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CONVERGENCE OF MANN'S ALTERNATING PROJECTIONS IN CAT(κ) SPACES

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Abstract

We study the convex feasibility problem in CAT(κ) spaces using Mann's iterative projection method. To do this, we extend Mann's projection method in normed spaces to CAT(κ) spaces with $\kappa \geq 0$, and then we prove the Δ-convergence of the method. Furthermore, under certain regularity or compactness conditions on the convex closed sets, we prove the strong convergence of Mann's alternating projection sequence in CAT(κ) spaces with $\kappa \geq 0$.

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1. Introduction

Mann introduced his iterative method in 1953 [14]. Mann's method is an important simple method to locate fixed points of a given map. More precisely, let V be a normed vector space. Given a closed convex set C in V and a map $T : C \to C$, *Mann's iterative method* is defined by the sequence $\{x_n\}$ where

$$x_{n+1} = (1 - t_n)T(x_n) + t_n x_n, \quad n = 0, 1, \ldots,$$

and x_0 is a given starting point in C. Mann's iterative method can be generalised to a general geodesic metric space (M, d) (see, for example, [8, 10, 12]). Given a closed convex subset C in M and a map $T : C \to C$, we can define the sequence $\{x_n\}$ by

$$x_{n+1} := T(x_n)\#_{t_n} x_n,$$

where $x\#_t y$ is the point $\gamma(t)$ on the geodesic $\gamma : [0, 1] \to M$ connecting $\gamma(0) = x$ and $\gamma(1) = y$ and x_0 is a given starting point in C. Many authors studied the convergence of Mann's iterative method for various classes of maps (for example, nonexpansive maps and quasi-nonexpansive maps) in geodesic metric spaces (for example, Banach

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space [11, 17, 19], CAT(0) space [8] and CAT(κ) space [10, 12]). In particular, in [10], He *et al.* studied Mann's iterative method for a nonexpansive mapping in a CAT(κ) space with $\kappa > 0$ and proved that the sequence Δ-converges to a point.

Mann's alternating projection method in a geodesic space is defined by a sequence $\{x_n\}$ given by

$$\begin{cases} x_{2n-1} = P_A(x_{2n-2}) \#_{t_{2n-2}} x_{2n-2}, \\ x_{2n} \;\;\;= P_B(x_{2n-1}) \#_{t_{2n-1}} x_{2n-1}, \end{cases} \quad n \in \mathbb{N}, \quad \{t_n\}_{n\geq 0} \subset [0,1), \tag{1.1}$$

where x_0 is a given starting point and A and B are closed convex sets. This is an important simple method to solve the convex feasibility problem, which is to find common elements in two given convex sets. This problem has been studied in many contexts (for example, Hilbert space [3, 7], CAT(0) space [2] and CAT(κ) space [5]). It is a generalisation of the alternating projection sequence introduced by von Neumann [15]. Indeed, if we take $t_n = 0$ for all $n \geq 0$ in (1.1), then we obtain the alternating projection method given by

$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}), \quad n \in \mathbb{N}, \tag{1.2}$$

where x_0 is a given point. Its convergence has been studied in a complete CAT(0) space [2], and in a complete CAT(κ) space for $\kappa > 0$ [5]. Mann's alternating method for two nonexpansive mappings in a p-uniformly convex metric space is studied in [6]. Note that for $\kappa \geq 0$, every CAT(κ) space (with diameter $\pi/(2\sqrt{\kappa})$ for $\kappa > 0$) is a p-uniformly convex metric space (see [16]). But, in general, the metric projection P_C for a convex closed subset C of a complete CAT(κ) space for $\kappa > 0$ need not be nonexpansive. In [12], it is shown that Mann's iterative method for a countable family of (quasi-)nonexpansive mappings in a CAT(1) space Δ-converges to a point.

The main purpose of this paper is to prove the Δ-convergence of Mann's alternating projection sequence (1.1) in a CAT(κ) space. Although the metric projection map P_C for a convex closed subset C of a complete CAT(κ) space is a quasi-nonexpansive mapping, our main result does not come from [12, Theorem 3.8]. Moreover, our result includes the alternating projection sequence given by (1.2), but the result in [12] does not (see Remark 3.8). Also, we prove the strong convergence of the sequence under certain regularity or compactness conditions on CAT(κ) spaces.

This paper is organised as follows. In Section 2 we briefly review the basic notions of CAT(κ) spaces and Δ-convergence. In Section 3 we first introduce Mann's alternating projection method in a CAT(κ) space with $\kappa \geq 0$, and prove its Δ-convergence (Theorem 3.6). In particular, we obtain the Δ-convergence of the alternating projection method in CAT(κ) spaces (Corollary 3.9). Then we prove the strong convergence of Mann's alternating projection method by assuming certain regularity or compactness conditions on the CAT(κ) spaces (Corollary 3.10).

2. Preliminaries

2.1. CAT(κ) spaces. In this subsection we recall some fundamental notions of a geodesic metric space and a CAT(κ) space with $\kappa \geq 0$. We basically follow [4].

Let (M, d) be a metric space and $x, y \in M$. A continuous map $\gamma : [0, 1] \to M$ is said to be a *geodesic* (path) connecting x and y if it satisfies the following properties:

$$\gamma(0) = x,\ \gamma(1) = y \quad \text{and} \quad d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| d(x, y) \text{ for all } t_1, t_2 \in [0, 1].$$

The image of the geodesic γ with $\gamma(0) = x$ and $\gamma(1) = y$ is called a *geodesic segment* connecting x and y and denoted by $[x, y]$.

A metric space (M, d) is called an *r-geodesic space* if for any $x, y \in M$ with $d(x, y) < r$, there exists a geodesic γ connecting x and y. If $r = \infty$, then (M, d) is called a *geodesic space*, that is, for any $x, y \in M$ there exists a geodesic γ connecting x and y.

Let $\mathbb{S}^n$ be the set of all elements in $\mathbb{R}^{n+1}$ such that $\langle x\,, x\rangle = 1$, where $\langle \cdot\,, \cdot\rangle$ is the Euclidean inner (scalar) product. Let $\rho : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function defined by

$$\rho(x, y) = \arccos\langle x\,, y\rangle.$$

Indeed, ρ is the great-circle distance. It is well known that $(\mathbb{S}^n, \rho)$ is a geodesic metric space. Note that for any $x, y \in \mathbb{S}^n$ with $\rho(x, y) < \pi$, there exists a unique geodesic γ with $\gamma(0) = x$ and $\gamma(1) = y$.

We always assume that $\kappa \geq 0$. Put $D_0 := \infty$ and $D_\kappa := \pi/\sqrt{\kappa}$ for $\kappa > 0$. Given a real number $\kappa \geq 0$, define the *model space* M^n_κ to be the following metric space:

(i) if $\kappa = 0$ then M^n_0 is n-dimensional Euclidean space $\mathbb{R}^n$;
(ii) if $\kappa > 0$ then M^n_κ is the geodesic metric space obtained from $(\mathbb{S}^n, \rho)$ by multiplying the function ρ by the constant $1/\sqrt{\kappa}$.

We use the symbol ρ_κ for the distance function of M^n_κ for each $\kappa \geq 0$. It is clear that M^n_κ is a geodesic metric space. Note that if $\rho_\kappa(x, y) < D_\kappa$ then there is a unique geodesic connecting x and y in M^n_κ (if $\kappa = 0$ then $D_\kappa = \infty$.)

Let (M, d) be a geodesic metric space. A *geodesic triangle* $\Delta := \Delta(x, y, z) \subseteq M$ consists of three points x, y, z in M and three geodesic segments $[x, y], [y, z]$ and $[x, z]$. Given a geodesic triangle $\Delta = \Delta(x, y, z) \subseteq M$, a geodesic triangle $\overline{\Delta} = \Delta(\overline{x}, \overline{y}, \overline{z}) \subseteq M^2_\kappa$ is said to be a *comparison triangle* for Δ if

$$d(x, y) = \rho_\kappa(\overline{x}, \overline{y}), \quad d(x, z) = \rho_\kappa(\overline{x}, \overline{z}) \quad \text{and} \quad d(y, z) = \rho_\kappa(\overline{y}, \overline{z}).$$

A point $\overline{p}$ in $[\overline{x}, \overline{y}] \subseteq \overline{\Delta}$ is a *comparison point* for p in $[x, y] \subseteq \Delta$ if $d(p, x) = \rho_\kappa(\overline{p}, \overline{x})$. Note that for a geodesic triangle $\Delta(x, y, z) \subseteq M$, if $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, then a comparison triangle $\overline{\Delta} \subseteq M^2_\kappa$ for Δ always exists (see [4]). For a geodesic triangle $\Delta = \Delta(x, y, z) \subseteq M$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, we say that $\triangle$ satisfies the CAT(κ) *inequality* if, for any $p, q \in \triangle$ and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}$,

$$d(p, q) \leq \rho_\kappa(\overline{p}, \overline{q}).$$

Definition 2.1. Let (M, d) be a metric space.

(i) (M, d) is a *CAT(0) space* if (M, d) is a geodesic space and all geodesic triangles in M satisfy the CAT(0) inequality.

(ii) (M, d) is a *CAT(κ) space* with $\kappa > 0$ if (M, d) is a D_κ-geodesic space and all geodesic triangles $\Delta(x, y, z) \subseteq M$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfy the CAT(κ) inequality.

Note that if (M, d) is a CAT(κ) space, there is a unique geodesic which connects each pair of points $x, y \in M$ whenever $d(x, y) < D_\kappa$ (see [4]).

If (M, d) is a CAT(κ) space, with $\mathrm{diam}(M) = \sup\{d(x, y) \mid x, y \in M\} < D_\kappa/2$ for $\kappa > 0$, then for any geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, any $z \in M$ and $t \in [0, 1]$, there exists a constant $c_M \in (0, 1]$ such that

$$d(z, \gamma(t))^2 \leq (1 - t)d(z, x)^2 + td(z, y)^2 - c_M t(1 - t)d(x, y)^2 \tag{2.1}$$

(see [16]). In particular, if (M, d) is a CAT(0) space, then $c_M = 1$ (see [8]).

A subset C of a geodesic metric space M is said to be *convex* if any two points $x, y \in C$ can be joined by a geodesic in M and the geodesic segment of every such geodesic is contained in C. Note that any ball in a CAT(κ) space for $\kappa > 0$ of radius smaller than $D_\kappa/2$ is convex. In particular, any ball in a CAT(0) space is convex.

For a nonempty subset S of a metric space (M, d), the *distance function* of S is defined by

$$d(x, S) = \inf\{d(x, s)\,;\ s \in S\} \quad \text{for } x \in M.$$

We now recall the notion of a projection map in a complete CAT(κ) space. Let (M, d) be a complete CAT(κ) space and $x \in M$ be given. Let C be a nonempty closed convex subset of M (with $d(x, C) < D_\kappa/2$ if $\kappa > 0$). It is well known that for given $x \in M$, there exists a unique point $P_C(x)$ in C such that

$$d(x, P_C(x)) = d(x, C). \tag{2.2}$$

(For the case of CAT(0) space, see [18], and for the case of CAT(κ) space ($\kappa > 0$), see [9].) By the uniqueness of $P_C(x)$ for all $x \in M$, we can define the *(metric) projection* P_C of M onto C by

$$P_C : M \ni x \longmapsto P_C(x) \in C.$$

PROPOSITION 2.2 [1]. *Let (M, d) be a complete* CAT(κ) *space with* $\mathrm{diam}(M) < D_\kappa/2$ *for $\kappa > 0$ and $C \subseteq M$ be a nonempty closed convex set. Then for all $x \in M$ and $z \in C$,*

$$d(z, P_C(x))^2 + c_M d(x, P_C(x))^2 \leq d(x, z)^2,$$

where c_M is given in (2.1). In particular, if (M, d) is a CAT(0) *space, then* $c_M = 1$ [18].

2.2. Δ-convergence in CAT(κ) spaces. We recall the notion of Δ- (or weak) convergence in CAT(κ) spaces. Let (M, d) be a complete CAT(κ) space and $\{x_n\} \subseteq M$ be a bounded sequence. For a given point $x \in M$, set

$$r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n).$$

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Then $r(\{x_n\}) := \inf_{y\in M} r(y, \{x_n\})$ is called the *asymptotic radius* of $\{x_n\}$. The *asymptotic centre* $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) := \{x \in M \mid r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is easily seen that

$$z \in A(\{x_n\}) \iff \limsup_{n\to\infty} d(z, x_n) \le \limsup_{n\to\infty} d(x, x_n) \quad \text{for any } x \in M.$$

A sequence $\{x_n\}$ is said to *Δ-converge* to $x \in M$ if for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, the point x is the unique asymptotic centre of $\{x_{n_k}\}$, and then x is called the *Δ-limit* of $\{x_n\}$. A point x in M is called a *Δ-cluster point* of a sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to x. This concept was first introduced by Lim [13] and has been studied by many authors (see, for example, [5, 6, 9, 10]). In Hilbert space, it is well known that the notion of Δ-convergence coincides with the notion of weak convergence.

Proposition 2.3 [9]. *Let M be a complete* CAT(κ) *space and $\{x_n\} \subseteq M$ be a bounded sequence with $r(\{x_n\}) < D_\kappa/2$ for $\kappa > 0$. Then*

(i) *$A(\{x_n\})$ has only one point;*
(ii) *$\{x_n\}$ has a Δ-cluster point $x \in M$, that is, $\{x_n\}$ has a Δ-convergent subsequence.*

Proposition 2.4 [10]. *Let (M, d) be a complete* CAT(κ) *space and let $z \in M$. If a sequence $\{x_n\} \subseteq M$ satisfies $r(z, \{x_n\}) < D_k/2$ for $\kappa > 0$ and $\{x_n\}$ Δ-converges to $x \in M$, then*

$$x \in \bigcap_{k=1}^{\infty} \overline{\mathrm{conv}}(\{x_k, x_{k+1}, \ldots\}),$$

where $\overline{\mathrm{conv}}(A) = \bigcap\{B \subseteq M \mid A \subseteq B$ and B is closed and convex$\}$, and

$$d(x, z) \le \liminf_{n\to\infty} d(x_n, z).$$

Remark 2.5. Under the same assumptions as in Proposition 2.4, if $r(z, \{x_n\}) < D_k/2$ for any z in a subset C of M, then

$$d(x, z) \le \liminf_{n\to\infty} d(x_n, z) \quad \text{for all } z \in C$$

(see also [2, Lemma 3.2] for the case of CAT(0) spaces).

3. Mann's alternating projections

We now recall the notion of Fejér monotone sequences in metric spaces M. Let $\{x_n\}$ be a sequence in M. For a nonempty subset $C \subseteq M$, a sequence $\{x_n\}$ is said to be *Fejér monotone* with respect to (w.r.t.) C if for any $z \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, z) \le d(x_n, z).$$

In what follows we recall some properties of Fejér monotone sequences.

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Proposition 3.1 [5]. *Let (M, d) be a complete metric space. Let $\{x_n\}$ be a sequence in M and let C be a nonempty closed convex subset of M. Suppose that $\{x_n\}$ is Fejér monotone w.r.t. C. Then*

(i) $\{x_n\}$ *is a bounded sequence;*
(ii) $d(x_{n+1}, C) \leq d(x_n, C)$ *for all* $n \in \mathbb{N}$;
(iii) $\{x_n\}$ *converges to some* $x \in C$ *if and only if* $d(x_n, C) \to 0$ *as* $n \to \infty$.

Lemma 3.2 [10]. *Let (M, d) be a complete* CAT(κ) *space and let $C \subset M$ be a nonempty set. Suppose that the sequence $\{x_n\} \subset M$ is Fejér monotone w.r.t. C and satisfies $r(\{x_n\}) < D_\kappa/2$ for $\kappa > 0$. Suppose also that any Δ-cluster point x of $\{x_n\}$ belongs to C. Then $\{x_n\}$ Δ-converges to a point in C.*

The same result as in Lemma 3.2 holds in a CAT(0) space (see [2, Proposition 3.3]).

Let A and B be closed convex subsets of a complete CAT(κ) space (M, d). *Mann's alternating projection method* produces a sequence $\{x_n\}$ by

$$\begin{cases} x_{2m-1} := P_A(x_{2m-2})\#_{t_{2m-2}} x_{2m-2}, \\ \quad x_{2m} := P_B(x_{2m-1})\#_{t_{2m-1}} x_{2m-1}, \end{cases} \quad m \in \mathbb{N}, \quad \{t_k\}_{k\geq 0} \subset [0, 1), \tag{3.1}$$

where x_0 is a given starting point and $x\#_t y$ is the point $\gamma(t)$ on a geodesic $\gamma : [0, 1] \to M$ connecting $\gamma(0) = x$ and $\gamma(1) = y$, called the t-weighted geometric mean of x and y.

Remark 3.3. If we take $t_k = 0$, for $k = 0, 1, \ldots$, in the sequence (3.1) constructed by the Mann's alternating projection method, then we have the sequence constructed by the *alternating projection method* [2, 5]:

$$x_{2m-1} = P_A(x_{2m-2}), \quad x_{2m} = P_B(x_{2m-1}), \quad n \in \mathbb{N}, \tag{3.2}$$

where x_0 is a given starting point, and A and B are closed convex subsets of a complete CAT(κ) space (M, d) for $\kappa \geq 0$.

Throughout this section, (M, d) denotes a complete CAT(κ) space for $\kappa \geq 0$. In the case of $\kappa > 0$, we also assume that $\operatorname{diam}(M) < D_\kappa/2$, unless specified otherwise.

Lemma 3.4. *Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. The sequence $\{x_n\} \subseteq M$ given in (3.1) with a starting point x_0 is Fejér monotone w.r.t. $A \cap B$.*

Proof. Let $z \in A \cap B$. For fixed $n \in \mathbb{N}$, without loss of generality we assume that $x_n = P_A(x_{n-1})\#_{t_{n-1}} x_{n-1}$. Then $x_{n+1} = P_B(x_n)\#_{t_n} x_n$. If $x_{n+1} = z$, then the proof is clear; indeed, $d(x_{n+1}, z) = 0 \leq d(x_n, z)$. Suppose that $x_{n+1} \neq z$. Then

$$d(x_{n+1}, z)^2 = d(P_B(x_n)\#_{t_n} x_n, z)^2 \leq t_n d(x_n, z)^2 + (1 - t_n) d(P_B(x_n), z)^2.$$

But, by Proposition 2.2,

$$c_M d(x_n, P_B(x_n))^2 + d(P_B(x_n), z)^2 \leq d(x_n, z)^2,$$

which implies that

$$d(P_B(x_n), z) \leq d(x_n, z).$$

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Therefore,

$$\begin{aligned} d(x_{n+1}, z)^2 &\le t_n d(x_n, z)^2 + (1 - t_n) d(P_B(x_n), z)^2 \\ &\le t_n d(x_n, z)^2 + (1 - t_n) d(x_n, z)^2 = d(x_n, z)^2, \end{aligned}$$

which completes the proof. □

LEMMA 3.5. *Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. Let x_0 be a given starting point and $\{x_n\}$ the sequence given in (3.1) with $\lim_{n\to\infty} t_n \neq 1$. Then*

$$\max\{d(x_n, A), d(x_n, B)\} \to 0 \quad \textit{as } n \to \infty. \tag{3.3}$$

PROOF. By using Proposition 2.2, for any $z \in A \cap B \subset A$,

$$d(x_{2m}, z)^2 \ge d(z, P_A(x_{2m}))^2 + c_M d(x_{2m}, P_A(x_{2m}))^2.$$

Taking the infimum for $z \in A \cap B$ and applying the projection property (2.2),

$$d(x_{2m}, A \cap B)^2 \ge d(A \cap B, P_A(x_{2m}))^2 + c_M d(x_{2m}, A)^2.$$

Therefore,

$$d(x_{2m}, A)^2 \le \frac{1}{c_M}(d(x_{2m}, A \cap B)^2 - d(A \cap B, P_A(x_{2m}))^2). \tag{3.4}$$

But, by (2.1),

$$d(x_{2m+1}, A \cap B)^2 \le t_{2m} d(x_{2m}, A \cap B)^2 + (1 - t_{2m}) d(P_A(x_{2m}), A \cap B)^2. \tag{3.5}$$

Combining (3.4) and (3.5),

$$\begin{aligned} d(x_{2m}, A)^2 &\le \frac{1}{c_M}\bigg(\bigg(1 + \frac{t_{2m}}{1 - t_{2m}}\bigg) d(x_{2m}, A \cap B)^2 - \bigg(\frac{1}{1 - t_{2m}}\bigg) d(x_{2m+1}, A \cap B)^2\bigg) \\ &\le \frac{1}{(1 - t_{2m}) c_M}(d(x_{2m}, A \cap B)^2 - d(x_{2m+1}, A \cap B)^2). \end{aligned}$$

Similarly, in the case where $z \in A \cap B \subset B$,

$$d(x_{2m-1}, B)^2 \le \frac{1}{(1 - t_{2m-1}) c_M}(d(x_{2m-1}, A \cap B)^2 - d(x_{2m}, A \cap B)^2).$$

By Lemma 3.4, the sequence $\{x_n\}$ is Fejér monotone w.r.t. $A \cap B$. Therefore, by (ii) in Proposition 3.1, the sequence $\{d(x_n, A \cap B)\}$ is a bounded and decreasing sequence in $\mathbb{R}$. Hence, $\{d(x_n, A \cap B)\}$ converges to some point in $\mathbb{R}$. Since $\lim_{n\to\infty} t_n \neq 1$, it follows that $d(x_{2m}, A) \to 0$ and $d(x_{2m-1}, B) \to 0$ as $m \to \infty$. On the other hand,

$$\begin{aligned} d(x_{2m-1}, x_{2m}) &= d(x_{2m-1}, P_B(x_{2m-1}) \#_{t_{2m-1}} x_{2m-1}) \\ &= (1 - t_{2m-1}) d(x_{2m-1}, P_B(x_{2m-1})) \\ &= (1 - t_{2m-1}) d(x_{2m-1}, B) \to 0, \end{aligned}$$

as $m \to \infty$, which implies that $d(x_{2m-1}, A) \to 0$ and $d(x_{2m}, B) \to 0$ as $m \to \infty$. Therefore, we conclude that

$$d(x_n, A) \to 0 \quad \text{and} \quad d(x_n, B) \to 0$$

as $n \to \infty$. Since $\max\{a_n, b_n\} = \frac{1}{2}(a_n + b_n + |a_n - b_n|)$ for real-valued sequences $\{a_n\}$ and $\{b_n\}$, we have proved (3.3). □

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We now recall the notion of regularity of sets in metric spaces (see [2]). Let A and B be two nonempty subsets of a metric space (M, d). We say that A and B are *boundedly regular* if for any bounded subset $V \subseteq M$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in V$ and $\max\{d(x, A), d(x, B)\} \leq \delta$,

$$d(x, A \cap B) \leq \epsilon.$$

The following theorem is the main result in this paper.

Theorem 3.6. *Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. Let x_0 be a given starting point and $\{x_n\}$ be the sequence given in (3.1) with* $\lim_{n\to\infty} t_n \neq 1$. *Then*

(i) *$\{x_n\}$ Δ-converges to a point $x \in A \cap B$;*
(ii) *$\{x_n\}$ converges to a point $x \in A \cap B$ whenever A and B are boundedly regular.*

Proof. (i) We only show the result for $\kappa > 0$. The proof for $\kappa = 0$ is similar. Let (M, d) be a complete CAT(κ) space for $\kappa > 0$. Let $\{x_n\} \subseteq M$ be Mann's alternating projection given in (3.1). By Lemma 3.4, the sequence $\{x_n\}$ is Fejér monotone w.r.t. $A \cap B$ which implies that $\{x_n\}$ is bounded with $r(\{x_n\}) < D_\kappa/2$ for $\kappa > 0$. Thus, by Proposition 2.3(ii), $\{x_n\}$ has a Δ-cluster point, x say, in M. By the definition of a Δ-cluster point, we can take a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which Δ-converges to x. Thus, by Remark 2.5 and (3.3) in Lemma 3.5,

$$d(x, A) = d(x, B) = 0,$$

which implies that $x \in A \cap B$. Since the sequence $\{x_n\}$ is Fejér monotone w.r.t. $A \cap B$, by Lemma 3.2, we conclude that $\{x_n\}$ Δ-converges to a point $x \in A \cap B$.

(ii) Suppose that A and B are boundedly regular. Then since $\{x_n\}$ is a bounded sequence, by (3.3) in Lemma 3.5,

$$d(x_n, A \cap B) \to 0 \quad \text{as } n \to \infty.$$

Therefore, by Proposition 3.1(iii), $\{x_n\}$ converges to a point $x \in A \cap B$. □

Remark 3.7. In [6], we can find the Δ-convergence of Mann's alternating projection method in a complete CAT(0) space. Indeed, the authors prove the Δ-convergence of Mann's alternating sequence for nonexpansive mappings on p-uniformly convex metric spaces. For any nonempty closed convex subset C of a complete CAT(0) space, the metric projection $P_C : M \to C \subset M$ is a nonexpansive map. But, in general, if $\kappa > 0$, then the metric projection map P_C for a nonempty closed convex subset C of a complete CAT(κ) space need not be nonexpansive.

Remark 3.8. Under some conditions for the given maps, a similar result to Theorem 3.6(i) can be found in [12], since the metric projection map P_C for a nonempty closed convex subset C of a complete CAT(κ) space is a quasi-nonexpansive mapping. However, Theorem 3.6 does not come from the result in [12]. Indeed, since the sequence of metric projections $\{P_A, P_B, P_A, P_B, \ldots\}$ does not converge in general, the projections P_A and P_B do not satisfy condition (C3) in [12, Theorem 3.8]. Also our result includes the alternating von Neumann sequence given in (3.2), but the result in [12] does not (see [12, Theorem 3.8]).

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Corollary 3.9 [2, 5]. *Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. Let x_0 be a given starting point and $\{x_n\}$ be the sequence given in (3.2) constructed by the alternating projection method. Then*

(i) $\{x_n\}$ *Δ-converges to a point $x \in A \cap B$;*
(ii) $\{x_n\}$ *converges to a point $x \in A \cap B$ whenever A and B are boundedly regular.*

Proof. The corollary follows from Remark 3.3 and Theorem 3.6. □

A metric space (M, d) is called *boundedly compact* if every bounded and closed subset of M is compact.

Corollary 3.10. *Suppose (M, d) is a boundedly compact complete $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(M) < D_\kappa/2$ for $\kappa > 0$. Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. Let x_0 be a starting point and $\{x_n\}$ be the sequence given in (3.1) constructed by Mann's alternating projection method with $\lim_{n\to\infty} t_n \neq 1$. Then $\{x_n\}$ converges to a point $x \in A \cap B$.*

Proof. By Theorem 3.6(i), $\{x_n\}$ Δ-converges to a point $x \in A \cap B$. Since the sequence $\{x_n\}$ is Fejér monotone w.r.t. $A \cap B$, the sequence $\{d(x_n, x)\}$ is bounded and decreasing in $\mathbb{R}$, and so $\{d(x_n, x)\}$ converges to a point in $\mathbb{R}$. Note that $\{x_n\} \subset \overline{\mathrm{conv}}(\{x_n\})$, where $\overline{\mathrm{conv}}(A) = \bigcap\{C \subseteq M \mid A \subseteq C$ and C is closed and convex$\}$. Since $\overline{\mathrm{conv}}(\{x_n\})$ is a closed and bounded subset in M, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $\widetilde{x} \in M$. Thus,

$$\lim_{k\to\infty} d(x_{n_k}, \widetilde{x}) = 0 \leq \lim_{k\to\infty} d(x_{n_k}, z) \quad \text{for all } z \in M,$$

which implies that $\widetilde{x} \in A(\{x_{n_k}\})$. By the uniqueness of the asymptotic centre, $x = \widetilde{x}$. Since $\{d(x_n, x)\}$ converges, $\{x_n\}$ converges to $x \in A \cap B$. □

By applying the Hopf–Rinow theorem (see [4]) and simple modifications of the proof of Corollary 3.10, we have the following corollary.

Corollary 3.11. *Let (M, d) be a locally compact complete $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(M) < D_\kappa/2$ for $\kappa > 0$. Let A and B be convex closed subsets of M with $A \cap B \neq \emptyset$. Let x_0 be a starting point and $\{x_n\}$ be the sequence given in (3.1) constructed by Mann's alternating projection method with $\lim_{n\to\infty} t_n \neq 1$. Then $\{x_n\}$ converges to a point $x \in A \cap B$.*

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