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TRAVELLING WAVE SOLUTIONS FOR RICH FLAMES OF REACTIVE SUSPENSIONS

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Abstract

The modelling of the combustion of dust suspensions leads to a nonlinear eigenvalue problem for a system of ordinary differential equations defined over an infinite interval. The equations contain a number of parameters. In this study, the shooting method is used to prove the existence of a solution. Linearisation is then used to provide an approximate solution, from which an estimate of the eigenvalue and its dependence on the given parameters can be obtained.

1. Introduction

Recently, Deshaies and Joulin [2] considered a steady, planar, isobaric combustion wave propagating in a homogeneous mixture consisting of a gaseous oxidiser, an inert gas, and monodisperse, spherical fuel particles. The chemical rate of heat release is modelled by a non-Arrhenius expression. In terms of a coordinate attached to the wave, the governing partial differential equations are reduced to a system of three ordinary differential equations defined on $(-\infty, \infty)$. The equations contain an unknown parameter, related to the wave speed, which is to be determined as part of the solution. Boundary conditions are imposed at the two ends of the infinite interval. Thus, the problem is a nonlinear eigenvalue problem for a system of ordinary differential equations over an infinite interval. Deshaies and Joulin used matched asymptotic expansions to study the problem.

The study of wave propagation for reactive systems has a large literature; see for example papers in [5]. The existence of a travelling wave is often proved by using phase plane methods. Recently, Berestycki, Nicolaenko, and Scheurer

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[1] used a shooting method to prove the existence of a travelling wave solution for a single equation modelling the deflagration for a compressible reacting gas. Relatively speaking, the shooting method is more constructive than phase plane methods. A number of authors have used the shooting method to construct existence proofs for fluid dynamical problems governed by systems of ordinary differential equations over a semi-infinite interval, among them Ho and Wilson [3], Serrin and McLeod [4] and Tam [6], [7]. In this paper, we use the shooting method to show that the Deshaies-Joulin model has a solution. We then use linearisation to construct an approximate solution and to obtain transcendental equations from which the wave speed can be estimated. The problem is formulated in Section 2. The existence proof is done in Section 3, and the linearisation carried out in Section 4.

2. Formulation and preliminary observations

The boundary value problem as formulated by Deshaies and Joulin is

$$\frac{d\Theta}{d\xi} + \frac{d\varphi}{d\xi} = y\tilde{F}(\Theta) \tag{2.1}$$

$$\frac{dy}{d\xi} = -y\tilde{F}(\Theta) \tag{2.2}$$

$$\frac{d^2\varphi}{d\xi^2} = \varphi + G(\Theta)\frac{d\Theta}{d\xi} \tag{2.3}$$

where

$$\tilde{F}(\Theta) = \frac{\lambda}{b}\frac{(1+S)}{\exp\left[\dfrac{\beta(1-\Theta)}{1+\alpha(\Theta-1)}\right]+S}\quad\frac{T_{ad}}{(T_{ad}-T_u)\Theta+T_u}$$

$$G(\Theta) = [1+\alpha(\Theta-1)]^3\beta b^{-1}$$

$$\Theta = \varphi = 1 - y = 0 \quad \text{at } \xi = -\infty \tag{2.4}$$

$$\Theta - 1 = y = 0 \quad \text{at } \xi = \infty. \tag{2.5}$$

The dependent variables Θ, φ and y are the temperature, radiant flux and oxidiser concentration, suitably non-dimensionalised. The independent variable ξ is related to the distance along flow line. The positive parameters $\lambda, S, \beta, T_{ad}, T_u$ and α are given with $T_{ad} > T_u, 0 < \alpha \equiv (T_{ad} - T_u)/T_u < 1$, and $\beta \gg 1$. The positive constant b is proportional to the speed of the wave, and is an eigenvalue of the problem, to be determined as part of the solution. In this formulation, the combustion wave travels from right to left.

We first observe that since the non-Arrhenius expression $\tilde{F}(\Theta)$ represents the rate of heat release, the parameters contained in it must be such that $\tilde{F}(\Theta)$ is non-negative, and is an increasing function of Θ for $0 < \Theta < 1$. This requirement is satisfied if $\beta \gg 1$. However, as given in (2.1), it presents a "cold boundary" difficulty at $\xi = -\infty$. Specifically, if the boundary conditions (2.4) hold, then the right hand side of (2.2) must vanish at $\xi = -\infty$. The expression $\tilde{F}(\Theta)$ does not satisfy this requirement even though $\tilde{F}(0) = \exp(-\beta/(1-\alpha))$ for $\beta \gg 1$. To remedy this situation, in place of the given $\tilde{F}(\Theta)$, we shall use $F(\Theta)$ defined by

$$F(\Theta) = \tilde{F}(\Theta) - \tilde{F}(0).$$

Equations (2.1) and (2.2) are accordingly replaced by

$$\frac{d\Theta}{d\xi} + \frac{d\varphi}{d\xi} = yF(\Theta) \tag{2.1a}$$

$$\frac{dy}{d\xi} = -yF(\Theta). \tag{2.2a}$$

In the following, we shall consider the BVP consisting of (2.1a), (2.2a), (2.3), (2.4) and (2.5). Combining equations (2.1a) and (2.2a), we obtain

$$\frac{d\Theta}{d\xi} + \frac{d\varphi}{d\xi} + \frac{dy}{d\xi} = 0, \tag{2.6}$$

and hence

$$\Theta + \varphi + y = 1, \tag{2.7}$$

where the integration constant is chosen to satisfy the boundary conditions (2.4) and (2.5).

To construct an existence proof using the shooting method, we need to divide the infinite interval into two semi-infinite intervals. As the governing equations are invariant under translation, the location of an initial point $\xi = 0$ is arbitrary. However, a suitable choice will facilitate significantly the construction of the proof. Using a prime to denote differentiation with respect to ξ, we have, from equation (2.2a), $y' < 0$ on $(-\infty, \infty)$, and

$$y = \exp\left\{-\int_{-\infty}^{\xi} F(\Theta(t))\, dt\right\}. \tag{2.8}$$

Since $F(\Theta)$ is non-negative, it follows that $y(-\infty) = 1$ and $y(\infty) = \text{const.} \geq 0$. Clearly, unless $\Theta = 0$, we have $0 \leq y(\infty) < 1$. Thus, there is a value ξ^* at which $y'(\xi^*) < 0$, $y''(\xi^*) = 0$, with $y' < 0$, $y'' < 0$ for $-\infty < \xi < \xi^*$.

We divide the infinite interval by placing the origin $\xi = 0$ at ξ^*. The conditions $y' < 0$, $y'' < 0$ in $(-\infty, 0)$ are crucial in the existence proof. We now have

$$y(\xi) = h \exp\left\{-\int_0^{\xi} F(\Theta(t))\, dt\right\}, \qquad 0 < \xi < \infty$$

where $0 < h < 1$. Of the two remaining functions Θ and φ, Θ occupies a more important role. The existence proof in the next section will therefore focus on Θ.

3. The existence proof

We first consider the initial value problem for y and Θ with $y(0) = h$, $\Theta(0) = k$, and show that there exist (h, k) such that the solution can be extended to $\xi = \infty$, satisfying the required conditions at $\xi = \infty$. We then consider the interval $(-\infty, 0)$ and show that the solution obtained in $(0, \infty)$ can be extended to $\xi = -\infty$, satisfying the required conditions at $\xi = -\infty$.

3a. The initial value problem in $(0, \infty)$**.**

With $y(0) = h$, $\Theta(0) = k$, we have $\varphi(0) = 1 - h - k$, $y'(0) = -hF(k)$, and $y''(0) = -y'(0)F(k) - ydF(k)/d\Theta\Theta'(0) = 0$ which implies

$$\Theta'(0) = F^2(k)/(dF(k)/d\Theta).$$

On the hk plane, let F denote the square $0 < h < 1$, $0 < k < 1$. We define the following subsets:

$$S^+ = \{(h,k) | \Theta(\xi^+) = 1,\ 0 < \Theta < 1 \text{ in } (0, \xi^+)\},$$
$$S^- = \{(h,k) | \Theta(\xi^-) = 0,\ 0 < \Theta < 1 \text{ in } (0, \xi^-)\}.$$

LEMMA 1. *S^+ and S^- are disjoint, non-empty, open sets.*

PROOF. S^+ and S^- are disjoint by definition. To show that they are non-empty, let $\overline{G}(\Theta)$ denote the indefinite integral

$$\int G(\Theta)\, d\Theta = (4\alpha b)^{-1}\beta(1 - \alpha + \alpha\Theta)^4.$$

From equation (2.3), we have

$$\varphi = \varphi(0) + \varphi'(0)\xi + \int_0^\xi \int_0^S \varphi(t)\, dt\, ds + \int_0^\xi [\overline{G(\Theta(t))} - \overline{G(k)}]\, dt.$$

Using (2.6) and (2.7), we have

$$\begin{aligned} \Theta = {} & k + h + \Theta'(0)\xi + g'(0)\xi - \xi^2/2 - y \\ & + \int_0^\xi \int_0^S [\Theta(t) + y(t)]\, dt\, ds - \int_0^\xi [\overline{G(\Theta(t))} - \overline{G(k)}]\, dt. \end{aligned} \tag{3.1}$$

For ξ sufficiently close to zero, we have

$$\begin{aligned} \Theta &= k + \theta'(0)\xi + \Theta''(0)\xi^2/2 + O(\xi^3), \\ y &= h + y'(0)\xi + O(\xi^3). \end{aligned} \tag{3.2}$$

Retaining only the dominant terms in Θ and y, we evaluate the integrals in (3.1) to obtain

$$\Theta = k + \frac{F^2(k)}{\frac{dF(k)}{d\Theta}}\xi - \tfrac{1}{2}\left\{(1-h-k) + \frac{\beta}{2b}\frac{F^2(k)}{\frac{dF(k)}{d\Theta}}(1-\alpha+k\alpha)^3\right\}\xi^2 + O(\xi^3). \quad (3.3)$$

For $k \sim 1$, and neglecting quantities of $O(e^{-\beta})$, we have

$$F(k) \sim \lambda/b,$$
$$\frac{dF(k)}{d\Theta} \sim \frac{\lambda}{b}\left(\frac{\beta-\alpha(1+S)}{1+S}\right).$$

Let $\varepsilon > 0$ be sufficiently small. If $k < 1$ is sufficiently close to 1, and k, h satisfy the inequality

$$\left(1+\frac{\varepsilon^2}{2}\right)k + \frac{\varepsilon^2}{2}h > 1 + \frac{\varepsilon^2}{2} + \frac{\lambda}{b}\frac{(1+S)\varepsilon}{\beta-\alpha(1+S)}\left(\frac{\varepsilon\beta}{4b}-1\right)$$

which describes a triangular region at the upper right corner of R, then it follows from (3.3) that $\Theta(\varepsilon) > 1$. Thus, there exists ξ^+ at which $\Theta(\xi^+) = 1$, implying that S^+ is non-empty.

For $k \sim 0$, we have

$$F(k) \sim 0,$$
$$\frac{dF(k)}{d\Theta} \sim \frac{\lambda(1+S)T_{ad}}{b[\exp(\beta/(1-\alpha))+S]T_u}\left\{\frac{\beta\exp(\beta/(1-\alpha))}{\exp(\beta/(1-\alpha))+S} - \alpha\right\}.$$

Let $\varepsilon > 0$ be sufficiently small. If $k > 0$ is sufficiently close to 0, and k, h satisfy the inequality

$$\left(1+\frac{\varepsilon^2}{2}\right)k + \frac{\varepsilon^2}{2}h < \frac{\varepsilon^2}{2},$$

then it follows from (3.3) that $\Theta(\varepsilon) < 0$. Thus, there exists ξ^- at which $\Theta(\xi^-) = 0$. The inequality describes a triangular region at the lower left corner of R. Clearly, this region is in S^- and so S^- is non-empty. Both S^+ and S^- are open since y and Θ depend continuously on the initial data.

It follows from Lemma 1 that the complement C of S^+ and S^- is non-empty. For $(h,k) \subset C$, Θ can be continued to $\xi = \infty$, satisfying $0 < \Theta < 1$ for $0 < \xi < \infty$.

LEMMA 2. *For $(h,k) \subset C$, we have $\lim_{\xi\to\infty} y = \delta$, where $0 \le \delta < h$.*

PROOF. Since $y = h\exp\{-\int_0^\xi F(\Theta(t))\,dt\}$ and $F(\Theta)$ is non-negative, y is monotonic decreasing. Thus, we have $\lim_{\xi\to\infty} y = \delta$, where $0 \le \delta < h$.

LEMMA 3. *For $(h,k) \subset C$, φ has no positive maximum.*

PROOF. Suppose $\tilde{\xi}$ is a local positive maximum for φ. Then we have $\varphi(\tilde{\xi}) > 0$, $\varphi'(\tilde{\xi}) = 0$, $\varphi''(\tilde{\xi}) \leq 0$. Since $y' < 0$ in $(0,\infty)$, it follows from $\Theta' + \varphi' + y' = 0$ that $\Theta'(\tilde{\xi}) > 0$. Equation (2.3) then implies $\varphi''(\tilde{\xi}) > 0$, contradicting $\varphi''(\tilde{\xi}) \leq 0$. Thus, φ has no positive maximum.

LEMMA 4. *For $(h,k) \subset C$, $\varphi < 0$ in $(0,\infty)$.*

PROOF. Suppose $\varphi > 0$ at some ξ. Since Lemma 3 shows that φ has no positive maximum, either φ becomes unbounded or φ tends to a positive constant as $\xi \to \infty$. Now, $(h,k) \in C$ implies that both Θ and y are bounded. The conservation statement $\Theta + \varphi + y = 1$ then implies that φ cannot become unbounded. Thus, φ must tend to a positive constant, which, together with Lemma 1, implies that Θ tends to a constant, and Θ' tends to zero. However, equation (2.3) is then violated. Thus, φ must be negative in $(0,\infty)$.

LEMMA 5. *For $(h,k) \subset C$, Θ has no local minimum*

PROOF. From $\theta'' + \varphi'' + y'' = 0$ and equation (2.3), we have

$$\Theta'' + G(\Theta)\Theta' = \varphi - y''. \tag{3.4}$$

Suppose $\bar{\xi}$ is a local minimum for Θ. Then $\Theta'(\bar{\xi}) = 0$, $\Theta''(\bar{\xi}) \geq 0$. Since $y''(\bar{\xi}) = y^2(\bar{\xi})F(\Theta(\bar{\xi})) > 0$, and $\varphi < 0$ in $(0,\infty)$, equation (3.4) implies that $\Theta''(\bar{\xi}) < 0$, contradicting the supposition that Θ has a local minimum at $\bar{\xi}$. Thus, θ has no local minimum.

LEMMA 6. *For $(h,k) \subset C$, we have $\lim_{\xi\to\infty} y = 0$, $\lim \Theta = 1$ and $\lim_{\xi\to\infty} \varphi = 0$.*

PROOF. Since Θ has no local minimum, Θ does not oscillate. Hence, Θ tends to a constant as ξ tends to ∞. Now, $y' < 0$ together with $\Theta' + \varphi' + y' = 0$ imply that $(\Theta' + \varphi)' > 0$ in $(0,\infty)$. Since $\Theta + \varphi < 1$, $\Theta + \varphi$, and hence φ, tend to constants. If φ does not tend to zero, equation (2.3) will be violated. Thus, $\lim_{\xi\to\infty} \varphi = 0$. Since $\lim_{\xi\to\infty} y = \delta < h < 1$, the conservation statement $\Theta + \varphi + y = 1$ implies that Θ must tend to a positive constant, which in turn implies that $F(\Theta)$ tends to a positive constant. Thus, it follows from $y = h\exp\{-\int_0^\xi F(\Theta(t))\,dt\}$, that $\lim_{\xi\to\infty} y = \delta = 0$, and hence $\lim_{\xi\to\infty} \Theta = 1$. It also follows from Lemma 5 that $\Theta' > 0$ in $(0,\infty)$.

The combined results of Lemma 1 to Lemma 6 prove the following theorem.

THEOREM 1. *There is at least one solution to the initial value problem consisting of equations* (2.1), (2.2a) *and* (2.3) *on the interval* $(0,\infty)$ *with initial conditions* $y(0) = h$, $\Theta(0) = k$, $\varphi(0) = 1 - h - k$, *where* $0 < h < 1$, $0 < k < 1$ *are suitably chosen.*

3b. The extension to $-\infty < \xi < 0$.

In extending the solution to $(-\infty, 0)$, we first recall that since

$$y(\xi) = \exp\left\{-\int_{-\infty}^{\xi} F(\Theta(t))\,dt\right\} = h\exp\left\{-\int_{0}^{\xi} F(\Theta(t))\,dt\right\}, \quad -\alpha < \xi < \infty,$$

we have $\lim_{\xi\to-\infty} y = 1$ regardless of the value of h, as long as $0 < h < 1$. Further, the location $\xi = 0$ was chosen to give $y''(0) = 0$, $y'' < 0$ for $-\infty < \xi < 0$. To proceed, we make the change of variables $\eta = -\xi$, so that equations (2.1a), (2.2a) and (2.3) become

$$\dot{\Theta} + \dot{\varphi} = -yF, \tag{3.5}$$

$$\dot{y} = yF, \tag{3.6}$$

$$\ddot{\varphi} = \varphi - G(\Theta)\Theta, \tag{3.7}$$

where the dot denotes differentiation with respect to η. The initial conditions are

$$\Theta(0) = k,\ y(0) = h,\ \varphi(0) = 1 - h - k. \tag{3.8}$$

We wish to show that the solution of the initial value problem (3.5) to (3.8) can be extended to $\eta = \infty$, satisfying the conditions

$$y = 1, \quad \Theta = 0, \quad \varphi = 0 \qquad \text{at } \eta = \infty. \tag{3.9}$$

As we have remarked, the condition $\lim_{\eta\to\infty} y = 1$ is satisfied, and also $\ddot{y} < 0$ in $(0,\infty)$. It remains to show that $\lim_{\eta\to\infty}\Theta = \lim_{\eta\to\infty}\varphi = 0$.

LEMMA 7. *φ has no positive maximum in $\Theta < \eta < \infty$.*

PROOF. The proof follows the same line as that for Lemma 3.

LEMMA 8. *Θ does not become negative.*

PROOF. We have $\Theta(0) = k$, $\dot{\Theta}(0) < 0$. If Θ vanishes at a finite point, say $\tilde{\eta}$, then the conditions $y', y'' < 0$, $0 < \eta < \infty$, $\lim_{\eta\to\infty} y = 1$ imply that $y(\tilde{\eta}) < 1$. The conservation statement $\Theta + \varphi + y = 1$ then implies $\varphi(\tilde{\eta}) > 0$. Since φ has no positive maximum, it follows that $\varphi(\eta) > \varphi(\tilde{\eta})$ for $\eta > \tilde{\eta}$. Now $y(\eta) \geq y(\tilde{\eta})$ for $\eta > \tilde{\eta}$ implies that $\Theta(\eta) < \Theta(\tilde{\eta})$ for $\eta > \tilde{\eta}$. Thus $F(\Theta) = 0$ for $\eta \geq \tilde{\eta}$, and hence $\dot{y} = 0$, $y = y(\tilde{\eta}) < 1$ for $\eta > \tilde{\eta}$, which contradicts $\lim_{\eta\to\infty} y = 1$. Hence, Θ does not vanish at a finite η.

LEMMA 9. $\lim_{\eta\to\infty} \Theta = \lim_{\eta\to\infty} \varphi = 0$.

PROOF. Since $\dot{\Theta}(0) < 0$ and Θ does not become negative, either Θ oscillates or $\lim_{\eta\to\infty} \Theta = \varepsilon \geq 0$. If Θ oscillates, it must have a local minimum. Suppose $\Theta(\bar{\eta}) > 0$, $\dot{\Theta}(\bar{\eta}) = 0$, $\ddot{\Theta}(\bar{\eta}) \geq 0$. From $\ddot{y} = \dot{y}F(\Theta) + y(dF/d\Theta)\dot{\Theta}$, we have $\ddot{y}(\bar{\eta}) = \dot{y}F(\Theta) > 0$, contradicting the condition $\ddot{y}(\eta) < 0$ for $0 < \eta < \infty$. Thus, Θ does not oscillate, and so $\lim_{\eta\to\infty} \Theta = \varepsilon \geq 0$. If $\varepsilon > 0$, then $\dot{y} = yF(\varepsilon) > 0$ as $\eta \to \infty$, contradicting $\lim_{\eta\to\infty} y = 1$. Hence $\varepsilon = 0$. The conservation statement $\Theta + \varphi + y = 1$ then implies $\lim_{\eta\to\infty} \varphi = 0$. Since φ has no positive maximum, $\lim_{\eta\to\infty} \varphi = 0$ implies that $\varphi < 0$ for $0 < \eta < \infty$.

Taken together, the Lemmas in 3-b prove that if $(h, k) \subset C$ as defined in 3-a, then the boundary value problem (2.1a), (2.2a), (2.3), (2.4) and (2.5) has a solution with the property that $0 < y < 1$, $y' < 0$, $0 < \Theta < 1$, $\Theta' > 0$, $\varphi = 1 - \Theta - y$, $\varphi < 0$ in $-\infty < \xi < \infty$. The eigenvalue b can be obtained by integrating equation (2.3) from $-\infty$ to $+\infty$. We have

$$b = \frac{\beta}{4\alpha}[1 - (1 - \alpha)^4] \left\{ - \int_{-\infty}^{\infty} \varphi \, d\xi \right\}^{-1}. \tag{3.10}$$

4. Linearisation and asymptotic approximations

While the existence proof by the shooting method is constructive in that it sheds some light on the behaviour of the solution, it offers no procedure for the determination of $y(0) = h$, $\Theta(0) = k$ and the eigenvalue b. To obtain an estimate of these qualities, we linearise the governing equations from which approximations of y, Θ and φ are constructed. The quantities h, k and b are then obtained as solutions of transcendental equations. If we make use of the property $\beta \gg 1$, we can simplify the transcendental equations to make the dependence of h, k and b on the other given parameters more discernible.

We first consider the interval $0 < \xi < \infty$. Since $\Theta' > 0$, we have $\Theta(0) = k < \Theta < 1 = \Theta(\infty)$. For $0 < \Theta < 1$, $F(\Theta)$ is an increasing function of Θ. Thus, we have

$$-yF(a) < y' = -yF(\Theta) < -yF(k)$$

and hence

$$h \exp\{-F(1)\xi\} < y < h \exp\{-F(k)\xi\}.$$

We could approximate y by $h \exp\{-F((k + 1)/2)\xi\}$, but the expression is quite cumbersome. Now, our choice of the location $\xi = 0$ where $y' < 0$, $y'' = 0$ suggests that the reactant y is being consumed in that neighbourhood, and the temperature Θ is close to its burned value. For the sake of simplicity, we therefore

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approximate y by

$$y = h\exp(-F_1\xi) \tag{4.1}$$

where $F(1)$ has been denoted by F_1. From equations (2.1), (2.2) and (2.3) we have

$$\Theta'' + G(\Theta)\Theta' - \Theta = y - 1 - y''. \tag{4.2}$$

To linearise equation (4.2), we first replace terms on the right side by corresponding terms obtained from (4.1), and then we have to linearise $G(\Theta)$. We observe that $(1-\alpha)^3\beta b^{-1} = G(0) < G(\Theta) < G(1) = \beta b^{-1}$. Thus, replacing $G(\Theta)$ by a positive constant does not alter its qualitative behavior. In the vicinity of $\xi = 0$, where the behavior of the solutions should be approximated as closely as possible, we have observed that Θ must be close to unity. Hence, we linearise (4.2) to

$$\Theta'' + G_1\Theta' - \Theta = h(1 - F_1^2)\exp(-F_1\xi) - 1, \tag{4.3}$$

where we have denoted $G(1)$ by G_1.

Solving equation (4.3), subject to $\Theta(0) = k$ and $\Theta(\infty) = 1$, we have

$$\Theta = 1 + A\exp(-\mu\xi) + B\exp(-F_1\xi) \tag{4.4}$$

where

$$A = k - 1 - h(b^2 - \lambda^2)/(\lambda^2 - \beta b - 1),$$
$$\mu = \frac{\beta}{2b}\{1 + (1 + (4b^2/\beta^2))^{1/2}\},$$
$$B = h(b^2 - \lambda^2)/(\lambda^2 - \beta b - 1).$$

From (4.4), we have

$$\Theta'(0) = -A\mu - BF_1.$$

To obtain a relation among b, h and k, we ask that $\Theta'(0)$ as determined from (4.4) be equal to its exact value; that is, we ask

$$-A\mu - BF_1 = F^2(k)\bigg/\frac{df(k)}{d\Theta}. \tag{4.5}$$

Next, we consider the interval $-\infty < \xi < 0$, or $0 < \eta < \infty$. Since we know how y should behave in this interval, we simply extend the solution for y obtained in $0 < \xi < \infty$ to $0 \le \eta < \infty$, by demanding that the slope at $\xi = 0$ be continuous.

We have

$$y = 1 - (1-h)\exp(-hF_1\eta/(1-h)). \tag{4.6}$$

In the same manner that we linearise equation (4.2) to (4.3), we now linearise the Θ-equation to

$$\ddot{\Theta} - G_1\dot{\Theta} - \Theta = [h^2F_1^2(1-h)^{-1} - (1-h)]\exp(-hF_1\eta/(1-h)) \tag{4.7}$$

whose solution subject to $\Theta(0) = k$ and $\Theta(\infty) = 0$, is

$$\Theta = (k - D)\exp(-\nu\eta) + D\exp(-hF_1\eta/(1-h)), \tag{4.8}$$

where

$$D = \frac{(1-h)[h^2F_1^2 - (1-h)^2]}{hF_1^2 + G_1F_1h(1-h) - (1-h)^2}$$

and

$$\nu = \tfrac{1}{2}[(G_1^2 + 4)^{1/2} - G_1].$$

From (4.8), we have

$$\dot{\Theta}(0) = -\nu(k - D) - hF_1D/(1-h). \tag{4.9}$$

To obtain a second relation among b, h and k, we ask that $\dot{\Theta}(0)$ as determined from (4.9) be equal to its exact value; that is, we ask

$$\nu(k - D) + hF_1D/(1-h) = F^2(k)\Big/\frac{dF(k)}{d\Theta}. \tag{4.10}$$

The conditions (4.5) and (4.10) ensure that Θ and its first derivative are continuous in $(-\infty, \infty)$. Using the above approximations for y and Θ, we have $\varphi = 1 - \Theta - y$, and a third relation among b, h and k is given by (3.9):

$$\frac{\beta}{4\alpha b}((1-\alpha)^4 - 1) = \frac{(1-h-D)(1-h)}{hF_1} - \frac{(k-D)}{\nu} - \frac{A}{\mu} - \frac{h+B}{F_1}. \tag{4.11}$$

The three transcendental equations (4.5), (4.10) and (4.11), when solved, yield the values of b, h and k.

The solution of the equations (4.5), (4.10) and (4.11) is non-trivial. To gain some understanding of the dependence of h, k and b on the given parameters, further simplifications must be made. Consistent with our supposition that the temperature Θ is close to its burned state at $\xi = 0$, we simplify (4.5) and (4.10) by replacing the term $F^2(k)/(dF(k)/d\Theta)$ by $F^2(1)/(dF(1)/d\Theta)$. If we use the information $\beta \gg 1$ to further simplify (4.5) by omitting terms that are $O(\beta^{-2})$, we have

$$k = 1 - h(b^2 - \lambda^2)/(1 + b\beta - \lambda^2). \tag{4.12}$$

It follows from (4.12) that $b > \lambda$ for $k < 1$. In the same manner, we simplify (4.10) and (4.11). After a fair amount of algebraic manipulation, and using $k \doteq 1$, we obtain from (4.10)

$$b^2 = \frac{1}{h(1-h)}\{h^2\lambda\beta - \lambda[(1+S)(1-h) - h^2\lambda]\} \tag{4.13}$$

and from (4.11)

$$b^2 = \frac{\lambda\beta}{2h(1-h)}\{[(1-h)^2 f^2(\alpha) + 6h^2(1-h)f(\alpha) + h^4]^{1/2} - [(1-h)f(\alpha) + h^2]\} \tag{4.14}$$

where $f(\alpha) = \alpha(\alpha^2 - 4\alpha + 6)/4$.

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It is clear from both (4.13) and (4.14) that $b^2 = O(\lambda\beta)$. Equating the right sides of equations (4.13) and (4.14), and keeping only terms that are $O(\beta)$, we have

$$2h^2 + h - 1 = 2(1-h)f(\alpha) \tag{4.15}$$

for the determination of h. For $0.75 < \alpha < 0.95$ which is a reasonable range for α, the value h^* determined from (4.15) is $0.644 < h^* < 0.650$. Thus, h^* is not sensitive to the value of α. Using $h^* = 0.65$, and substituting into (4.13), we have

$$b^2 = 1.86\lambda\beta - 4.4\lambda[0.35(1+S) - 0.42]. \tag{4.16}$$

We emphasize that inasmuch as linearisation is a somewhat subjective procedure, the above result is intended to provide only a qualitative description of the solution. However, since the linearised equation does not alter the nature of the nonlinear equation, and yields a solution consistent with the features exhibited by the true solution, as demonstrated in the existence proof, we can have some confidence in its validity as an approximation.

5. Concluding remarks

We have a shooting method to construct an existence proof to a nonlinear eigenvalue problem over an infinite interval. The qualitative information obtained in the course of the proof was used to construct approximate solutions by linearisation. Asymptotic consideration based on $\beta \gg 1$ then yields information on the dependence of the eigenvalue on the given parameters. We believe this procedure may be useful in similar problems.

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