty$ and the theorem is proved.

The authors would like to acknowledge several helpful comments of a referee.

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